LOCAL COHOMOLOGY WITH SUPPORT IN A PARAMETER IDEAL

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Abstract. Motivated in part by an attempt to understand better the notion of parameter-like sequence introduced in [Ho3, (2.2)], we study results concerning the heights of the annihilators and the finiteness of the dimension of the socle in certain local cohomology modules with support in a parameter ideal. We obtain positive results under certain hypotheses of low dimension or codimension, but we also find examples that show, in the general case, that support in a parameter ideal does not restrict the behavior of local cohomology much more than support in an arbitrary ideal. The results obtained here strongly suggest that it would be worthwhile to seek a modification of the notion of parameter-like sequence introduced in [Ho3].

0. Introduction

All rings in this paper are assumed to be commutative, with unity and Noetherian, unless otherwise specified. Given a local ring, say R, if no other convention is made its maximal ideal is denoted by $\mathfrak{m}_R$ and its residue class field, i.e., $R/\mathfrak{m}_R$, is denoted by k_R. However, by $(R, \mathfrak{m}, k)$ we indicate that R is local with its maximal ideal being $\mathfrak{m}$ and its residue field being $k = R/\mathfrak{m}$. For a module M over a local ring $(R, \mathfrak{m}, k)$, the socle of M is defined as $(0 :_M \mathfrak{m})$ and is denoted by $\mathrm{soc}_R(M)$ or simply $\mathrm{soc}(M)$ if R is understood. The notation $H^i_I(M)$ is used for the i th local cohomology module of the module M with support in the ideal I. For background on local cohomology theory, we refer the reader to [BrSh], [GrHa], and [LC].

Given a part of system of parameters $\underline{x} = x_1, x_2, \ldots, x_n$ of R, we want to study the height of $\mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x})}(M))$, as well as the dimension of $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ as a vector space over the residue field k. In case R is complete and equidimensional, then by a result which may be found in [Ho3], R is a module-finite extension of a Gorenstein domain A that contains $x_1, x_2, \ldots, x_n$ as part of a system of parameters of A (while A may be chosen to be regular if R contains a field). For this reason, we may, in many cases, focus our attention on Gorenstein domains.

We list some of the results that are obtained in Section 4. We make the convention that the height of the unit ideal is $+\infty$. By *k-dimension* we mean dimension as a vector space over a field k.

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Theorem (See Theorem 4.1, Theorem 4.2)**.** *Let $(R, \mathfrak{m}, k)$ be a complete local domain or a local Gorenstein domain and M a finitely generated torsion-free R-module. Assume* $\dim(R) = d$. *Let $\underline{x} = x_1, x_2, \dots, x_n$ be a subsystem of parameters of R. Then*

(1) If $n = 0, 1, 2, d-1$ or d, $\mathrm{soc}_R(H^i_{(\underline{x})}(M))$ *has finite k-dimension for all $i \in \mathbb{N}$.*
(2) If M satisfies $\mathbf{S_{d-3}}$, then $\mathrm{soc}_R(H^i_{(\underline{x})}(M))$ *has finite k-dimension for all $i \in \mathbb{N}$.*
(3) If $d \le 4$, then $\mathrm{soc}_R(H^i_{(\underline{x})}(M))$ *has finite k-dimension for all $i \in \mathbb{N}$.*
(4) $\mathrm{Ann}_R(H^n_{(\underline{x})}(M))$ *has height* 0.
(5) If $n = 0, 1, 2, d-1$ or d, then $\mathrm{Ann}_R(H^i_{(\underline{x})}(M))$ *has height at least* 2 *for all $i \le n-1$.*

One of our main motivations for this paper is to better understand the notion of *parameter-like sequence* introduced in [Ho3], where it is was introduced for the purpose of defining an analogue of tight closure in mixed characteristic smaller than the notion of solid closure introduced in [Ho2]. Solid closure was shown to be too large to have the right properties in equal characteristic 0 in [Ro3]. A proof of the existence of a closure with suitable properties would, for example, settle the local homological conjecctures (cf. [Du], [Ho1], [Ro1], [Ro2]) for background on these).

Recall that the definition of parameter-like sequence involves a condition that certain local cohomology modules have annihilators with heights that are "large enough," which means that, in a sense, the local cohomology modules themselves are "small." Details are given below. Also recall that an R-module M is said to have *pure dimension d* is all its nonzero submodules have dimension d. In fact, M has pure dimension d if and only if $\dim(R/P) = d$ for all $P \in \mathrm{Ass}_R(M)$.

Definition 0.1 ([Ho3, (2.2)])**.** Let $(R, \mathfrak{m})$ be a complete local ring of pure dimension d, S an R-algebra, and $\underline{x} = x_1, \dots, x_n$ a (partial or full) system of parameters of R. Then let $\mathcal{T}_0(S)$ be the quotient of S by the ideal of all elements that have an annihilator of positive height in R, and recursively, if $\mathcal{T}_i(S)$ has been defined for $i < n$, then let $\mathcal{T}_{i+1}(S)$ be the quotient of $\mathcal{T}_i(S)/x_{i+1}\mathcal{T}_i(S)$ by the ideal of all elements $u \in \mathcal{T}_i(S)/x_{i+1}\mathcal{T}_i(S)$ such that $\dim(Ru) < n-(i+1)$. Then we call $\underline{x}$ *parameter-like* in S if $\mathcal{T}_n(S) \ne 0$, and for all $i = 0, 1, \dots, n-1$, the height of $\mathrm{Ann}_R\left(\mathrm{H}^{n-1-i}_{(\underline{x})}(\mathcal{T}_i(S))\right)$ (in R) is at least $i+2$. (Recall that the height of the unit ideal is $+\infty$.)

With this notion of parameter-like sequence, Hochster showed the following result in [Ho3, (2.3)]: If R is a complete local domain and $R \to S$ is a module-finite extension of domains, then every full system of parameters of R is a parameter-like sequence in S.

More generally, suppose that $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local extension of complete local domains such that $\mathrm{ht}(\mathfrak{m}S) = \mathrm{ht}(\mathfrak{m})$, i.e., some (or equivalently, every) system of parameters of R is a partial system of parameters of S. One might also hope that any partial or full system of parameters $\underline{x} = x_1, \dots, x_n$ of R (with $n \le \dim(R)$) is parameter-like in S and hence, in particular, that the height of $\mathrm{Ann}_R\left(\mathrm{H}^{n-1}_{(\underline{x})}(S)\right)$ is at least 2. However, the answer is not clear even in the case where $R = S$. Studying this type of question was one of the main motivations for the work in this paper. While we were able to obtain certain positive results, the examples in Section 5 show that this is not true in general: in fact, the annihilator $\mathrm{Ann}_R\left(\mathrm{H}^{n-1}_{(\underline{x})}(S)\right)$ can be 0 (see Example 5.3).

1. Preliminaries

Remark 1.1. It is straightforward to check that, given ideals I, J of a local ring $(R, \mathfrak{m})$ such that $I + J$ is $\mathfrak{m}$-primary, $\mathrm{H}^j_J(\mathrm{H}^i_I(M)) \cong \mathrm{H}^j_{\mathfrak{m}}(\mathrm{H}^i_I(M))$ for all $i, j \in \mathbb{N}$. In fact, for any ideals I, J of any Noetherian ring R and for any R-module M, we have $\mathrm{H}^j_J(\mathrm{H}^i_I(M)) \cong \mathrm{H}^j_{I+J}(\mathrm{H}^i_I(M)) \cong \mathrm{H}^j_{\sqrt{I+J}}(\mathrm{H}^i_I(M))$ for all $i, j \in \mathbb{N}$. To see this, it suffices to show $\mathrm{H}^j_J(\mathrm{H}^i_I(M)) \cong \mathrm{H}^j_{I+J}(\mathrm{H}^i_I(M))$. Then, by induction on the number of generators of I, it suffices to show, for any $x \in I$, $\mathrm{H}^j_J(\mathrm{H}^i_I(M)) \cong \mathrm{H}^j_{(x)+J}(\mathrm{H}^i_I(M))$ for all $i, j \in \mathbb{N}$. But this is a consequence of the following exact sequence

$$\cdots \to \mathrm{H}^{j-1}_J(\mathrm{H}^i_I(M)_x) \to \mathrm{H}^j_{(x)+J}(\mathrm{H}^i_I(M)) \to \mathrm{H}^j_J(\mathrm{H}^i_I(M)) \to \mathrm{H}^j_J(\mathrm{H}^i_I(M)_x) \to \cdots$$

and the fact that $\mathrm{H}^{j-1}_J(\mathrm{H}^i_I(M)_x) = \mathrm{H}^j_J(\mathrm{H}^i_I(M)_x) = 0$.

Lemma 1.2. *Let $(R, \mathfrak{m}, k)$ be a local ring and $0 \to M_1 \to M_2 \to M_3$ be an exact sequence of R-modules. If both $\mathrm{soc}(M_1)$ and $\mathrm{soc}(M_3)$ have finite dimension over k, then so does $\mathrm{soc}(M_2)$.*

Proof. This follows from the fact that $\mathrm{Hom}_R(\frac{R}{\mathfrak{m}}, \_)$ is left exact. $\square$

Lemma 1.3. *Let $(R, \mathfrak{m}, k)$ be a local ring, $\underline{x} = x_1, x_2, \ldots, x_n \in R$, and M be a finitely generated R-module. Then*

(1) For a fixed i, if $\mathrm{H}^j_{(\underline{x})}(M) = 0$ for all $j < i$, then $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}^i_{(\underline{x})}(M)) \cong \mathrm{H}^i_{\mathfrak{m}}(M)$ and thus $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite dimension as a k-vector space.

(1′) For $i \in \{0, 1\}$, $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite dimension as a k-vector space.

(2) If $n = 0, 1$ or if $\dim(R/(\underline{x})) = 0, 1$, then $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension for all i.

(3) If $\dim(R) \leq 3$ and $\underline{x}$ is part of a system of parameters of R, then the socle of $\mathrm{H}^i_{(\underline{x})}(M)$ has finite dimension as a k-vector space for all i.

Proof. Choose $\underline{y} = y_1, y_2, \ldots, y_c \in R$ such that their images form a full system of parameters of $R/(\underline{x})$. Form Cech complexes $C_{(\underline{x})}(M)$ and $C_{(\underline{y})}(R)$. Then

$$\mathrm{H}^i\left(C_{(\underline{x})}(M) \otimes_R C_{(\underline{y})}(R)\right) = \mathrm{H}^i_{\mathfrak{m}}(M)$$

for all i. One of the spectral sequences of the double complex $C_{(\underline{x})}(M) \otimes_R C_{(\underline{y})}(R)$ has $E_2^{p,q} = \mathrm{H}^q_{(\underline{y})}(\mathrm{H}^p_{(\underline{x})}(M))$ with maps $d_2^{p,q} : E_2^{p,q} \to E_2^{p-1,q+2}$ for all p, q.

(1) This follows from the above spectral sequence: By the assumption, we know that $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}^i_{(\underline{x})}(M)) = E_2^{i,0} = E_\infty^{i,0} \cong \mathrm{H}^i_{\mathfrak{m}}(M)$.

(1′) If $i = 0$, this is a special case of (1). If $i = 1$, then observe that $\mathrm{H}^1_{\underline{x}}(M) \cong \mathrm{H}^1_{(\underline{x})}(M/\mathrm{H}^0_{(\underline{x})}(M))$. Now apply (1) to $\mathrm{H}^1_{(\underline{x})}(M/\mathrm{H}^0_{(\underline{x})}(M))$ as $\mathrm{H}^0_{(\underline{x})}(M/\mathrm{H}^0_{(\underline{x})}(M)) = 0$.

(2) The cases where $n = 0$ and $\dim(R/(\underline{x})) = 0$ are straightforward. The case where $n = 1$ follows from (1′). The case where $\dim(R/(\underline{x})) = 1$ follows from the

above spectral sequence (mapping cone in this case): We have exact sequences

$$0 \to \mathrm{H}^1_{(y_1)}(\mathrm{H}^{i-1}_{(\underline{x})}(M)) \to \mathrm{H}^i_{\mathfrak{m}}(M) \to \mathrm{H}^0_{(y_1)}(\mathrm{H}^i_{(\underline{x})}(M)) \to 0$$

for all i. Since $\mathrm{H}^i_{\mathfrak{m}}(M)$ is an Artinian R-module and, by Remark 1.1, $\mathrm{H}^0_{(y_1)}(\mathrm{H}^i_{(\underline{x})}(M)) \cong \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}^i_{(\underline{x})}(M))$ for every i, we have that $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}^i_{(\underline{x})}(M))$ is Artinian over R. Hence $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension for every i.

(3) This follows immediately from (2). □

Before stating Lemma 1.5, we make a remark. For any ring R and any ideal I of R, the height of I is denoted by $\mathrm{ht}_R(I)$.

Remark 1.4. Let R be a ring, $P \in \mathrm{Spec}(R)$, $E(R/P)$ the injective hull of R/P over R (or over R_P), and M a finitely generated R-module. Notice that

$$\mathrm{Ann}_{R_P}(\mathrm{Hom}_{R_P}(M_P, E(R/P))) \supseteq \mathrm{Ann}_{R_P}(M_P) = (\mathrm{Ann}_R(M))_P.$$

This implies that $\mathrm{Ann}_R(\mathrm{Hom}_{R_P}(M_P, E(R/P))) \supseteq \mathrm{Ann}_R(M)$, which then implies

$$\begin{aligned}\mathrm{Ann}_{R_P}(\mathrm{Hom}_{R_P}(M_P, E(R/P))) &\supseteq (\mathrm{Ann}_R(\mathrm{Hom}_{R_P}(M_P, E(R/P))))_P \\ &\supseteq (\mathrm{Ann}_R(M))_P.\end{aligned}$$

By Matlis duality, we have $\mathrm{Hom}_{R_P}(\mathrm{Hom}_{R_P}(M_P, E(R/P)), E(R/P)) \cong \widehat{M_P}$, in which $\widehat{M_P}$ stands for the P_P-adic completion of M_P. This shows

$$\mathrm{Ann}_{R_P}(\mathrm{Hom}_{R_P}(M_P, E(R/P))) \subseteq \mathrm{Ann}_{R_P}(\widehat{M_P}) = \mathrm{Ann}_{R_P}(M_P) = (\mathrm{Ann}_R(M))_P.$$

Combining the above, we get

$$\begin{aligned}\mathrm{Ann}_{R_P}(\mathrm{Hom}_{R_P}(M_P, E(R/P))) &= (\mathrm{Ann}_R(\mathrm{Hom}_{R_P}(M_P, E(R/P))))_P \\ &= (\mathrm{Ann}_R(M))_P.\end{aligned}$$

Finally, we see

$$\mathrm{ht}_{R_P}\left(\mathrm{Ann}_{R_P}(\mathrm{Hom}_{R_P}(M_P, E(R/P)))\right) = \mathrm{ht}_R\left(\mathrm{Ann}_R(\mathrm{Hom}_{R_P}(M_P, E(R/P)))\right).$$

(Indeed, if an ideal I of R satisfies $I = \{r \in R \,|\, r/1 \in I_P\}$, then $\mathrm{ht}_{R_P}(I_P) = \mathrm{ht}_R(I)$.)

The next lemma will be used in proving Theorem 4.2. Notice that the local version of the lemma was proved in [Ho3, Lemma 2.1(b)].

Lemma 1.5 (Compare with [Ho3, Lemma 2.1(b)])**.** *Let R be a Gorenstein ring, S be a multiplicatively closed subset of R with $\dim(S^{-1}R) = n$, $\underline{x} = x_1, x_2, \dots, x_n$ be a sequence of elements in R such that $\mathrm{ht}_{S^{-1}R}(S^{-1}(\underline{x})) = n$, and M be a finitely generated torsion-free R-module. Then $\mathrm{ht}_R(\mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x})}(S^{-1}M))) \geq 2$ for every $i \leq n-1$.*

Proof. Say that $P_1, P_2, \dots, P_r$ are all the minimal prime ideals over $(\underline{x})R$ that do not intersect with S. Then $S^{-1}P_1, S^{-1}P_2, \dots, S^{-1}P_r$ are all the minimal prime ideals over $S^{-1}(\underline{x})$, which are also maximal ideals of $S^{-1}R$. Then, for every i, we have a natural isomorphism

$$\mathrm{H}^i_{(\underline{x})}(S^{-1}M) \cong \bigoplus_{j=1}^{r} \mathrm{H}^i_{(\underline{x})}(M_{P_j})$$

since every element in $\mathrm{H}^i_{(\underline{x})}(S^{-1}M)$ is killed by a power of $S^{-1}(\underline{x})$. Hence

$$\mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x})}(S^{-1}M)) = \bigcap_{j=1}^{r} \mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x})}(M_{P_j})).$$

Thus, for each $i \leq n-1$, it suffices to show that $\mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x})}(M_{P_j}))$ (as an ideal of R) has height ≥ 2 for every $1 \leq j \leq r$. By Remark 1.4, it suffices to show that $\mathrm{Ann}_{R_{P_j}}(\mathrm{H}^i_{(\underline{x})}(M_{P_j}))$ (as an ideal of R_{P_j}) has height ≥ 2 for every $i \leq n-1$ and every $1 \leq j \leq r$.

Therefore we may assume that R is Gorenstein local and $\underline{x}$ is a system of parameters without loss of generality. For completeness, we provide a proof of this local case, although it is essentially the same as that of [Ho3, Lemma 2.1(b)]. Say that our local Gorenstein ring is $(R, \mathfrak{m}, k)$ and $E := E_R(k)$ is the injective hull of the residue field k. By local duality, we have $\mathrm{H}^i_{(\underline{x})}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M,R), E)$. Therefore it is enough to prove $\mathrm{Ann}_R(\mathrm{Ext}^{n-i}_R(M,R))$ has height ≥ 2 for every $i \leq n-1$. Suppose, on the contrary, that $\mathrm{Ann}_R(\mathrm{Ext}^{n-i}_R(M,R)) \subseteq P \in \mathrm{Spec}(R)$ with $\mathrm{ht}(P) \leq 1$ for some $i \leq n-1$. Then $\mathrm{Ext}^{n-i}_{R_P}(M_P, R_P) \neq 0$, which contradicts the fact that $\mathrm{Ext}^{n-i}_{R_P}(M_P, R_P) = 0$ since M_P is torsion-free over R_P and $n - i \geq 1$ while R_P has injective dimension equal to $\dim(R_P) \leq 1$. □

2. Partial $\mathbf{S_2}$-ification

Let $A \subseteq (R, \mathfrak{m})$ be an extension of domains such that R is local satisfying $\mathbf{S_2}$ and M be a finitely generated torsion-free R-module. Denote $M^* := \mathrm{Hom}_R(M, R)$. Then the natural R-homomorphism $h : M \to M^{**}$ is injective. We will identify M with the R-submodule $h(M)$ of M^{**}.

Let $\underline{x} = x_1, x_2, \ldots, x_n \in A$, where $n \geq 2$, be a sequence such that $\mathrm{ht}((\underline{x})R) = n$. We will construct what we call a *partial* $\mathbf{S_2}$*-ification* of M on $I = (\underline{x})A$, which will be denoted by $M_I^{**\mathrm{p}}$ or simply $M^{**\mathrm{p}}$ if I is understood. Before constructing $M^{**\mathrm{p}}$ explicitly, we need a definition. We say that $y, z \in A$ are *special* in I if there exist $x'_1, \ldots, x'_{n-2} \in A$ such that $\sqrt{(x'_1, \ldots, x'_{n-2}, y, z)A} = \sqrt{IA}$. Now we define $M_I^{**\mathrm{sp}}$ as the R-submodule of M^{**} generated by

$$\{\alpha \in M^{**} \,|\, y\alpha, z\alpha \in M \text{ for some } y, z \text{ special in } I\}.$$

Notice that $(M_I^{**\mathrm{sp}})^{**} = M^{**}$, which, by induction, shows that

$$M_I^{**\mathrm{sp}} \subseteq (M_I^{**\mathrm{sp}})_I^{**\mathrm{sp}} \subseteq ((M_I^{**\mathrm{sp}})_I^{**\mathrm{sp}})_I^{**\mathrm{sp}} \subseteq \cdots (\subseteq M^{**})$$

form an ascending chain of R-submodules of M^{**}. This chain stabilizes by the Noetherian assumption. We call the stable submodule, denoted by $M_I^{**\mathrm{p}}$, of M^{**}, the *partial* $\mathbf{S_2}$-ification of M on I. (The subring A could be as small as the subring of R generated over the prime subring by $x_1, x_2, \ldots, x_n$ and could also be as large as R itself. Our notion of partial $\mathbf{S_2}$-ification depends strongly on the choice of A. Given $x_1, x_2, \ldots, x_n$, the larger the ring A is, the larger $M_I^{**\mathrm{p}}$ is.)

Lemma 2.1. *Keeping the above assumptions and notations, we have*

(1) If $y, z \in A$ are special in I, then y, z form a regular sequence on $M_I^{**\mathrm{p}}$.

(2) $\mathrm{H}^n_I(M) \cong \mathrm{H}^n_I(M_I^{**\mathrm{p}})$.

Proof. (1) Given $y, z \in A$ which are special in I, it suffices to show that if $yu = zv$ for some $u, v \in M_I^{**\mathrm{p}}$ then $v \in yM_I^{**\mathrm{p}}$. Since R satisfies $\mathbf{S_2}$, y, z form a regular sequence on M^{**}. Therefore $v = yw, u = zw$ for some $w \in M^{**}$, which forces $w \in (M_I^{**\mathrm{p}})_I^{**\mathrm{sp}}) = M_I^{**\mathrm{p}}$ by the construction of $M_I^{**\mathrm{p}}$.

(2) It is enough to show $\mathrm{H}^n_I(M) \cong \mathrm{H}^n_I(M_I^{**\mathrm{sp}})$, which, by the construction of $M_I^{**\mathrm{sp}}$, reduces to showing $\mathrm{H}^n_I(M) \cong \mathrm{H}^n_I(M + R\alpha)$ with $\alpha \in M^{**}$ such that $y\alpha, z\alpha \in M$ for some y, z special in I. For this, consider the short exact sequence

$$0 \to M \to M + R\alpha \to \frac{M + R\alpha}{M} \to 0,$$

which gives a long exact sequence

$$\cdots \to \mathrm{H}^{n-1}_I\left(\tfrac{M+R\alpha}{M}\right) \to \mathrm{H}^n_I(M) \to \mathrm{H}^n_I(M + R\alpha) \to \mathrm{H}^n_I\left(\tfrac{M+R\alpha}{M}\right) \to 0.$$

Since there exist $x'_1, \dots, x'_{n-2} \in A$ such that $\sqrt{(x'_1, \dots, x'_{n-2}, y, z)A} = \sqrt{IA}$ and $y, z \in \mathrm{Ann}\left(\frac{M+R\alpha}{M}\right)$, we have

$$\mathrm{H}^i_I\left(\tfrac{M+R\alpha}{M}\right) \cong \mathrm{H}^i_{(x'_1,\dots,x'_{n-2})}\left(\tfrac{M+R\alpha}{M}\right) = 0$$

for $i = n-1, n$. Thus $\mathrm{H}^n_I(M) \cong \mathrm{H}^n_I(M + R\alpha)$. ☐

3. Weak syzygies

In the situation of Lemma 3.1 below, we think of M_r as a *weak syzygy* of M_0 in a somewhat technical sense.

Lemma 3.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and I be an ideal of R. For integers $r > 0$, $e \geq 0$ and finitely generated R-modules M_0, M_r, suppose there exists an exact sequence*

$$0 \to M_r \to G_{r-1} \to G_{r-2} \to \cdots \to G_1 \to G_0 \to M_0 \to 0$$

of finitely generated R-modules such that $\operatorname{depth}_I(G_i) \geq e + i + 1$ for all $0 \leq i \leq r-1$. Then

(1) $\mathrm{H}^e_I(M_0)$ is isomorphic to an R-submodule of $\mathrm{H}^{e+r}_I(M_r)$.

(2) $\operatorname{soc}(\mathrm{H}^e_I(M_0))$ has finite dimension over k if and only if $\operatorname{soc}(\mathrm{H}^{e+r}_I(M_r))$ does.

Proof. For every $1 \leq i \leq r-1$, there exists $M_i \subseteq G_{i-1}$ such that

$$0 \to M_i \to G_{i-1} \to M_{i-1} \to 0$$

is exact for every $i = 1, 2, \dots, r$. Then, using the long exact sequence for local cohomology and the fact that $\mathrm{H}^{e+i-1}_I(G_{i-1}) = 0$, we have an exact sequence

$$0 \to \mathrm{H}^{e+i-1}_I(M_{i-1}) \to \mathrm{H}^{e+i}_I(M_i) \to \mathrm{H}^{e+i}_I(G_{i-1})$$

for every $i = 1, 2, \dots, r$. Part (1) follows immediately.

(2) By part (1), we see that if $\operatorname{soc}(\mathrm{H}^{e+r}_I(M_r))$ has finite dimension over k, then so does $\operatorname{soc}(\mathrm{H}^e_I(M_0))$. On the other hand, for each $i = 1, 2, \dots, r$, we notice that $\mathrm{H}^j_I(G_{i-1}) = 0$ for all $j < e+i$, which implies that $\operatorname{soc}(\mathrm{H}^{e+i}_I(G_{i-1}))$ has finite dimension over k by Lemma 1.3 (1). Therefore the other direction ("only if") of the conclusion follows from applying Lemma 1.2 repeatedly. ☐

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. Let $\underline{x} = x_1, x_2, \ldots, x_n$ be part of a system of parameters of R. Then, for any finitely generated R-module N and any $e \in \mathbb{N}$, $\mathrm{H}^e_{(\underline{x})}(N)$ is isomorphic to an R-submodule of $\mathrm{H}^n_{(\underline{x})}(M)$ for some finitely generated R-module M.*

Proof. We may assume $0 \le e < n$. Let M be a $(n-e)$-th syzygy of N. By the above lemma, we have that $\mathrm{H}^e_{(\underline{x})}(N)$ is isomorphic to an R-submodule of $\mathrm{H}^n_{(\underline{x})}(M)$. □

Remark 3.3. Let $(R, \mathfrak{m}, k)$ be a complete equidimensional local ring (e.g., a complete domain) and $\underline{x} = x_1, x_2, \ldots, x_n \in R$ be a part of a system of parameters of R. By a result in [Ho3, (2.2) (a)], R is module-finite extension of a Gorenstein local subring A containing $\underline{x}$ as a part of a system of parameters. For this reason, we may do the following:

(1) If we want to investigate whether $\mathrm{H}^e_{(\underline{x})}(M)$ has height 2 annihilator, we may assume without loss of generality that R is Gorenstein (hence Cohen-Macaulay).
(2) If we want to investigate whether $\mathrm{H}^e_{(\underline{x})}(M)$ has a finite dimensional socle, we may assume without loss of generality that R is Gorenstein (hence Cohen-Macaulay) and (by the preceding corollary) it suffices to study the same questions for $\mathrm{H}^n_{(\underline{x})}(M)$, the highest cohomology, for all R-modules M.

4. Two Main Results

Theorem 4.1. *Let $(R, \mathfrak{m}, k)$ be a complete local domain or a local Gorenstein domain and M a finitely generated R-module. Assume* $\dim(R) = d$. *Let $\underline{x} = x_1, x_2, \ldots, x_n$ be part of a system of parameters of R. Then*

(1) If $n = 0, 1, d-1$ or d, then $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension for all i.
(2) If, for some j, M satisfies $\mathbf{S_{j-1}}$, then $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension for all $i \le j$.
(2′) If M satisfies $\mathbf{S_{n-1}}$, then $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension for all i.
(2″) If M is torsion-free over R and $n = 2$, then $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension for all i.
(3) If M satisfies $\mathbf{S_{d-3}}$, then $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension for all i.
(4) If $d \le 4$ and M is torsion-free over R, then $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite dimension as a k-vector space for all i.

Proof. This reduces to the Gorenstein case. Moreover, we may restrict our attention to the cases when $i \le n$.

(1) This follows from Lemma 1.3 (2) immediately.

(2) It is enough to prove the case when $i = j$, which follows from the fact that M is a $(j-1)$-th syzygy by [EG, Theorem 3.8, p. 51]. Indeed, since M is a $(j-1)$-th syzygy of a finitely generated R-module, say N, then, by Lemma 3.1, it suffices to show that $\mathrm{H}^1_{(\underline{x})}(N)$ has a finite dimensional socle. But this is covered in Lemma 1.3 (1′).

(2′) This is a special case of part (2) above.

(2″) This case is a special case of (2′) above. But it also follows from partial $\mathbf{S_2}$-ification of M on $I = (x_1, x_2)R$ and Lemma 1.3 (1): Indeed, we have an exact sequence $0 \to M \to M_I^{**_\mathrm{p}} \to N \to 0$ of finitely generated R-modules with $\sqrt{\mathrm{Ann}_R(N)} \supseteq I$.

Thus we have $\mathrm{H}^1_I(N) = \mathrm{H}^2_I(N) = 0 = \mathrm{H}^0_I(M_I^{**\mathrm{p}}) = \mathrm{H}^1_I(M_I^{**\mathrm{p}})$, which gives $\mathrm{H}^2_I(M) \cong \mathrm{H}^2_I(M_I^{**\mathrm{p}})$. But $\mathrm{H}^2_I(M_I^{**\mathrm{p}})$ has a finite dimension socle by Lemma 1.3 (1). When $i < 2$, it follows from Lemma 1.3 (1′) that $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has finite k-dimension.

(3) This follows from parts (1) and (2′) combined. Indeed, by (1), we only need to consider the case where $n \le d-2$. But $n \le d-2$ and $\mathbf{S_{d-3}}$ imply $\mathbf{S_{n-1}}$ on M, which, by (2′), shows that $\mathrm{soc}_R(\mathrm{H}^i_{(\underline{x})}(M))$ is a finite-dimensional k-vector space for all i.

(4) This follows from (1) and (2″) combined, and also from (3). □

Theorem 4.2. *Let R be a domain with $\dim(R) = d$, M a finitely generated torsion-free R-module, and $\underline{x} = x_1, x_2, \ldots, x_n$ elements of R. Then*

1. *If R is Gorenstein and $\mathrm{ht}((\underline{x})R) = h$, then $\mathrm{Ann}_R(\mathrm{H}^h_{(\underline{x})}(M))$ has height 0.*
2. *If $(R, \mathfrak{m}, k)$ be a complete local domain and $\underline{x}$ is part of a system of parameters of R, then $\mathrm{Ann}_R(\mathrm{H}^n_{(\underline{x})}(M))$ has height 0.*
3. *If $\underline{x}$ is part of a system of parameters of R and $n = 0, 1, 2, d-1$ or d, then $\mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x})}(M))$ has height at least 2 for $i \le n-1$.*

Proof. (1) Choose $P \in \mathrm{Spec}(R)$ such that $(\underline{x})R \subseteq P$ and $\mathrm{ht}\, P = h$. Then it follows from the proof of [Ho3, (2.1) (c)] that $\mathrm{Ann}_{R_P}(\mathrm{H}^h_{(\underline{x})}(M_P))$, which contains $(\mathrm{Ann}_R(\mathrm{H}^h_{(\underline{x})}(M)))_P$, has height 0 in R_P. Hence $\mathrm{Ann}_R(\mathrm{H}^h_{(\underline{x})}(M))$ has height 0 in R.

(2) This reduces to the (complete) Gorenstein case by [Ho3, (2.1) (a)]. Then this is a special case of (1).

(3) Denote $I = (\underline{x})R$. The cases where $n = 0, 1$ are straightforward while the case where $n = d$ is covered in [Ho3] (also in Lemma 1.5). If $n = 2$, then the exact sequence

$$0 \to M \to M_I^{**\mathrm{p}} \to N \to 0$$

as in the proof of Theorem 4.1 (2″) implies that $\mathrm{H}^1_I(M) \cong \mathrm{H}^0_I(N) = N$, whose annihilator has height ≥ 2 in R. Finally, suppose $n = d-1$. Choose $x \in \mathfrak{m}$ such that $\underline{x}' = x_1 \ldots, x_{d-1}, x$ is a system of parameters for R. Then, for each i, we have an exact sequence (by using a spectral sequence (as in the proof of Lemma 1.3) or by using the mapping cone)

$$\cdots \to \mathrm{H}^i_{(\underline{x}')}(M) \to \mathrm{H}^i_{(\underline{x})}(M) \to \mathrm{H}^i_{(\underline{x})}(M_x) \to \cdots .$$

As we already know that $\mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x}')}(M))$ has height ≥ 2 for every $i \le d-1$ (which is the case of $n = d$), it suffices to prove that $\mathrm{Ann}_R(\mathrm{H}^i_{(\underline{x})}(M_x)$ (as an ideal of R) has height ≥ 2 for every $i \le n-1 = d-2$. But the latter statement is evident by applying Lemma 1.5 to R, $S = \{x^m \mid m \in \mathbb{N}\}$, $\underline{x}$ and M. □

5. Examples

Example 5.1 (Hartshorne)**.** Let $k[[U, V, X, Y]]$ be a formal power series ring over a field k in variables U, V, X, Y. Then $\mathrm{H}^2_{(X,Y)}\left(\frac{k[[U,V,X,Y]]}{(UX-VY)}\right)$ has infinite dimensional socle.

(For a quick proof of this, we notice that $\frac{k[[U,V,X,Y]]}{(UX-VY)}$ may be identified with the subring $k[[X,Y,XT,YT]]$ of $k[[X,Y,T]]$ and hence

$$\operatorname{soc}\Big(\mathrm{H}^2_{(X,Y)}\Big(\frac{k[[U,V,X,Y]]}{(UX-VY)}\Big)\Big) \cong (0:_{\mathrm{H}^2_{(X,Y)}(k[[X,Y,XT,YT]])}(X,Y,XT,YT)).$$

We know that

$$\mathrm{H}^2_{(X,Y)}(k[[X,Y,XT,YT]]) \cong \frac{k[[X,Y,XT,YT]]_{XY}}{k[[X,Y,XT,YT]]_X + k[[X,Y,XT,YT]]_Y}.$$

For any $n \in \mathbb{N}$, let

$$a_n = \frac{X^n(YT)^n}{(XY)^{n+1}} = \frac{T^n}{XY} \in k[[X,Y,XT,YT]]_{XY} \subset k[[X,Y,T]]_{XY}$$

and

$$\alpha_n \in \frac{k[[X,Y,XT,YT]]_{XY}}{k[[X,Y,XT,YT]]_X + k[[X,Y,XT,YT]]_Y}$$

be the class of a_n.

Now it is straightforward to check that $\operatorname{Ann}(\alpha_n) = (X,Y,XT,YT)$ for all n and $\{\alpha_n \mid n \in \mathbb{N}\}$ is independent over k.)

Example 5.2. Let $R = k[[U,V,X,Y,Z]]$ be a formal power series ring over a field k in variables U,V,X,Y,Z and $P = (UX-VY,Z)$. Then $\mathrm{H}^2_{(X,Y,Z)}(\frac{R}{P}) \cong \mathrm{H}^2_{(X,Y)}\big(\frac{k[[U,V,X,Y]]}{(UX-VY)}\big)$ and $\mathrm{H}^2_{(X,Y,Z)}(\frac{R}{P})$ embeds into $\mathrm{H}^3_{(X,Y,Z)}(P)$, in which the embedding follows from the long exact sequence induced from the short exact sequence $0 \to P \to R \to R/P \to 0$. From Example 5.1, we immediately see that $\mathrm{H}^3_{(X,Y,Z)}(P)$ also has infinite dimensional socle. Now let $S = R \oplus X^{\frac{1}{2}}P$, which is a domain under the obvious addition and multiplication. Actually, S is module-finite over R and, as R-modules, $S = R \oplus X^{\frac{1}{2}}P \cong R \oplus P$. Therefore, S is complete local with its maximal ideal equal to $\mathfrak{m}_S = \mathfrak{m}_R + X^{\frac{1}{2}}P$ and residue field equal to k. Also, we have $\mathfrak{m}_S^2 \subset \mathfrak{m}_R S$, so that $\mathfrak{m}_R S$ is an $\mathfrak{m}_S$-primary ideal. Since

$$\operatorname{soc}_R(\mathrm{H}^3_{(X,Y,Z)}(S)) \cong \operatorname{soc}_R(\mathrm{H}^3_{(X,Y,Z)}(R)) \oplus \operatorname{soc}_R(\mathrm{H}^3_{(X,Y,Z)}(P))$$

has infinite dimension, we deduce that $\operatorname{soc}_S(\mathrm{H}^3_{(X,Y,Z)}(S))$ also has infinite dimension as a k-vector space. (If, on the contrary, $(0:_{\mathrm{H}^3_{(X,Y,Z)}(S)} \mathfrak{m}_S)$ has finite dimension, then $\mathrm{H}^0_{\mathfrak{m}_S}(\mathrm{H}^3_{(X,Y,Z)}(S))$ will be an Artinian S-module. Consequently, $(0:_{\mathrm{H}^3_{(X,Y,Z)}(S)} \mathfrak{m}_S^2)$ and hence $(0:_{\mathrm{H}^3_{(X,Y,Z)}(S)} \mathfrak{m}_R)$ would have finite dimension, a contradiction.) Also observe that X,Y,Z form part of a system of parameters of S while $\dim(S) = 5$.

Example 5.3. Let $A = k[[U,V,W,X,Y,Z]]$ be a power series ring in 6 variables over a field k and

$$Q = (UX-VY, UZ-WX, VZ-WY)A = I_2\big(\begin{smallmatrix} U & V & W \\ X & Y & Z \end{smallmatrix}\big).$$

Then A/Q is isomorphic to the subring $B = k[[X,Y,Z,XT,YT,ZT]]$ of $k[[X,Y,Z,T]]$. Let $R = k[[X,Y,Z]]$ with maximal ideal $\mathfrak{m} = \mathfrak{m}_R = (X,Y,Z)R$. Then, for every

$n \in \mathbb{N}$, we have $B \cong \mathfrak{m}^n T^n \oplus W_n$ (for some W_n) as R-modules. Clearly $\mathfrak{m}^n \cong \mathfrak{m}^n T^n$ as R-modules and, hence,

$$\operatorname{Ann}_R(\mathrm{H}^1_{(X,Y,Z)}(B)) \subseteq \operatorname{Ann}_R(\mathrm{H}^1_{(X,Y,Z)}(\mathfrak{m}^n))$$

for all $n \in \mathbb{N}$. Moreover, as

$$\operatorname{Ann}_R(\mathrm{H}^1_{(X,Y,Z)}(\mathfrak{m}^n)) = \operatorname{Ann}_R(\mathrm{H}^0_{(X,Y,Z)}(R/\mathfrak{m}^n)) = \operatorname{Ann}_R(R/\mathfrak{m}^n) = \mathfrak{m}^n$$

for every n, we conclude that

$$\operatorname{Ann}_R(\mathrm{H}^1_{(X,Y,Z)}(B)) \subseteq \cap_{n=1}^{\infty} \mathfrak{m}^n = 0.$$

This implies that $\operatorname{Ann}_R(\mathrm{H}^2_{(X,Y,Z)}(Q)) = 0$ since

$$\mathrm{H}^2_{(X,Y,Z)}(Q) \cong \mathrm{H}^1_{(X,Y,Z)}(A/Q) \cong \mathrm{H}^1_{(X,Y,Z)}(B).$$

Now let $S = A \oplus X^{\frac{1}{2}} Q$, which is a complete local domain module finite over A. We see that $\operatorname{Ann}_R(\mathrm{H}^2_{(X,Y,Z)}(S)) = 0$ since $S \cong A \oplus Q$ as A-modules (hence as R-modules). Observe that X, Y, Z is part of a system of parameters of S and $\dim(S) = 6$.

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