GRADED RINGS OF RATIONAL TWIST IN PRIME CHARACTERISTIC

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Abstract. We study the generating function associated to complexity sequence of the twisted construction of a $\mathbb{N}$-graded ring. We regard this as an object reflecting the properties of the ring and its grading and perform a detailed analysis of the case of the polynomial ring with general $\mathbb{N}$-grading. Applications to the Frobenius complexity of determinantal rings are provided.

1. Introduction

This paper is dedicated to the study of certain generating functions associated naturally to a novel concept in commutative algebra in positive characteristic, the twisted construction of a $\mathbb{N}$-graded ring, where $\mathbb{N}$ denotes the set of all non-negative integers. Katzman, Schwede, Singh and Zhang have introduced this concept in their paper [KSSZ]. The importance of this construction is illustrated by its applications to the study of the ring of Frobenius operators on the injective hull of the residue field of a local ring $(R, \mathfrak{m}, k)$ in positive characteristic. In this paper, we highlight some combinatorial features of graded rings that are inherently present in positive characteristic and are due to the twisted construction.

Unless otherwise noted, in this paper we let R be a commutative ring of prime characteristic $p > 0$. Let $\mathscr{R}$ be an $\mathbb{N}$-graded commutative ring with $\mathscr{R}_0 = R$.

Definition 1.1. Define the twisted construction on $\mathscr{R}$ by

$$T(\mathscr{R}) := \oplus_{e \geqslant 0} \mathscr{R}_{p^e-1},$$

which is an $\mathbb{N}$-graded ring by

$$a * b = ab^{p^e}$$

for all $a \in \mathscr{R}_{p^e-1}$, $b \in \mathscr{R}_{p^{e'}-1}$. The degree e piece of $T(\mathscr{R})$ is $T_e(\mathscr{R}) = \mathscr{R}_{p^e-1}$.

Our research is centered around the following generating function.

Definition 1.2. Let $\{c_e\}_{e \geqslant 0}$ be the complexity sequence for $T = T(\mathscr{R})$. Let

$$\mathcal{C}_{\mathscr{R}}(z) := \sum_{e=0}^{\infty} c_e z^e \in \mathbb{Q}[\![z]\!],$$

which we call the *twisted generating function* of $\mathscr{R}$.

We are going to show that, for some classes of rings, this generating function is rational and it carries additional interesting features connected to the grading of the ring. This includes the class of polynomial rings, a case that brings forward some features connected

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to the Ehrhart polynomial of an integral convex polytope which are interesting in their own right. We will state our main results now, but first we need to introduce more terminology.

Definition 1.3. We say that $\mathscr{R}$ has a grading with *rational twist*, or simply that the graded ring $\mathscr{R}$ has *rational twist*, if $\mathcal{C}(z) = \mathcal{C}_{\mathscr{R}}(z) = \frac{P(z)}{Q(z)}$ with $P(z)$, $Q(z) \in \mathbb{Q}[z]$. Furthermore, with $\mathcal{C}(z) = \frac{P(z)}{Q(z)}$ where $P(z)$, $Q(z) \in \mathbb{Q}[z]$ do not have common roots in $\mathbb{C}$, if $Q(z)$ has a unique simple root $1/\gamma$ of minimal absolute value, we say that the graded ring $\mathscr{R}$ has rational twist with *dominant eigenvalue* (or, with *dominant eigenvalue* γ).

With $\mathscr{R}$ having rational twist with dominant eigenvalue γ, Theorem 3.1 gives, for $e \gg 0$,

$$c_e = \rho\gamma^e + \text{lower order terms } o(\gamma^e).$$

In this case, we call the number ρ is the *twisted complexity multiplicity* of $\mathscr{R}$, or simply *t-multiplicity*.

Our main result is the following theorem, see Corollary 2.11.

Theorem 1.4. *Let R be a commutative ring of prime characteristic p.*

(1) *Let $\mathscr{R} = R[x_1, \ldots, x_m]$ graded with $\deg(x_i) = d_i > 0$, $i = 1, \ldots, m$. Then $\mathscr{R}$ has rational twist.*
(2) *Let $\mathscr{R} = V_r(R[x_1, \ldots, x_m])$, where $r \geqslant 1$, be the rth Veronese subring of the standard graded polynomial ring $R[x_1, \ldots, x_m]$. Then $\mathscr{R}$ has rational twist.*

This statement is part of Theorem 2.10 which, in fact, is substantially more general.

Our interest stems from applications to the Frobenius complexity of local rings of prime characteristic via the ring of Frobenius operators on the injective hull of the residue field. So, let us review briefly these concepts.

1.1. **Frobenius operators and Frobenius complexity.** For an R-module M, an eth Frobenius action, $\phi : M \to M$ is an R-additive map $\phi : M \to M$ such that $\phi(rm) = r^q\phi(m)$, for all $r \in R, m \in M$. Let $\mathscr{F}^e(M)$ be the collection of all eth Frobenius operators on M.

Definition 1.5. We define *the algebra of Frobenius operators* on M by

$$\mathscr{F}(M) = \oplus_{e\geqslant 0}\mathscr{F}^e(M),$$

with the multiplication on $\mathscr{F}(M)$ determined by composition of functions; that is, if $\phi \in \mathscr{F}^e(M), \psi \in \mathscr{F}^{e'}(M)$ then $\phi\psi := \phi \circ \psi \in \mathscr{F}^{e+e'}(M)$. Hence, in general, $\phi\psi \neq \psi\phi$.

The ring of Frobenius operators on the injective hull $E_R(k)$ of the residue field of a local ring of positive characteristic has been studied by many researchers in commutative algebra. The twisted construction appears naturally in this context as shown by the theorem stated below.

Theorem 1.6 ([KSSZ]). *Let $(R, \mathfrak{m}, k)$ be a normal, complete local ring of positive characteristic. Let ω^{-1} denote the inverse of the canonical module of R, $E = E_R(k)$ the injective hull of k and $\mathscr{R}(\omega^{-1}) = \oplus_{n\geqslant 0}\omega^{(-n)}$ the anticanonical cover of R. Then we have an isomorphism of graded rings*

$$\mathscr{F}(E) \cong T(\mathscr{R}(\omega^{-1})).$$

Let $(R, \mathfrak{m}, k)$ be a local ring and $E = E_R(k)$ the injective hull of k. One of the most important investigations on $\mathscr{F}(E)$ regards its generation as ring over R. To this end, the notion of Frobenius complexity of R, $\mathrm{cx}_F(R)$, has been introduced in [EY1].

We are now in position to state the definition of the Frobenius complexity of a local ring of prime characteristic.

Definition 1.7. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p. Denote by E the injective hull of the residue field of R. Denote $k_e(R) := k_e(\mathscr{F}(E))$, for all e, and call these numbers the *Frobenius growth sequence* of R, as per Definition 1.10. Then $c_e = c_e(R) := k_e(R) - k_{e-1}(R)$ defines the *Frobenius complexity sequence* of R.

The *complexity* of $\mathscr{F}(E)$ is

$$\inf\{n \in \mathbb{R}_{>0} : c_e = O(n^e)\}$$

and it is denoted by $\mathrm{cx}(\mathscr{F}(E))$. Therefore, if there is no $n > 0$ such that $c_e(\mathscr{F}(E)) = O(n^e)$, then $\mathrm{cx}(\mathscr{F}(E)) = \infty$.

We define the *Frobenius complexity* of the ring R by

$$\mathrm{cx}_F(R) = \log_p(\mathrm{cx}(\mathscr{F}(E))),$$

if $\mathrm{cx}(\mathscr{F}(E))$ is nonzero and finite. If the Frobenius growth sequence of the ring R is eventually constant (i.e., $\mathrm{cx}(\mathscr{F}(E)) = 0$), then the Frobenius complexity of R is set to be $-\infty$. If $\mathrm{cx}(\mathscr{F}(E)) = \infty$, the Frobenius complexity if R is set to be ∞.

Corollary 1.8. *Let $(R, \mathfrak{m}, k)$ be a normal, complete local ring of prime characteristic p. If the graded ring $\mathscr{R}(\omega^{-1})$ has rational twist with dominant eigenvalue γ, then*

$$\mathrm{cx}_F(R) = \log_p(\gamma).$$

In [EY2], we observed that the complexity of the twisted construction for the anticanonical cover of the determinantal rings of 2×2 minors in a matrix of indeterminates is given by the twisted construction of the Veronese ring of the polynomial ring. Hence our computations in Section 3 provide insight in how to compute the Frobenius complexity of determinantal rings, using methods different from those in the aforementioned paper. In particular, we obtain the following result, according to Example 3.19(1).

Theorem 1.9. *Let K be a field of characteristic p and $m \geqslant 3$ be an integer. Consider the determinantal ring of 2×2 minors in a matrix of indeterminates of size $m \times (m-1)$ over K, and denote by S_m the completion of the ring at its maximal homogenous ideal. Let γ be the dominant eigenvalue of $K[x_1, \dots, x_m]$ considered with standard grading. Then*

(1) $\mathrm{cx}_F(S_m) = \log_p(\gamma)$,
(2) $p^{m-2} < \gamma < p^{m-1}$,
(3) $\lim_{p\to\infty}(\mathrm{cx}_F(S_m)) = m - 1$,
(4) $\lim_{p\to\infty} \frac{\gamma}{p^{m-1}} = 1 - \frac{1}{(m-1)!}$.

Proof. This follows from Example 3.19(1) and [EY1, Theorem 5.6] as well as [EY2, Theorem 4.1]. It is proved in [EY1, Theorem 5.6] and [EY2, Theorem 1.20] that the Frobenius complexity of S_m is directly related to the complexity of $T(K[x_1, \dots, x_m])$. Specifically, if the dominant eigenvalue of $K[x_1, \dots, x_m]$ is γ, then $\mathrm{cx}_F(S_m) = \log_p \gamma$. □

Let us put our results in a larger context. In our previous work ([EY1, EY2]), we examined the dominant eigenvalue of the polynomial ring with standard grading and the Veronese ring of a polynomial ring with standard grading, although without using this terminology and with different methods. Boix, in his thesis [B], and Álvarez Montaner, in [A], have associated various generating functions to Cartier algebras or rings of Frobenius operators. In particular, Álvarez Montaner has defined the generating function associated to the complexity sequence for a skew algebra and studied it for the ring of Frobenius operators on the injective hull of the residue field of a local ring for the examples in the literature where the complexity sequence was understood. Our point of view is that the twisted construction of a graded ring is what should be investigated combinatorially, and hence the systematic investigation of the fundamental case of polynomial rings in relation to this concept. Theorem 1.6 connects the two objects and explains, from our point of view, the reason why the generating functions of rings of Frobenius operators have remarkable structure (as they are directly connected to the twisted construction). The fact that the generating function of twisted construction is rational in full generality for polynomial rings shows that this behavior goes potentially beyond that of rings of Frobenius operators. For the convenience of the reader, we add here that the complexity sequence for the ring of Frobenius operators on a Stanley-Reisner ring has been investigated, although not using this terminology, by Álvarez Montaner, Boix and Zarzuela in [ABZ], and, subsequently, Boix and Zarzuela in [BZ], and, more recently, Ilioaea in [I].

The bulk of our paper surrounds the complexity concept which we will review now.

1.2. **Complexity of skew algebras.** Let us review the definition of the complexity of a graded ring. A detailed introduction, with proofs, can be found in [EY1].

Definition 1.10. Let $A = \oplus_{e \geqslant 0} A_e$ be a $\mathbb{N}$-graded ring, not necessarily commutative.

(1) Let $G_e(A) = G_e$ be the subring of A generated by the elements of degree less or equal to e. We agree that $G_{-1} = A_0$.
(2) We use $k_e = k_e(A)$ to denote the minimal number of homogeneous generators of G_e as a subring of A over A_0. (So $k_{-1} = k_0 = 0$.) We say that A is *degree-wise finitely generated* if $k_e < \infty$ for all $e \geqslant 0$.
(3) For a degree-wise finitely generated ring A, we say that a set X of homogeneous elements of A *minimally generates* A if for all e, $X_{\leqslant e} = \{a \in X : \deg(a) \leqslant e\}$ is a minimal set of generators for G_e with $k_e = \|X_{\leqslant e}\|$ for every $e \geqslant 0$. (Here $\|\cdot\|$ denotes cardinality in the sequel.) Also, let $X_e = \{a \in X : \deg(a) = e\}$.

Proposition 1.11. *Let A be a degree-wise finitely generated $\mathbb{N}$-graded ring and X a set of homogeneous elements of A. Then*

(1) *The minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule is $k_e - k_{e-1}$ for all $e \geqslant 0$.*
(2) *If X generates A as a ring over A_0 then $\|X_e\| \geqslant k_e - k_{e-1}$ for all $e \geqslant 0$.*

Let $f(n)$ and $g(n)$ be real-valued functions defined on the set of natural numbers. We say that $f(n) = O(g(n))$ if there exists $M > 0$ and a nonnegative integer n_0 such that $|f(n)| \leqslant M \cdot |g(n)|$ for all $n \geqslant n_0$.

Definition 1.12. Let A be a degree-wise finitely generated ring. The sequence $\{k_e\}_e$ is called the *growth* sequence for A. The *complexity sequence* is given by $\{c_e(A) = k_e - k_{e-1}\}_{e\geqslant 0}$. The *complexity* of A is

$$\inf\{n \in \mathbb{R}_{>0} : c_e(A) = O(n^e)\}$$

and it is denoted by $\mathrm{cx}(A)$. Therefore, if there is no $n > 0$ such that $c_e(A) = O(n^e)$, then $\mathrm{cx}(A) = \infty$.

In fact, for a particular type of algebras, called R-skew algebras, one can say even more. Let us review this concept.

Definition 1.13. Let A be a $\mathbb{N}$-graded ring such that there exists a ring homomorphism $R \to A_0$, where R is a commutative ring. We say that A is a (left) *R-skew algebra* if $aR \subseteq Ra$ for all homogeneous elements $a \in A$. A right R-skew algebra can be defined analogously. In this paper, our R-skew algebras will be left R-skew algebras and therefore we will drop the adjective 'left' when referring it to them.

One should note here that $\mathscr{F}(E)$ is a left R-skew algebra.

Proposition 1.14. *Let A be a degree-wise finitely generated R-skew algebra such that $R = A_0$. Then $c_e(A)$ equals the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as a left R-module for all e.*

2. Rational twist

Let $\mathscr{R}$ be an $\mathbb{N}$-graded commutative ring of prime characteristic p with $\mathscr{R}_0 = R$. As before, let $T(\mathscr{R}) := \oplus_{e\geqslant 0}\mathscr{R}_{p^e-1}$ denote the twisted construction associated to $\mathscr{R}$. Denote the following:

- $T := T(\mathscr{R})$.
- $G_e := G_e(T)$.
- $T_e := T_e(\mathscr{R}) = \mathscr{R}_{p^e-1}$.

We also let $\{c_e\}_{e\geqslant 0}$ equal the complexity sequence of T, and consider the generating function

$$\mathcal{C}_{\mathscr{R}}(z) := \sum_{e=0}^{\infty} c_e z^e = \sum_{e=1}^{\infty} c_e z^e \in \mathbb{Q}[\![z]\!],$$

which we call the *twisted generating function* of $\mathscr{R}$.

Definition 2.1. Let $\mathscr{R}$ be an $\mathbb{N}$-graded commutative ring of prime characteristic p, finitely generated over $\mathscr{R}_0 = R$. We say that graded ring $\mathscr{R}$ has *rational twist* (or that the grading on $\mathscr{R}$ has *rational twist*) if

$$\mathcal{C}_{\mathscr{R}}(z) := \sum_{e=0}^{\infty} c_e z^e = \sum_{e=1}^{\infty} c_e z^e$$

is a rational function, that is, $\mathcal{C}_{\mathscr{R}}(z) = P(z)/Q(z)$ for some $P(z)$, $Q(z) \in \mathbb{Q}[x]$.

The leading question of our research is

Question 2.2. Assume that $\mathscr{R}$ is finitely generated as an R-algebra. When is $\mathcal{C}(z)$ a rational function in z? That is, when does $\mathscr{R}$ have rational twist?

2.1. **Affine semigroup rings with rational twist.** Let R be a commutative ring of prime characteristic p, and let m, $d_1, \ldots, d_m$ be positive integers. Also let A be a finitely generated semigroup of $(\mathbb{N}^m, +)$, with the assumption $(0, \ldots, 0) \in A$. Consider the polynomial ring $R[x_1, \ldots, x_m]$, with general grading $\deg(x_i) = d_i$ for all $i = 1, \ldots, m$. Let $R[A]$ denote the semigroup ring of A over R, so $R[A]$ is a graded subring of $R[x_1, \ldots, x_m]$. In this section, we study $T(R[A])$ and, (sometimes, in particular, $T(R[x_1, \ldots, x_m])$). We are interested in the sequence $\{c_e\}_{e \geqslant 0}$ of this ring, as well as its generating function, i.e., the twisted generating function.

In addition, we plan to study the dominant eigenvalue and t-multiplicity, when they exist, and consider their behavior when $p \to \infty$, in the case where $d_1, \ldots, d_m$ are fixed.

To simplify notation, denote the following (with R, p, m, $d_1, \ldots, d_m$ and A understood):

- $\mathscr{R} := R[A]$, with its grading described as above.
- $d := (d_1, \ldots, d_m)$.
- Let $\alpha_1, \ldots, \alpha_h$ be a mininal generating set for A, with total degrees $f_1, \ldots, f_h$. That is, $|\alpha_i| = f_i$, where for $\alpha = (a_1, \ldots, a_m) \in \mathbb{N}^m$, $|\alpha| := |\alpha|_d := a_1d_1 + \cdots + a_md_m$.
- For every $i \in \mathbb{N}$, denote $A_i = \{\alpha \in A : |\alpha| = i\}$.
- $T := T(\mathscr{R})$.
- $G_e := G_e(T)$.
- $T_e := T_e(\mathscr{R}) = T_e(R[A]) = \mathscr{R}_{p^e-1} = (R[A])_{p^e-1}$. As there are several gradings going on, when we say the degree of a monomial, we agree that it refers to its (total) degree in $\mathscr{R} = R[A]$. Thus a monomial in T_e is a monomial of (total) degree $p^e - 1$. For every $e \in \mathbb{N}$, denote the set of all monomials in T_e by $\mathcal{T}_e$, and denote the set of all monomials in $(G_{e-1})_e$ by $\mathcal{G}_e$.
- For every $e \geqslant 0$, denote $\mathcal{C}_e := \mathcal{T}_e \setminus \mathcal{G}_e$. Note that $\mathcal{C}_0 = \emptyset$ and $\mathcal{C}_1 = \mathcal{T}_1$. Clearly, $\mathcal{T}_e$, $\mathcal{G}_e$ and $\mathcal{C}_e$ give rise to the monomial bases of $T_e = \mathscr{R}_{p^e-1}$, $(G_{e-1})_e$ and $\frac{T_e}{(G_{e-1})_e}$ respectively. Thus $c_e = c_e(T) = \|\mathcal{C}_e\|$, the cardinality of $\mathcal{C}_e$. Note that $c_0(T) = 0$, $t_0(T) = \operatorname{rank}_R(T_0) = \operatorname{rank}_R(R) = 1$ and $c_1(T) = t_1(T) = \operatorname{rank}_R(T_1) = \operatorname{rank}_R(\mathscr{R}_{p-1})$.
- The twisted generating function of $R[A]$ is
$$\mathcal{C}(z) = \mathcal{C}_{R[A]}(z) := \sum_{e=0}^{\infty} c_e z^e = \sum_{e=1}^{\infty} c_e z^e$$
- We also denote
$$\mathcal{T}(z) = \mathcal{T}_{R[A]}(z) := \sum_{e=0}^{\infty} t_e z^e \in \mathbb{Q}[\![z]\!].$$

Notation 2.3. As in [EY1], we will also use the following notation in the sequel: For an integer $a \in \mathbb{N}$, if $a = c_np^n + \cdots + c_1p + c_0$ with $0 \leqslant c_i \leqslant p-1$ for all $0 \leqslant i \leqslant n$, then we use $a = \overline{c_n \cdots c_0}$ to denote the base p expression of a. Also, we write $a|_e$ to denote the remainder of a when dividing to p^e. Thus, if $a = \overline{c_n \cdots c_0}$ and $n \geqslant e-1$ then $a|_e = \overline{c_{e-1} \cdots c_0}$, which we refer to as the eth truncation of a. Put differently, $a|_e = a - \left\lfloor \frac{a}{p^e} \right\rfloor p^e$, in which $\left\lfloor \frac{a}{p^e} \right\rfloor$ is the floor function of $\frac{a}{p^e}$. When adding up integers $a_i \in \mathbb{N}$ with $1 \leqslant i \leqslant m$, all written in base p expressions, we can talk about the carry over to the digit corresponding to p^e, which is simply $\left\lfloor \frac{a_1|_e+\cdots+a_m|_e}{p^e} \right\rfloor$. All the notation depends on the choice of p, which should be clear from the context.

Definition 2.4. Let A be an affine semigroup, that is $A \subseteq \mathbb{N}^m$, for some $m \in \mathbb{N}$.

(1) We say that A is CD if A is *closed under differences* in the following sense:
$$(\alpha \in A) \wedge (\beta \in A) \wedge (\alpha - \beta \in \mathbb{N}^m) \implies \alpha - \beta \in A.$$

(2) We say that A satisfies the CLTD condition (or A is CLTD) is A is *closed under left twisted differences* in the following sense: for all $e' > 0$, $e'' > 0$,
$$(\alpha' \in A_{p^{e'}-1}) \wedge (\alpha'' \in \mathbb{N}^m) \wedge (\alpha' + p^{e'}\alpha'' \in A_{p^{e'+e''}-1}) \implies \alpha'' \in A.$$

(3) We say that A satisfies the CRTD condition (or A is CRTD) is A is *closed under right twisted differences* in the following sense: for all $e' > 0$, $e'' > 0$,
$$(\alpha' \in \mathbb{N}^m) \wedge (\alpha'' \in A_{p^{e''}-1}) \wedge (\alpha' + p^{e'}\alpha'') \in A_{p^{e'+e''}-1} \implies \alpha' \in A.$$

Lemma 2.5. *Every affine semigroup A that is CD satisfies CRTD.*

Proof. With the notation as in the Definition 2.4 part (3) above, if $\alpha' + p^{e'}\alpha'' \in A$, $\alpha' \in \mathbb{N}^m$ and $\alpha'' \in A$ (hence $p^{e'}\alpha'' \in A$), then $\alpha' = (\alpha' + p^{e'}\alpha'') - p^{e'}\alpha'' \in A$ by CD. ☐

Example 2.6. (1) It is clear that $\mathbb{N}^m$ satisfies CD, CRTD and CLTD.

(2) Let $V_r = \{\alpha \in \mathbb{N}^m : r \mid |\alpha|\}$, the r-Veronese sub-semigroup of $\mathbb{N}^m$. Then V_r satisfies CD, CRTD and CLTD. Indeed, as V_r satisfies CD, it satisfies CRTD. For CLTD, we consider the following two cases: If $p \mid r$, then there is no $\alpha \in V_r$ such that $|\alpha| = p^e - 1$ with $e > 0$ and hence the CLTD condition is trivially satisfied. Now assume $p \nmid r$, e', $e'' > 0$, $\alpha' \in V_r$ with $|\alpha'| = p^{e'} - 1$, $\alpha'' \in \mathbb{N}^m$ such that $|\alpha' + p^{e'}\alpha''| = p^{e'+e''} - 1$ (and hence $|\alpha''| = p^{e''} - 1$) and $\alpha' + p^{e'}\alpha'' \in V_r$. Then
$$p^{e'}|\alpha''| = |p^{e'}\alpha''| = |\alpha' + p^{e'}\alpha''| - |\alpha'|,$$
which is a multiple of r. Hence $r \mid |\alpha''|$, since $\gcd(p, r) = 1$.

Let us consider an affine semigroup $A \subseteq \mathbb{N}^m$. Let $\alpha = (a_1, \ldots, a_m) \in A$ such that $|\alpha| := |\alpha|_d := a_1 d_1 + \cdots + a_m d_m = p^e - 1$, which corresponds to the fact that $x^\alpha := x_1^{a_1} \cdots x_m^{a_m} \in \mathcal{T}_e$. Fix any positive integers e', e'' such that $e = e' + e''$. Then $x^\alpha \in T_{e'} * T_{e''}$ if and only if it can be decomposed as
$$x^\alpha = x^{\alpha'} * x^{\alpha''} = x^{\alpha' + p^{e'}\alpha''}$$
for some $x^{\alpha'} \in \mathcal{T}_{e'}$, $x^{\alpha''} \in \mathcal{T}_{e''}$, if and only if there is an equation
$$\alpha = \alpha' + p^{e'}\alpha''$$
for some $\alpha' \in A_{p^{e'}-1}$ and $\alpha'' \in A_{p^{e''}-1}$, which is equivalent to the existence of equations
$$a_i = a_i' + p^{e'}a_i'' \quad \text{for all} \quad i \in \{1, \ldots, m\}$$
for some $(a_1', \ldots, a_m') \in A_{p^{e'}-1}$ and $(a_1'', \ldots, a_m'') \in A_{p^{e''}-1}$. We denote this situation by $(*)$. Observe that, when $(*)$ holds, a_i' and a_i'' are uniquely determined by $a_i' = a_i|_{e'}$ and $a_i'' = (a_i - a_i|_{e'})/p^{e'}$ for all $i = 1, \ldots, m$. We have two cases:

(1) If A satisfies CLTD, then the above $(*)$ holds if and only if
$$\alpha|_{e'} := (a_1, \ldots, a_m)|_{e'} := (a_1|_{e'}, \ldots, a_m|_{e'}) \in A_{p^{e'}-1}$$
which simply means
$$(a_1, \ldots, a_m)|_{e'} \in A \text{ and } |(a_1, \ldots, a_m)|_{e'}| := \sum_{i=1}^{m} (a_i|_{e'})d_i = p^{e'} - 1$$
(see Notation 2.3 for the meaning of $a_i|_{e'}$) if and only if
$$x^{\alpha|_{e'}} := x_1^{a_1|_{e'}} \cdots x_m^{a_m|_{e'}} \in \mathcal{T}_{e'}.$$
(2) If A satisfies CRTD, then the above $(*)$ holds if and only if
$$(\alpha - \alpha|_{e'})/p^{e'} \in A_{p^{e''}-1}$$
if and only if
$$x^{(\alpha-\alpha|_{e'})/p^{e'}} \in \mathcal{T}_{e''}.$$

With the argument above, we establish the following results.

Proposition 2.7. *Consider $T = T(R[A])$, in prime characteristic p. Let $e, e', e'' > 0$ such that $e = e' + e''$ and let $x^\alpha = x_1^{a_1} \cdots x_m^{a_m} \in \mathcal{T}_e$.*

(1) *If A satisfies CLTD, then $x^\alpha \in T_{e'} * T_{e''} \iff x^{\alpha|_{e'}} \in \mathcal{T}_{e'}$. If this is the case, then x^α can be expressed as a twisted product of a unique monomial in $\mathcal{T}_{e'}$ (namely $x^{\alpha|_{e'}}$) and a unique monomial in $\mathcal{T}_{e''}$ (namely $x^{(\alpha-\alpha|_{e'})/p^{e'}}$).*

(2) *If A satisfies CRTD, then $x^\alpha \in T_{e'} * T_{e''} \iff x^{(\alpha-\alpha|_{e'})/p^{e'}} \in \mathcal{T}_{e''}$. If this is the case, then x^α can be expressed as a twisted product of a unique monomial in $\mathcal{T}_{e'}$ (namely $x^{\alpha|_{e'}}$) and a unique monomial in $\mathcal{T}_{e''}$ (namely $x^{(\alpha-\alpha|_{e'})/p^{e'}}$).*

Proof. The equivalences have been proved in the discussion above. The uniqueness (in both (1) and (2)) follows from how twisted multiplication works in light of base p expressions of the components of $\alpha = (a_1, a_2, \ldots, a_m)$. □

Thus we have the following:

Proposition 2.8. *Let $T = T(R[A])$ in prime characteristic p and $e \geqslant 1$. Then*

(1) *If A satisfies CLTD then $\mathcal{T}_e = \biguplus_{e'=1}^{e} \mathcal{C}_{e'} * \mathcal{T}_{e-e'}$, in which $\biguplus$ denotes disjoint union.*
(2) *If A satisfies CRTD then $\mathcal{T}_e = \biguplus_{e'=1}^{e} \mathcal{T}_{e-e'} * \mathcal{C}_{e'}$.*
(3) *If A satisfies CLTD or CRTD then $t_e = \sum_{e'=1}^{e} c_{e'} t_{e-e'}$.*

Proof. (1) For any $x^\alpha = x_1^{a_1} \cdots x_m^{a_m} \in \mathcal{T}_e$, denote $e' = \min\{i \geqslant 1 \mid x^\alpha \in \mathcal{T}_i * \mathcal{T}_{e-i}\}$, which exists and is at most e. So
$$x^\alpha = x^{\alpha|_{e'}} * x^{(\alpha-\alpha|_{e'})/p^{e'}}$$
with $x^{\alpha|_{e'}} \in \mathcal{T}_{e'}$ and $x^{(\alpha-\alpha|_{e'})/p^{e'}} \in \mathcal{T}_{e-e'}$. By the minimality of e', we see $x^{\alpha|_{e'}} \in \mathcal{C}_{e'}$. Thus $\mathcal{T}_e = \bigcup_{e'=1}^{e} \mathcal{C}_{e'} * \mathcal{T}_{e-e'}$.

It remains to show that $\bigcup_{e'=1}^{e} \mathcal{C}_{e'} * \mathcal{T}_{e-e'}$ is a disjoint union. Suppose that there exist $1 \leqslant e_1 < e_1 + e_2 \leqslant e$ and $x^\alpha \in \mathcal{T}_e$ such that
$$x^\alpha \in (\mathcal{C}_{e_1} * \mathcal{T}_{e-e_1}) \cap (\mathcal{C}_{e_1+e_2} * \mathcal{T}_{e-e_1-e_2}).$$

By Proposition 2.7(1), we see that $x^{\alpha|_{e_1}} \in \mathcal{C}_{e_1}$ and $x^{\alpha|_{e_1+e_2}} \in \mathcal{C}_{e_1+e_2}$. Observing that $x^{(\alpha|_{e_1+e_2})|_{e_1}} = x^{\alpha|_{e_1}} \in \mathcal{T}_{e_1}$, we apply Proposition 2.7(1) to $x^{\alpha|_{e_1+e_2}} \in \mathcal{T}_{e_1+e_2}$ to conclude

$$x^{\alpha|_{e_1+e_2}} \in \mathcal{T}_{e_1} * \mathcal{T}_{e_2},$$

which contradicts $x^{\alpha|_{e_1+e_2}} \in \mathcal{C}_{e_1+e_2}$. Thus $\bigcup_{e'=1}^{e} \mathcal{C}_{e'} * \mathcal{T}_{e-e'}$ is a disjoint union.

(2) For any $x^\alpha = x_1^{a_1} \cdots x_m^{a_m} \in \mathcal{T}_e$, denote $e' = \min\{i \geqslant 1 \,|\, x^\alpha \in \mathcal{T}_{e-i} * \mathcal{T}_i\}$, which exists and is at most e. So

$$x^\alpha = x^{\alpha|_{e-e'}} * x^{(\alpha-\alpha|_{e-e'})/p^{e-e'}}$$

with $x^{\alpha|_{e-e'}} \in \mathcal{T}_{e-e'}$ and $x^{(\alpha-\alpha|_{e-e'})/p^{e-e'}} \in \mathcal{T}_{e'}$. By the minimality of e', we see $x^{(\alpha-\alpha|_{e-e'})/p^{e-e'}} \in \mathcal{C}_{e'}$. Thus $\mathcal{T}_e = \bigcup_{e'=1}^{e} \mathcal{T}_{e-e'} * \mathcal{C}_{e'}$. The assertion that $\bigcup_{e'=1}^{e} \mathcal{T}_{e-e'} * \mathcal{C}_{e'}$ is a disjoint union can be proved similarly to (1) above, relying on applications of Proposition 2.7(2).

(3) If A has CLTD, then in light of (1) above, we have

$$t_e = \|\mathcal{T}_e\| = \sum_{e'=1}^{e} \|\mathcal{C}_{e'} * \mathcal{T}_{e-e'}\| = \sum_{e'=1}^{e} \|\mathcal{C}_{e'}\| \, \|\mathcal{T}_{e-e'}\| = \sum_{e'=1}^{e} c_{e'} t_{e-e'},$$

in which the equality $\|\mathcal{C}_{e'} * \mathcal{T}_{e-e'}\| = \|\mathcal{C}_{e'}\| \, \|\mathcal{T}_{e-e'}\|$ holds because the map from $\mathcal{C}_{e'} \times \mathcal{T}_{e-e'}$ to $\mathcal{C}_{e'} * \mathcal{T}_{e-e'}$ defined by twisted multiplication is bijective. The proof for the case of A being CRTD is similar, which relies on (2) above. □

Proposition 2.8 allows us to state the following:

Theorem 2.9. *Let $T = T(R[A])$ be as above. Assume that A satisfies CLTD or CRTD. Then*

$$\mathcal{C}(z)\mathcal{T}(z) = \mathcal{T}(z) - 1 \quad \text{or equivalently} \quad \mathcal{C}(z) = 1 - \frac{1}{\mathcal{T}(z)}.$$

So $\mathcal{C}(z)$ is rational if and only if $\mathcal{T}(z)$ is rational.

Proof. By Proposition 2.8, we have

$$\mathcal{C}(z)\mathcal{T}(z) = \left(\sum_{e=1}^{\infty} c_e z^e\right)\left(\sum_{e=0}^{\infty} t_e z^e\right) = \sum_{e=1}^{\infty} t_e z^e = \mathcal{T}(z) - 1.$$ □

Theorem 2.10. *Let $T = T(R[A])$ be as above (with R having prime characteristic p, A an affine semigroup in $\mathbb{N}^m$, and $\deg(x_i) = d_i \geqslant 1$). Then $\mathcal{T}(z)$ is rational. If A satisfies CLTD or CRTD, then $\mathcal{C}(z)$ is rational hence the graded ring $\mathscr{R} = R[A]$ has rational twist.*

Proof. We may assume that R is a field without loss of generality. Because of Theorem 2.9, it suffices to show that $\mathcal{T}(z)$ is rational. As $T = T(\mathscr{R}) = T(R[A])$, we see $t_e = \operatorname{rank}_R(\mathscr{R}_{p^e-1})$ for all $e \geqslant 0$.

Let $\alpha_1, \ldots \alpha_h$ be a minimal generating set for A with total degrees $f_1, \ldots, f_h$. Let $D = \operatorname{lcm}(f_1, \ldots, f_h)$. For every $n \in \mathbb{N}$, let $h(n) := \dim_R((R[A])_n)$, the Hilbert function of $\mathscr{R} = R[A]$, with the grading induced by the total degree. It is known [BI] that there exist polynomials of degree at most $m-1$

$$h_i(x) \in \mathbb{Q}[x], \quad i = 0, \ldots, D-1,$$

such that

$$h(n) = \operatorname{rank}_R(\mathscr{R}_n) = h_i(n) \quad \text{if} \quad n \equiv i \mod D.$$

for all $n \gg 0$.

In the remaining of the proof, we assume

$$h(n) = \operatorname{rank}_R(\mathscr{R}_n) = h_i(n) \quad \text{if} \quad n \equiv i \mod D$$

for all $n \geqslant 0$ (which is the case when $\mathscr{R} = R[x_1, \dots, x_m]$), as we can change finitely many values of t_e without affecting the rationality of $\mathcal{T}(z)$.

The sequence $\{p^e - 1 \mod D\}_{e \geqslant 0}$ is *eventually* periodic (which is actually periodic when $p \nmid D$). Let E be the eventual period. For each $i = 0, \dots, E-1$, let $k(i)$ be the unique integer such that $0 \leqslant k(i) \leqslant D-1$ and $p^{nE+i} - 1 \equiv k(i) \mod D$ for $n \gg 0$ (in the case of $p \nmid D$, we actually have $p^{nE+i} - 1 \equiv k(i) \mod D$ for any $n \geqslant 0$, e.g., $n = 0$), and define the function

$$\eta_i(x) := h_{k(i)}(p^i x - 1) =: \sum_{j=0}^{m-1} a_{ij} x^j \in \mathbb{Q}[x].$$

Now, for all $e \gg 0$ (for all $e \geqslant 0$ when $p \nmid D$) such that $e \equiv i \mod E$ with $0 \leqslant i \leqslant E-1$, we have

$$t_e = h(p^e - 1) = h_{k(i)}(p^e - 1) = h_{k(i)}(p^i(p^{e-i}) - 1) = \eta_i(p^{e-i}).$$

Without affecting the rationality of $\mathcal{T}(z)$, we assume that $\{p^e - 1 \mod D\}_{e \geqslant 0}$ is periodic with E being the period (which is actually the case when $p \nmid D$).

Finally, given what has been covered above, we have

$$\begin{aligned}
\mathcal{T}(z) = \sum_{e=0}^{\infty} t_e z^e &= \sum_{e=0}^{\infty} h(p^e - 1) z^e \\
&= \sum_{i=0}^{E-1} \sum_{n=0}^{\infty} h(p^{nE+i} - 1) z^{nE+i} \\
&= \sum_{i=0}^{E-1} \sum_{n=0}^{\infty} \eta_i(p^{nE}) z^{nE+i} \\
&= \sum_{i=0}^{E-1} \sum_{n=0}^{\infty} \sum_{j=0}^{m-1} a_{ij} (p^{nE})^j z^{nE+i} \\
&= \sum_{i=0}^{E-1} \sum_{j=0}^{m-1} a_{ij} z^i \sum_{n=0}^{\infty} (p^{jE} z^E)^n \\
&= \sum_{i=0}^{E-1} \sum_{j=0}^{m-1} \frac{a_{ij} z^i}{1 - p^{jE} z^E} = \sum_{j=0}^{m-1} \frac{\sum_{i=0}^{E-1} a_{ij} z^i}{1 - p^{jE} z^E},
\end{aligned}$$

which is rational. □

In particular, we obtain the following

Corollary 2.11. *Let R be a commutative ring of prime characteristic p.*

(1) *Let $\mathscr{R} = R[x_1, \dots, x_m]$ graded with $\deg(x_i) = d_i$, $i = 1, \dots, m$. Then the graded ring $\mathscr{R}$ has rational twist.*
(2) *Let $\mathscr{R} = V_r(R[x_1, \dots, x_m])$, where $r \geqslant 1$, be the rth Veronese subring of the graded polynomial ring $R[x_1, \dots, x_m]$ with $\deg(x_i) = d_i$, $i = 1, \dots, m$. Then the graded ring $\mathscr{R}$ has rational twist.*

Proof. Both (1) and (2) follow from Theorem 2.10, in light of Example 2.6. □

3. Dominant eigenvalue

In this section let R be a commutative ring of prime characteristic p and $\mathscr{R}$ be an $\mathbb{N}$-graded R-skew left algebra.

We remind the reader a standard result on generating functions and recurrence relations necessary in this paper.

Theorem 3.1 (see [St], page 464, Theorem 4.1.1)**.** *Let $\alpha_1, \dots, \alpha_d$ be complex numbers, $d \geqslant 1$ and $\alpha_d \neq 0$. Let $f : \mathbb{N} \to \mathbb{C}$ a function. The following assertions are equivalent:*

(1) *The generating function of the sequence f satisfies*
$$\sum_{n \geqslant 0} f(n) x^n = \frac{P(x)}{Q(x)},$$
where $Q(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$ and $P(x)$ is a polynomial of degree less than d.
(2) *For all $n \geqslant 0$,*
$$f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0.$$
(3) *For all $n \geqslant 0$,*
$$f(n) = \sum_{i=1}^{k} P_i(n) \gamma_i^n,$$
where $1 + \alpha_1 x + \cdots + \alpha_d x^d = \prod_{i=1}^k (1 - \gamma_i x)^{m_i}$, the γ_i's are distinct and nonzero, and $P_i(n)$ is a polynomial of degree less than m_i.

Remark 3.2. Let $\sum_{n \geqslant 0} f(n) x^n = \frac{P(x)}{Q(x)}$, $Q(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d = \prod_{i=1}^k (1 - \gamma_i x)^{m_i}$ and $f(n) = \sum_{i=1}^k P_i(n) \gamma_i^n$ satisfy the conditions in Theorem 3.1 above. Further assume that $P(z)$ and $Q(z)$ do not have common roots in $\mathbb{C}$. We claim that each $P_i(n)$ must have degree $m_i - 1$. (Here we agree that a polynomial has degree -1 if and only if the polynomial is zero.) Indeed, if $\deg(P_j) < m_j - 1$ for some j, then Theorem 3.1 will imply that $\sum_{n \geqslant 0} f(n) x^n = \frac{\tilde{P}(x)}{\tilde{Q}(x)}$ where $\tilde{Q}(x) = \prod_{i=1}^k (1 - \gamma_i x)^{\deg(P_i)+1}$ with $\deg(\tilde{Q}(x)) < \deg(Q(x))$, which is a contradiction.

Remark 3.3. Let $\sum_{n\geqslant 0} f(n)x^n = \frac{P(x)}{Q(x)}$ satisfy the conditions in Theorem 3.1 above. Further assume that $P(z)$ and $Q(z)$ do not have common roots in $\mathbb{C}$. From Theorem 3.1(2), we see

$$\begin{pmatrix} f(n+1) \\ f(n+2) \\ \vdots \\ f(n+d-2) \\ f(n+d-1) \\ f(n+d) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -\alpha_d & -\alpha_{d-1} & -\alpha_{d-2} & \dots & -\alpha_2 & -\alpha_1 \end{pmatrix} \begin{pmatrix} f(n) \\ f(n+1) \\ f(n+2) \\ \vdots \\ f(n+d-2) \\ f(n+d-1) \end{pmatrix}$$

for all $n \geqslant 0$. Note that the $d \times d$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -\alpha_d & -\alpha_{d-1} & -\alpha_{d-2} & \dots & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

is precisely the (transpose of the) companion matrix of the polynomial

$$C(x) := x^d + \alpha_1 x^{d-1} + \cdots + \alpha_{d-1}x + \alpha_d = x^d Q(1/x).$$

In fact, $C(x)$ is called the *characteristic polynomial* of the (recursive) sequence f in the literature. Similarly, we may call the roots of the characteristic polynomial $C(x)$, which are the reciprocals of the roots of $Q(x)$, the *eigenvalues* of the sequence f.

With Theorem 3.1 and Theorem 2.10 in mind, we give the following definition.

Definition 3.4. We say that $\mathscr{R}$ has rational twist with *dominant eigenvalue* if $\mathcal{C}(z)$ is a rational function of the form $\mathcal{C}(z) = \frac{P(z)}{Q(z)}$, where $P(z) \in \mathbb{Q}[z]$ and $Q(z) \in \mathbb{Q}[z]$ do not have common roots in $\mathbb{C}$, such that either $Q(z)$ is constant or $Q(z)$ has a unique simple root $1/\gamma$ of minimal absolute value.[1]

In the case where $Q(z)$ has a unique simple root $1/\gamma$ of minimal absolute value, Theorem 3.1 gives, for $e \gg 0$,

$$c_e = \rho\gamma^e + \text{lower order terms } o(\gamma^e).$$

We call the number ρ is the *twisted complexity multiplicity* of $\mathscr{R}$, or simply *t-multiplicity*, and we call γ the *dominant eigenvalue* of $\mathscr{R}$. [2]

[1]Since $c_i = 0$ for all $i < 0$, it is guaranteed that $Q(0) \neq 0$.

[2]More generally, assume that $Q(z) = \prod_{i=1}^{k}(1-\gamma_i z)^{m_i}$ with $\gamma := \gamma_1$ having maximal absolute value (i.e., $|\gamma| > |\gamma_i|$ for all $i \geqslant 2$) and $m_1 \geqslant 1$. Theorem 3.1 indicates that $c_e = \sum_{i=1}^{k} P_i(e)\gamma_i^e = O(P_1(e)\gamma^e)$ as $e \to \infty$, in which each P_i is a polynomials of degree $m_i - 1$ by Remark 3.2. Let ρ be the leading coefficient of P_1. Since the sequence $\{c_e\}$ is non-negative and not eventually zero, it follows that $\rho > 0$ and $\gamma = \mathrm{cx}(\mathscr{R}) > 0$. In this case γ is the unique root (with multiplicity m_1) of the characteristic polynomial of the sequence $\{c_e\}_{e\in\mathbb{N}}$ of maximal absolute value.

The following question is long reaching and refines the leading Question 2.2. See also section 5 in [A].

Question 3.5. Let R be a Noetherian ring and assume that $\mathscr{R}$ is a finitely generated $\mathbb{N}$-graded R-algebra with $\mathscr{R}_0 = R$.

(1) Does the graded ring $\mathscr{R}$ have a rational twist?
(2) If graded ring $\mathscr{R}$ has rational twist, is it with dominant eigenvalue?

Remark 3.6. (1) When $Q(z)$ is of degree 1 and has the form $Q(x) = 1 - \gamma x$, then $\rho = P(1/\gamma)$. Indeed, $P(z) = (1-\gamma z)R(z) + r$, where $r \in \mathbb{C}$ and so $\mathcal{C}(z) = R(z) + \frac{r}{1-\gamma z}$, which gives $\mathcal{C}(z) = R(z) + \sum_{e\geqslant 0} r\gamma^e z^e$. So, for $e > \deg(R)$, $c_e = r\gamma^e$ and hence $\rho = r$. But $r = P(1/\gamma)$.

(2) If $Q(z) = (1-\gamma x)(1-\delta x)$ with $|\gamma| > |\delta|$, we write the decomposition of $\mathcal{C}(z)$ in partial fractions

$$\mathcal{C}(z) = \frac{P(z)}{Q(z)} = R(z) + \frac{b}{1-\delta z} + \frac{\rho}{1-\gamma z},$$

where ρ is indeed the t-multiplicity. So, $P(z) = R(z)(1-\delta z)(1-\gamma z) + b(1-\gamma z) + \rho(1-\delta z)$ and hence $P(1/\gamma) = \rho(1-\frac{\delta}{\gamma})$. Therefore,

$$\rho = \frac{P(1/\gamma)}{1-\frac{\delta}{\gamma}}.$$

In general, for a local normal complete ring $(R, \mathfrak{m}, k)$, the study of the Frobenius complexity of R reduces to the study of the complexity sequence for the anticanonical cover $\mathscr{R}$ of R. The Frobenius complexity of R is given by $\log_p(\gamma)$, where γ is the dominant eigenvalue of $\mathscr{R}$.

When the anticanonical cover of R is finitely generated, then it is a homomorphic image of a polynomial ring over R in finitely many variables with non-standard grading. Therefore, it is natural to study in detail the twisted generating function of the polynomial ring over R with nonstandard grading.

In our previous papers [EY1, EY2], we examined the dominant eigenvalue of the polynomial ring with standard grading and the Veronese ring of a polynomial ring with standard grading.

Theorem 3.7 ([EY2, Corollary 3.13, Subsection 3.2])**.** *Let R be a a Noetherian ring, $r \geqslant 1$ and $m \geqslant r+2$ be integers, and $\mathscr{R} = V_r(R[x_1, \dots, x_m])$ be the r-th Veronese ring of $R[x_1, \dots, x_m]$ with standard grading. Then, for large p, $\mathscr{R}$ has rational twist with dominant eigenvalue. If γ_p denotes the dominant eigenvalue of $\mathscr{R}$ in characteristic p, then $\lim_{p\to\infty} \log_p(\gamma_p) = m-1$.*

3.1. **The polynomial case.** Let $\mathscr{R} = R[x_1, \dots, x_m]$ with $\deg(x_1) = d_1, \dots, \deg(x_m) = d_m$. Under the considerations in Section 2 about semigroups, this means that we take $A = \mathbb{N}^m$. As before, let $D = \operatorname{lcm}(d_1, \dots, d_m)$. Further assume $p \nmid D$, which implies that the congruence class of $p \pmod D$ is contained in $\mathbb{Z}_D^\times$, the group of units of $\mathbb{Z}_D$. We will assume without loss of generality that R is a field.

We use the same notation as in the proof of Theorem 2.10. For every $n \in \mathbb{N}$, let $h(n) := \operatorname{rank}_R(\mathscr{R}_n)$. There exist $h_i(x) \in \mathbb{Q}[x]$, $i = 0, \ldots, D-1$, such that

$$h(n) = \operatorname{rank}_R(\mathscr{R}_n) = h_i(n) \quad \text{if} \quad n \equiv i \mod D.$$

for all $n \geqslant 0$. Let E be the order of p in $\mathbb{Z}_D^\times$. For $i = 0, \ldots, E-1$, let

$$\eta_i(x) = h_{k(i)}(p^i x - 1)$$

where $0 \leqslant k(i) \leqslant D-1$ and $p^i - 1 \equiv k(i) \mod D$. Write

$$\eta_i(x) = \sum_{j=0}^{m-1} a_{ij} x^j \in \mathbb{Q}[x], \quad i = 0, \ldots, E-1.$$

Finally, for $j = 0, \ldots, m-1$, let

$$\zeta_j(z) := \sum_{i=0}^{E-1} a_{ij} z^i.$$

These are polynomials of degree at most $E-1$, with leading coefficient $a_{E-1,j}$, when nonzero.

We state a couple of useful facts about the quasi-polynomial $h(n)$; see Proposition 12 and Proposition 18 in [CM].

Proposition 3.8. *Assume that* $\gcd(d_1, \ldots, d_m) = 1$.

(1) *For* $i = 0, \ldots, D-1$, *the leading term of* $h_i(n)$ *is* $\frac{1}{(m-1)!D} \cdot n^{m-1}$.

(2) *For* $k = 0, \ldots, m-1$, *let*

$$\delta_k = \operatorname{lcm}(\gcd(d_i : i \in I) : \|I\| = k+1, I \subseteq \{1, \ldots, m\}).$$

If $i - j \equiv \delta_k \mod D$, *then the coefficients of* n^k *in* h_j *and* h_i *are equal to each other.*

The following result plays a crucial role in investigating the rational twist of $\mathscr{R}$, for small values of E, the order of p in $\mathbb{Z}_D^\times$.

Proposition 3.9. *The rational function* $\mathcal{T}(z)$ *can be written in the form* $\dfrac{Q(z)}{R(z)}$ *where*

$$R(z) = \prod_{j=0}^{m-1}(1 - p^{jE} z^E) \qquad \text{and} \qquad Q(z) = \sum_{j=0}^{m-1} \Big[\zeta_j(z) \prod_{i \neq j}(1 - p^{iE} z^E)\Big].$$

Moreover,

(1) $Q(z) = az^{mE-1} + \cdots + 1$, *where* $a = (-1)^{m-1} p^{(m-1)mE/2} \eta_{E-1}(1/p^E)$, *and* $Q(0) = 1$. *In particular* Q *has degree* $mE-1$ *if and only if* $1/p^E$ *is not a root of* η_{E-1}.

(2) $R(z)$ *has degree* mE *and its roots are* $\epsilon \cdot \frac{1}{p^j}$, *where* ϵ *is any* E*th roots of unity, for any* $j = 0, \ldots, m-1$.

(3) $R(z)$ *and* $Q(z)$ *do not have common factor if and only if* $\zeta_j(\epsilon \cdot \frac{1}{p^j}) \neq 0$ *for any* E*th root of unity* ϵ *and* $j = 0, \ldots, m-1$.

(4) $\mathcal{C}(z) = \dfrac{Q(z) - R(z)}{Q(z)}$ *is an irreducible rational function if and only if* $\zeta_j(\epsilon \cdot \frac{1}{p^j}) \neq 0$ *for any* E*th root of unity* ϵ *and any* $j = 0, \ldots, m-1$.

Proof. As in the proof of Theorem 2.10, we have

$$\mathcal{T}(z)=\sum_{j=0}^{m-1}\frac{\sum_{i=0}^{E-1}a_{ij}z^i}{1-p^{jE}z^E}=\sum_{j=0}^{m-1}\frac{\zeta_j(z)}{1-p^{jE}z^E}=\frac{\sum_{j=0}^{m-1}\left[\zeta_j(z)\prod_{i\neq j}(1-p^{iE}z^E)\right]}{\prod_{j=0}^{m-1}(1-p^{jE}z^E)}=\frac{Q(z)}{R(z)}.$$

The coefficient of z^{mE-1} in $Q(z)$ equals

$$\begin{aligned}\sum_{j=0}^{m-1}a_{E-1,j}(-1)^{m-1}p^{\sum_{i\neq j}iE}&=(-1)^{m-1}p^{(m-1)mE/2}\sum_{j=0}^{m-1}a_{E-1,j}\frac{1}{p^{jE}}\\&=(-1)^{m-1}p^{(m-1)mE/2}\eta_{E-1}(1/p^E).\end{aligned}$$

Hence, the degree of $Q(z)$ is $mE-1$ if and only if $\eta_{E-1}(1/p^E)\neq 0$. Also,

$$Q(0)=\sum_{j=0}^{m-1}\zeta_j(0)=\sum_{j=0}^{m-1}a_{0,j}=\eta_0(1)=h_0(0)=1.$$

The rest of the proof is clear. □

Recall that $c_0=0$, $t_0=1$, $c_1=t_1$ and, in Proposition 2.8, we have shown that

$$t_e=c_et_0+c_{e-1}t_1+\cdots+c_1t_{e-1},\ \text{for all } e\geqslant 1.$$

Also, $t_e=h(p^e-1)$, for all $e\geqslant 0$.

Proposition 3.10. *We have*

(1) *If $m=1$, then $c_E=1$ and $c_e=0$ for all $e\neq E$.*
(2) *If $m=2$ and $d_1=d_2=1$, then $c_e=0$ for $e\geqslant 2$.*

Proof. (1). Note that $t_{kE}=1$ for any nonnegative integer k while $t_e=0$ for all other e. A computation based on Proposition 2.8 shows that $c_E=1$ while $c_e=0$ for all $e\neq E$.

(2). Note that $D=1$ and $E=1$. Also, $h_0(x)=x+1$ which gives $\eta_0(x)=x$, leading to $Q(z)=1-z$ and $R(z)=(1-z)(1-pz)$ which further gives $\mathcal{C}(z)=pz$ (cf. Theorem 2.9), hence establishing the claim that $c_e=0$ for all $e\geqslant 2$. □

Proposition 3.11. *Let $\mathcal{C}(z)=\sum_{e\geqslant 0}c_ez^e$ be the twisted generating function. Assume that $\gcd(d_1,\ldots,d_m)=1$. As the prime number p varies, we treat t_e and c_e as functions of p by writing $t_e=:t_e(p)$ and $c_e=:c_e(p)$. Let i be an integer such that $0\leqslant i<D$ and $\gcd(i,D)=1$, and let $P(i)$ denote the set of all prime numbers p such that $p\equiv i \mod D$.*

Then, for every fixed $e\geqslant 1$, $c_e(p)$ is a polynomial in p of degree $e(m-1)$, for all $p\in P(i)$, with leading coefficient equal to

$$\frac{1}{(m-1)!D}\left(1-\frac{1}{(m-1)!D}\right)^{e-1};$$

or equivalently, $c_e(p)$ is a polynomial of degree e in $\left(1-\frac{1}{(m-1)!D}\right)\cdot p^{m-1}$, for all $p\in P(i)$, with leading coefficient

$$\frac{1}{(m-1)!D}\cdot\left(1-\frac{1}{(m-1)!D}\right)^{-1}=\frac{1}{(m-1)!D-1}.$$

Therefore, for every given e, $\lim_{p\to\infty} \frac{c_e(p)}{p^{e(m-1)}} = \frac{1}{(m-1)!D}\left(1 - \frac{1}{(m-1)!D}\right)^{e-1}$.

Proof. With the restriction $p \in P(i)$, we prove the first claim about $c_e(p)$ by induction on $e \geqslant 1$. When $e = 1$, the claim is a consequence of Proposition 3.8, as $c_1 = t_1 = h(p-1)$. For $e \geqslant 2$, note that $c_e = t_e - (c_{e-1}t_1 + \cdots + c_1 t_{e-1})$. By Proposition 3.8, $t_{e'} = h(p^{e'} - 1)$ is a polynomial in p of degree $e'(m-1)$ with leading coefficient $\frac{1}{(m-1)!D}$, for all $p \in P(i)$. Thus the degree of $c_e(p)$ as a polynomial in p is $e(m-1)$ and the leading coefficient is

$$\frac{1}{(m-1)!D} - \sum_{e'=1}^{e-1} \frac{1}{(m-1)!D} \cdot \left(1 - \frac{1}{(m-1)!D}\right)^{e'-1} \cdot \frac{1}{(m-1)!D},$$

which equals to $\frac{1}{(m-1)!D}\left(1 - \frac{1}{(m-1)!D}\right)^{e-1}$ after some computation. Thus, with the restriction $p \in P(i)$, we have $\lim_{p\to\infty} \frac{c_e(p)}{p^{e(m-1)}} = \frac{1}{(m-1)!D}\left(1 - \frac{1}{(m-1)!D}\right)^{e-1}$, which is independent of i.

Without the restriction of $p \in P(i)$, the claim $\lim_{p\to\infty} \frac{c_e(p)}{p^{e(m-1)}} = \frac{1}{(m-1)!D}\left(1 - \frac{1}{(m-1)!D}\right)^{e-1}$ follows from the fact that there are only finitely many $P(i)$ to be considered. □

Remark 3.12. In the proof above, one can notice that c_e is a polynomial in p which is determined by the polynomial expression of $h(p^i - 1)$ as a polynomial in p, with $1 \leqslant i \leqslant e$. This expression is hence dependent on the congruence class of p modulo D.

3.2. **The polynomial case when** $E = \mathrm{ord}(p) = 1$ **in** $\mathbb{Z}_D^\times$**.** The following are the assumptions we make in this subsection.

Assumption 1. (1) *$m = 2$ and $D > 1$, or $m \geqslant 3$.*
(2) *$E = 1$, or equivalently, $p \equiv 1 \mod D$.*
(3) *The positive integers $d_1, \dots, d_m$ are pairwise relatively prime.*
(4) *The coefficients of $\eta_0(x) = h_0(x-1) = \sum_{j=0}^{m-1} a_{0,j} x^j$ are nonnegative.*

Note that when $E = 1$, the polynomials $\zeta_j(z) = a_{0,j}$ are constants for all $j = 0, \dots, m-1$. In addition, $\eta_0(x) = h_0(x-1) = \sum_{j=0}^{m-1} a_{0,j} x^j$ is a polynomial with coefficients that do not depend on p.

We introduce first a few other notations that will be useful a little later. Let $\{a_0, \dots, a_l\}$ be positive numbers and $b_0 < b_1 < \cdots < b_l$. Define

$$F(z) = \sum_{j=0}^{l} \left(a_j \prod_{\substack{0 \leqslant i \leqslant l \\ i \neq j}} (b_i - z) \right).$$

Lemma 3.13. *The polynomial $F(z)$ has degree l and has l many distinct roots $z_1, \dots, z_l \in \mathbb{R}$ with $b_0 < z_1 < b_1 < \cdots < z_{l-1} < b_{l-1} < z_l < b_l$. Moreover,*

$$z_1 + \cdots + z_l = \frac{\sum_{j=0}^{l} \left(a_j \sum_{i \neq j} b_i\right)}{\sum_{j=0}^{l} a_j}.$$

Proof. For each $i = 0, \dots l-1$, we have $F(b_i)F(b_{i+1}) < 0$. $\square$

More generally, let $a_0, \dots, a_{m-1}$ be nonnegative numbers and $b_0 < b_1 < \cdots < b_{m-1}$. Let $\Gamma = \{j : a_j \neq 0\} = \{j_0, \dots, j_l\} \neq \emptyset$, with $j_0 < \dots < j_l$. Define

$$F_\Gamma(z) = \sum_{j \in \Gamma} \left(a_j \prod_{i \in \Gamma \setminus \{j\}} (b_i - z) \right).$$

Lemma 3.14. *Using the notation above, we have*

$$\sum_{j=0}^{m-1} \left(a_j \prod_{\substack{0 \leqslant i \leqslant m-1 \\ i \neq j}} (b_i - z) \right) = F_\Gamma(z) \prod_{i \notin \Gamma} (b_i - z).$$

Moreover, $F_\Gamma(z)$ has degree l and has l many distinct roots in $\mathbb{R}$.

Recall that, as in Proposition 3.9, we set $Q(z) = \sum_{j=0}^{m-1} \left[\zeta_j(z) \prod_{i \neq j} (1 - p^{iE} z^E) \right]$. With this, as well as Assumption 1, in mind, we state the following

Corollary 3.15. *Let $\Gamma = \{j : a_{0,j} \neq 0\} = \{j_0, \dots, j_l\}$ with $j_0 < \dots < j_l$. Let*

$$F_\Gamma(z) = \sum_{j \in \Gamma} \left[a_{0,j} p^{-j} \prod_{i \in \Gamma \setminus \{j\}} (1/p^i - z) \right].$$

Note that $a_{0,m-1} = \frac{1}{(m-1)D!}$ and hence $j_l = m-1$. The following assertions hold:

(1) *We have*
$$Q(z) = p^{(m-1)m/2} F_\Gamma(z) \prod_{i \notin \Gamma} (1/p^i - z);$$

(2) *$F_\Gamma(z)$ has degree l with leading coefficient $(-1)^{l-1}\eta_0(1/p)$ when $\eta_0(1/p) \neq 0$, $F_\Gamma(0) = p^{-h}$, with $h = \sum_{j \in \Gamma} j$, and has l many real simple roots $1/\gamma_1, \dots, 1/\gamma_l$ such that $p^{j_h} < \gamma_{l-h} < p^{j_{h+1}}$, for all $h = 0, \dots, l-1$;*

(3) *The irreducible form of $\mathcal{C}(z)$ is $\dfrac{F_\Gamma(z) - \prod_{j \in \Gamma}(1/p^j - z)}{F_\Gamma(z)}$.*

(4) *We have $\lim_{p \to \infty} \dfrac{\gamma_1}{p^{m-1}} = 1 - \frac{1}{(m-1)D!}$. Here, when we take limit as $p \to \infty$, we subject p to the restriction that $p \equiv 1 \mod D$.*

Proof. (1) In light of Lemma 3.14, we have

$$\begin{aligned} Q(z) = \sum_{j=0}^{m-1} \left[a_{0,j} \prod_{i \neq j} (1 - p^i z) \right] &= p^{(m-1)m/2} \cdot \sum_{j=0}^{m-1} \left[a_{0,j} p^{-j} \prod_{i \neq j} (1/p^i - z) \right] \\ &= p^{(m-1)m/2} \cdot F_\Gamma(z) \cdot \prod_{i \notin \Gamma} (1/p^i - z)). \end{aligned}$$

(2) Since $1 = Q(0) = p^{(m-1)m/2} \cdot F_\Gamma(0) \cdot p^{-\sum_{i\notin\Gamma} i}$, we see that $F_\Gamma(0) = p^{-h}$, with $h = \sum_{j\in\Gamma} j$. The rest follows from Lemma 3.13.

(3) For the second part, $\mathcal{C}(z) = \dfrac{Q(z) - R(z)}{Q(z)}$ and

$$R(z) = \prod_{j=0}^{m-1}(1 - p^j z) = p^{(m-1)m/2} \cdot \prod_{j=0}^{m-1}(1/p^j - z).$$

Hence, after simplifying, we conclude

$$\mathcal{C}(z) = \frac{F_\Gamma(z) - \prod_{j\in\Gamma}(1/p^j - z)}{F_\Gamma(z)}.$$

To verify that we get an irreducible form of $\mathcal{C}(z)$, note that, for all $j \in \Gamma$,

$$F_\Gamma(1/p^j) = a_{0,j}/p^j \prod_{i\in\Gamma\setminus\{j\}}(1/p^i - 1/p^j) \neq 0.$$

(4) The condition $\eta_0(1/p) \neq 0$ holds for all but finitely many values of p, so we will work under this assumption, since we are considering p approaching infinity. Consider $\sum_{j\in\Gamma}\left(a_{0,j}\prod_{i\in\Gamma\setminus\{j\}}(p^i - z)\right) = p^{(l-1)l/2} \cdot z^l F_\Gamma(1/z)$, which has degree l and has $\gamma_1, \ldots, \gamma_l$ as distinct roots. We have

$$\gamma_1 + \cdots + \gamma_l = \frac{\sum_{j\in\Gamma}\left(a_{0,j} \cdot \sum_{i\in\Gamma\setminus\{j\}} p^i\right)}{\sum_{j\in\Gamma} a_{0,j}} = \sum_{j\in\Gamma}\left(a_{0,j} \cdot \sum_{i\in\Gamma\setminus\{j\}} p^i\right),$$

since $\sum_{j\in\Gamma} a_{0,j} = \eta_0(1) = 1$. Note that $j_l = m-1$, since η_0 is a polynomial of degree $m-1$ with leading coefficient $a_{0,m-1} = \frac{1}{(m-1)D!}$. Moreover, we observe that $\lim_{p\to\infty}\gamma_j/p^{m-1} = 0$, since $\gamma_j/p^{m-1} \leqslant 1/p$, for $j = 2, \ldots, l$. So, from what is proved above, we see

$$\lim_{p\to\infty}\frac{\gamma_1}{p^{m-1}} = \lim_{p\to\infty}\frac{\gamma_1 + \cdots + \gamma_l}{p^{m-1}} = \sum_{j\in\Gamma\setminus\{l\}} a_{0,j} = 1 - a_{0,j_l} = 1 - \frac{1}{(m-1)D!}. \qquad \square$$

We are ready now to state the main results about the T-construction of the polynomial ring with general grading when the order of p is 1 in $\mathbb{Z}_D^\times$.

Theorem 3.16. *Recall that $\mathscr{R} = R[x_1, \ldots, x_m]$ with grading given by $\deg(x_i) = d_i$, for $i = 1, \ldots, m$. Under the conditions stated in Assumption 1 the following assertions hold.*

(1) *The complexity sequence is given by*

$$c_e = \rho_1\gamma_1^e + \cdots + \rho_l\gamma_l^e,$$

with $p^{j_h} < \gamma_{l-h} < p^{j_{h+1}}$, for all $h = 0, \ldots, l-1$, for all $e \geqslant 1$.

(2) *$\mathscr{R}$ has rational twist of dominant eigenvalue.*

(3) *The dominant eigenvalue is γ_1 with $\gamma_1 < p^{m-1}$ and $\lim_{p\to\infty}\dfrac{\gamma_1}{p^{m-1}} = 1 - \frac{1}{(m-1)D!}$. In general, for $p \gg 0$, we have that $p^{m-2} < \gamma_1$. Additionally, if $a_{0,m-2} \neq 0$, then $p^{m-2} < \gamma_1$, for all p. Here all considerations regarding p are for values of p such that $p \equiv 1 \mod D$.*

Proof. (1) Theorem 3.1 and Corollary 3.15 immediately give the statement

(2) Corollary 3.15 shows that γ_1 is positive and strictly greater than the other γ_j, with $j = 2, \dots, l$.

(3) This is also immediate from Corollary 3.15 (4) and (2), noticing that $1 - \frac{1}{(m-1)D!} > \frac{1}{p}$ for $p \gg 0$ and that $j_{l-1} = m - 2$, if $a_{0,m-2} \neq 0$. □

3.3. **The polynomial case when $E = \operatorname{ord}(p) = 2$ in $\mathbb{Z}_D^\times$.** We are now assuming that $\operatorname{ord}(p) = 2$ in $\mathbb{Z}_D^\times$. We will further assume that $d_1, \dots, d_m$ are pairwise relatively prime. Since $E = 2$, we have two polynomials $\eta_0(x)$ and $\eta_1(x)$.

Note that $\eta_0(x) = h_0(x-1) = \sum_{j=0}^{m-1} a_{0,j} x^j$ is independent of p while $\eta_1(x) = h_k(px-1) = \sum_{j=0}^{m-1} a_{1,j} x^j$, where $p - 1 \equiv k \mod D$ with $0 \leqslant k \leqslant D-1$, and hence it does depend on p.

The following are the assumptions we make for the remainder of this section:

Assumption 2. (1) *$m \geqslant 2$.*

(2) *The positive integers $d_1, \dots, d_m$ are pairwise relatively prime.*

(3) *$E = 2$ (which forces $D > 1$).*

(4) *The coefficients of $\eta_0(x) = h_0(x-1) = \sum_{j=0}^{m-1} a_{0,j} x^j$ are positive, and moreover $a_{0,0} = \eta_0(0) > a_{1,0} = \eta_1(0) \geqslant 0$.*

Lemma 3.17. *There exists a constant C depending upon p such that*

$$\eta_1(x) - \eta_0(px) = C,$$

so we have that $p^j \cdot a_{0,j} = a_{1,j}$ for all $j = 1, \dots, m-1$.

Proof. Since $d_1, \dots, d_m$ are pairwise relatively prime, this is a consequence of Proposition 3.8 which shows that all h_k have the same coefficients in each degree, with the possible exception of their constant terms. The conclusion can be verified via studying η_0 and η_1. □

Let us recall that when $E = 2$ we have

$$\mathcal{T}(z) = \sum_{i=0}^{1} \sum_{j=0}^{m-1} \frac{a_{ij} z^i}{1 - p^{2j} z^2} = \sum_{j=0}^{m-1} \frac{a_{0,j} + a_{1,j} z}{1 - p^{2j} z^2}.$$

Since Lemma 3.17 gives $a_{1,j} = p^j \cdot a_{0,j}$ for $j = 1, \dots, m-1$, we get

$$T(z) = \frac{a_{0,0} + a_{1,0} z}{1 - z^2} + \sum_{j=1}^{m-1} \frac{a_{0,j}}{1 - p^j z} = \frac{q(z)}{p(z)},$$

where

$$p(z) = (1 - z^2) \cdot \prod_{j=1}^{m-1} (1 - p^j z),$$

$$q(z) = (a_{0,0} + a_{1,0} z) \prod_{j=1}^{m-1} (1 - p^j z) + \sum_{j=1}^{m-1} a_{0,j} \prod_{\substack{1 \leqslant i \leqslant m-1 \\ i \neq j}} (1 - p^i z)(1 - z^2).$$

Proposition 3.18. *With the same notation as above, the following assertions hold:*

(1) *The polynomials $p(z)$ and $q(z)$ are relatively prime.*
(2) $\mathcal{C}(z) = \dfrac{q(z) - p(z)}{q(z)}$ *is irreducible.*
(3) *The polynomial $q(z)$ has degree m and simple roots $1/\gamma_1, \ldots, 1/\gamma_m$ where $p^{m-j-1} < \gamma_j < p^{m-j}$ for $j = 1, \ldots, m-1$ and $-1 < \gamma_m < 0$.*
(4) *The ring $\mathscr{R}$ has rational twist of dominant eigenvalue. The dominant eigenvalue is γ_1 with $p^{m-2} < \gamma_1 < p^{m-1}$ and $\lim_{p\to\infty} \dfrac{\gamma_1}{p^{m-1}} = 1 - \frac{1}{(m-1)D!}$. Here the limit is for values of p such that $E = 2$ (or rather, for $p^2 \equiv 1 \mod D$).*
(5) *The complexity sequence is given by*

$$c_e = \rho_1\gamma_1^e + \cdots + \rho_m\gamma_m^e,$$

for all $e \geqslant 1$.

Proof. (1) We see that $p(z)$ and $q(z)$ do not have common roots, because $a_{0,0} > a_{1,0} \geqslant 0$ and $a_{0,j} > 0$ for all $j = 0, 1, \ldots, m-1$.

(2) This is an easy consequence.

(3) We see that $q(0) = \eta_0(1) = h_0(0) = 1$ and $q(-1) > 0$. The leading term of $q(z)$ is

$$(-1)^{m-1}\left(a_{1,0}p^{\sum_{j=1}^{m-1} j} + \sum_{j=1}^{m-1} a_{0,j}p^{\left(\sum_{i=1}^{m-1} i\right)-j}\right)z^m.$$

As such, we see that $q(z)$ is negative for $z \ll 0$. Therefore, $q(z)$ has a real root that is strictly less than -1. Since the coefficients of $\eta_0(x) = \sum_{j=0}^{m-1} a_{0,j}x^j$ are assumed positive and $a_{1,0} \geqslant 0$, the sign of $q(1/p^j)$ is the same as $(-1)^{m-j-1}$, for each $j = 0, \ldots, m-1$.

Summarizing, the roots of $q(z)$ are $\lambda_1, \ldots, \lambda_{m-1}, \lambda_m$, with

$$\frac{1}{p^{m-j}} < \lambda_j < \frac{1}{p^{m-j-1}} \quad \text{and} \quad \lambda_m < -1,$$

where $j = 1, \ldots, m-1$. If we write $\gamma_j = \lambda_j^{-1}$ for $j = 1, \ldots, m$, we obtain the statement.

(4) Note that $\gamma_1, \ldots, \gamma_m$ are the roots of

$$z^m q(1/z) = (a_{0,0}z + a_{1,0})\prod_{j=1}^{m-1}(z - p^j) + \sum_{j=1}^{m-1}\left(a_{0,j}\prod_{\substack{1\leqslant i\leqslant m-1\\ i\neq j}}(z - p^i)(z^2 - 1)\right).$$

The first Viéte relation for the above polynomial (with leading coefficient $q(0) = 1$) gives

$$\gamma_1 + \cdots + \gamma_m = -a_{1,0} + a_{0,0}\sum_{j=1}^{m-1} p^j + \sum_{j=1}^{m-1}\left(a_{0,j}\sum_{\substack{1\leqslant i\leqslant m-1\\ i\neq j}} p^i\right).$$

Finally, we see

$$\lim_{p\to\infty}\frac{\gamma_1}{p^{m-1}} = \lim_{p\to\infty}\frac{\gamma_1+\gamma_2+\cdots+\gamma_m}{p^{m-1}} = a_{0,0}+\cdots+a_{0,m-2}$$
$$= \eta_0(1)-a_{0,m-1} = h_0(0)-a_{0,m-1} = 1-a_{0,m-1} = 1-\frac{1}{(m-1)!D}. \qquad \square$$

Next, we are going to carry out some computations of $\mathcal{T}(z)$ and $\mathcal{C}(z)$. The examples below involve the cases where $E=1$ or $E=2$.

Example 3.19. Consider the following examples.

(1) Let $d_1=d_2=\cdots=d_m=1$. So $D=1$ and $E=1$. If $m=1$ or 2, Proposition 3.10 shows that $c_e=0$ for $e\geqslant 2$. Assume now that $m\geqslant 3$. Then $h_0(x)=\frac{(x+m-1)\cdots(x+2)(x+1)}{(m-1)!}$ and hence

$$\eta_0(x)=\frac{(x+m-2)\cdots(x+1)x}{(m-1)!}.$$

Note that the coefficients of η_0 are all nonnegative. In fact, using the notations from Corollary 3.15, we have $l=m-2$, $j_l=m-1$, and $a_{0,m-1}=\frac{1}{(m-1)!}$. Hence $\lim_{p\to\infty}\frac{\gamma_1}{p^{m-1}}=1-\frac{1}{(m-1)!}$.

(2) Let us start with the case $m=2$, $\gcd(d_1,d_2)=1$, $D=d_1d_2$ and $p\equiv 1 \mod D$. So $E=1$. One can see that $h_0(x)=\frac{1}{D}x+1$ since h_0 is a polynomial of degree one with leading coefficient $1/D$ and $h_0(0)=1$. Hence $\eta_0(x)=h_0(x-1)=\frac{1}{D}x+\frac{D-1}{D}$ and this leads to

$$\mathcal{T}(z)=\frac{D-(pD-(p-1))z}{D(1-z)(1-pz)}$$

and

$$\mathcal{C}(z)=\frac{D-(pD-(p-1))z-D(1-z)(1-pz)}{D-(pD-(p-1))z}.$$

Therefore the root of the denominator of $\mathcal{C}(z)$ is $\frac{D}{pD-(p-1)}$, which shows that we have dominant eigenvalue $\gamma=\frac{p(D-1)+1}{D}$ and the complexity sequence $\{c_e\}_e$ is given by $c_e=\rho\cdot(\frac{p(D-1)+1}{D}))^e$ for $e\geqslant 2$, where ρ is a constant. Note that $\lim_p\log_p(\gamma)=1$ as well as $\lim_p(\frac{\gamma}{p})=1-\frac{1}{D}$, as $p\to\infty$ with the restriction $p\equiv 1 \mod D$.

To compute the t-multiplicity, i.e. the value of ρ, we will consider $1-\frac{1}{\mathcal{T}(z)}$ and write this rational fraction to exhibit denominator $Q(z)=1-\frac{pD-(p-1)}{D}z$. The numerator is $P(z)=-(1-z)(1-pz)$. Since the degree of $Q(z)$ is 1, we note that the t-multiplicity equals $P(1/\gamma)$, by Remark 3.6, and so it is equal to

$$-\left(1-\frac{D}{pD-(p-1)}\right)\left(1-p\cdot\frac{D}{pD-(p-1)}\right),$$

which equals

$$\frac{(D-1)(p-1)^2}{(pD-(p-1))^2}.$$

Note that this equals 0 if $D = 1$. As $p \to \infty$, with the restriction $p \equiv 1 \mod D$, we have $\lim_p(\rho)$ equals $\frac{1}{D-1}$, when $D \neq 1$.

(3) Let us examine the case $m = 3$ and $p \equiv 1 \mod D$ in detail. In general, for $m \geqslant 2$ and $E = 1$, we have seen in the proof of Theorem 2.10 that

$$\mathcal{T}(z) = \sum_{j=0}^{m-1} \frac{a_{0j}}{1 - p^j z}.$$

Therefore, we need to find the coefficients of the polynomial $\eta_0(x) = \sum_{j=0}^{2} a_{0j} x^j$, or, equivalently, of $h_0(x) = \eta_0(x+1)$, which can be determined from the relation

$$h_0(kD) = \operatorname{rank}(\mathscr{R}_{kD}) \quad \text{for} \quad k \geqslant 0.$$

In other words, $h_0(kD)$ equals the number of nonnegative vectors $(n_1, \ldots, n_m)$ such that $\sum n_i d_i = kD$. These numbers are difficult to compute in general for arbitrary m, so let us specialize further.

Let $m = 3$, $d_1 = 2$, $d_2 = 3$, $d_5 = 5$ and $p = 31$. Note $D = 30$. By direct inspection, we see $h_0(0) = 1$, $h_0(30) = 21$ and $h_0(60) = 71$. Then $h_0(x) = \frac{1}{60}x^2 + \frac{1}{6}x + 1$ and $\eta_0(x) = \frac{1}{60}x^2 + \frac{2}{15}x + \frac{17}{20}$. Hence

$$T(z) = \frac{1 - 972z + 25451z^2}{(1-z)(1-31z)(1-961z)}$$

and the denominator of $C(z)$ is $25451z^2 - 972z + 1$ with simple roots approximately equal to 0.001058 and 0.037133. So the dominant eigenvalue γ is approximately equal to $1/0.001058 \approx 945.179$. Compare this value to $(59/60) \cdot (31)^2 \approx 944.983$ obtained by specializing

$$\left(1 - \frac{1}{(m-1)!D}\right) \cdot p^{m-1},$$

per Theorem 3.16. Note that $\log_{31}(945.179) \approx 1.995166$.

Let us consider now the general case where $m = 3$ and $p \equiv 1 \mod D$. As before,

$$\mathcal{T}(z) = \sum_{j=0}^{2} \frac{a_{0j}}{1 - p^j z},$$

where $\eta_0(x) = h_0(x-1) = \sum_{j=0}^{2} a_{0j} x^j$. Now the leading coefficient of h_0 is known to equal $1/(2D)$. Let $h_0(D) = s$, and with this notation one can compute h_0 and get

$$h_0(x) = \frac{1}{2D}x^2 + \frac{2s - D - 2}{2D}x + 1,$$

and furthermore

$$\eta_0(x) = \frac{1}{2D}(x^2 + (2s - D - 4)x + 3D + 3 - 2s).$$

So,

$$\mathcal{T}(z) = \frac{1}{2D} \cdot \left(\frac{3D + 3 - 2s}{1 - z} + \frac{2s - D - 4}{1 - pz} + \frac{1}{1 - p^2 z}\right).$$

Then the denominator of $\mathcal{C}(z)$ is

$$\big((3D+3-2s)p^3+(2s-D-4)p^2+p\big)\,z^2-bz+2D,$$

where

$$b=p^2(2D-1)+p(3D+4-2s)+2s-D-3.$$

If we denote the roots of $C(z)$ by $1/\gamma$ and $1/\delta$, where $\gamma>|\delta|$ and γ is hence the dominant eigenvalue, then a direct computation confirms that

$$\lim_{p\to\infty}\log_p\gamma=2.$$

(4) Let us consider an example with $E=2$. Let us take $m=3$, $d_1=2, d_2=3, d_3=5$ and $p=29$. Since $D=30$, we see that $E=2$ for this choice of the prime p.

In the preceding example, we computed $h_0(x)=\frac{1}{60}x^2+\frac{1}{6}x+1$ and $\eta_0(x)=\frac{1}{60}x^2+\frac{2}{15}x+\frac{17}{20}$. To compute $\eta_1(x)=h_{28}(29x-1)$, we need to find first $h_{28}(x)$. We know that $h_{28}(x)$ differs from $h_0(x)$ by a constant which can be determined by finding the value of $h(28)=h_{28}(28)=\|\{(a_1,a_2,a_3)\in\mathbb{Z}_{\geqslant 0}:2a_1+3a_2+5a_3=28\}\|$. By direct inspection, we get $h(28)=18$, and so $h_{28}(x)=\frac{1}{60}x^2+\frac{1}{6}x+\frac{4}{15}$. Then $\eta_1(x)=h_{28}(29x-1)=\frac{841}{60}x^2+\frac{58}{15}x+\frac{7}{60}$. All the conditions in Assumption 2 are satisfied. We have $q(z)=2958z^3+20629z^2-852z+1$, with roots approximately equal to -7.0150, 0.0012091 and 0.039857. The dominant eigenvalue is $\gamma_1\approx 1/0.0012091\approx 827.0614$. The reader should note that this is greater than $(59/60)\cdot(29)^2\approx 826.98$, which is obtained by specializing

$$\left(1-\frac{1}{(m-1)!D}\right)\cdot p^{m-1},$$

the value appearing in the leading term of the polynomial describing c_e obtained in Proposition 3.11; also see Proposition 3.18 concerning the ratio of the dominant eigenvalue γ_1 to p^{m-1}. Moreover, $\log_{29}(827.0614)\approx 1.9950368$.

4. Further Questions

Let $\mathscr{R}=R[x_1,\,\dots,\,x_m]$, with $\deg(x_1)=d_1,\,\dots,\,\deg(x_m)=d_m$ and $\gcd(d_1,\dots,d_m)=1$. The computations in the previous section make the case for the following conjecture.

Conjecture 4.1. The grading on $\mathscr{R}$ has rational twist with dominant eigenvalue. If $\gamma(p)$ is the dominant eigenvalue of $\mathscr{R}$ in prime characteristic p, then $\lim_{p\to\infty}\log_p\gamma(p)=m-1$.

We have already established that the grading on $\mathscr{R}$ has rational twist in Theorem 2.10. Let $D=\operatorname{lcm}(d_1,\dots,d_m)$. Let E be the order of p in $\mathbb{Z}_D^\times$. Our investigations in Section 3 show that, under additional assumptions on p, D and E, the conjecture is true. These assumptions have in common one statement that we would like to highlight in this discussion. We need to recall the notations in that section.

For every $n\in\mathbb{N}$, let $h(n):=\operatorname{rank}_R(\mathscr{R}_n)$, and denote

$$h_i(x)\in\mathbb{Q}[x],\quad i=0,\,\dots,\,D-1,$$

such that

$$h(n) = \mathrm{rank}_R(\mathscr{R}_n) = h_i(n) \quad \text{if} \quad n \equiv i \mod D.$$

for all $n \geqslant 0$. For $i = 0, \dots, E-1$, let

$$\eta_i(x) = h_k(p^i x - 1)$$

where $0 \leqslant k \leqslant D-1$ and $p^i - 1 \equiv k \mod D$. Write

$$\eta_i(x) = \sum_{j=0}^{m-1} a_{ij} x^j \in \mathbb{Q}[x], \quad i = 0, \dots, E-1.$$

For $j = 0, \dots, m-1$, let

$$\zeta_j(z) = \sum_{i=0}^{E-1} a_{ij} z^i.$$

These are polynomials of degree at most $E-1$, with leading coefficient $a_{E-1,j}$, when nonzero.

The following statement was essential in our investigations and we highlight it now as a conjecture.

Conjecture 4.2. The polynomial $\eta_0(x) = h_0(x-1)$ has positive coefficients.

Conjecture 4.2 implies the following conjecture

Conjecture 4.3. The polynomial $h_0(x)$ has positive coefficients.

Conjecture 4.3 appears as if it is a statement that should have been settled by now in the literature, since the polynomial $h_0(x)$ sits at the confluence of combinatorics, commutative algebra and discrete geometry. However, perhaps due to our own limitations, we do not know of any relevant results in this direction.

One can reformulate this two conjectures in terms of Ehrhart polynomials, which might put the statement in a different light. We close with a few comments along these lines.

Let $\mathcal{P}$ be the convex polytope with vertices at the following lattice points in $\mathbb{R}^m$:

$$v_1 = (D/d_1, 0, 0 \dots, 0, 0), v_2 = (0, D/d_2, 0, \dots, 0, 0), \dots, v_m = (0, 0, 0, \dots, 0, D/d_m).$$

Let $E_{\mathcal{P}}(t)$ be the Ehrhart polynomial associated to $\mathcal{P}$. Since $h_0(x)$ has the property that

$$h_0(nD) = \|\{(a_1, \dots, a_m) \in \mathbb{N}^m : a_1 d_1 + \dots + a_m d_m = nD\}\|,$$

we see that $E_{\mathcal{P}}(x) = h_0(xD)$, or $h_0(x) = E_{\mathcal{P}}(\frac{x}{D})$. Furthermore, $\eta_0(x) = E_{\mathcal{P}}(\frac{x-1}{D})$.

Using this interpretation, J. Louis has written a Sage procedure in [L] that checks the validity of Conjectures 4.2 and 4.3 for specific values of m, $d_1, \dots, d_m$. In all examples that we considered, the statements were verified. In addition, using the formulas from [BR] on page 15 for the general expression of h_0 for $m \leqslant 3$, stated in terms of the restricted Frobenius partition function, one can see that these conjectures hold in these cases. Using similar ideas, in [J], J. Jaramillo has derived the formula for $h_0(x)$ for $m = 4$ (which is routine) and $m = 5$; and the conjectures can be verified in the case $m = 4$ as well, while for $m = 5$ the general formula proved to complicated to be used in direct computation. We think that the connections of our work to this concept in discrete geometry will be found intriguing by others as well.

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