

INJECTIVE CAPACITY AND COGENERATION

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ABSTRACT. Let M and N be modules over a commutative ring R with N Noetherian. We define the *injective capacity of M with respect to N over R* to be the supremum of the values t for which $N^{\oplus t}$ embeds into M . In a dual fashion, we deem the *number of cogenerators of N with respect to M over R* to be the infimum of the numbers t for which N embeds into $M^{\oplus t}$. We demonstrate that the global injective capacity is the infimum of its local analogues and that the global number of cogenerators is the supremum of the corresponding local invariants. We also prove enhanced versions of these statements and consider the graded case.

1. INTRODUCTION

Our goal in this paper is to prove various refinements of the following assertion:

Theorem 1.1 (Main Theorem). *If M and N are modules over a commutative ring R with N Noetherian, then the existence of an injection in $\text{Hom}_R(N, M)$ is a local property.*

There are two reasons why we find this result worthy of report. One is the spectacular failure of the analogous statement for surjections, and the other is the voluminous response to this failure that has, at least to our knowledge, excluded consideration of the problem that we present here. We open with a survey of the related literature on epimorphisms and proceed to illustrate that, inasmuch as a comparison can be made, the monic case affords stronger conclusions with weaker hypotheses.

Theorems on factor modules relevant to our discussion come in two types. Findings of the first type deliver values t for which $\text{Hom}_R(M, N^{\oplus t})$ harbors a surjection, where M and N are modules over a ring R . For example, in [1], the first author shows that, if R is a d -dimensional commutative Noetherian ring and M and N are finitely generated R -modules such that $N^{\oplus(t+d)}$ is locally a factor of M , then $N^{\oplus t}$ is globally a factor of M [1, Theorem 1.1(1)]. The same statement holds if we replace *factor* with *direct summand* [1, Theorem 1.1(2)]. Specializations of these results are due to several authors: Serre [12, *Théorème 1*] addresses the setting in which M is projective and $N = R$; Bass [3, Theorem 8.2] removes Serre's projective condition assuming that $t = 1$; and De Stefani–Polstra–Yao [7, Theorem 3.13] extends Bass by allowing t to be an arbitrary positive integer. Additional variations on Serre's Splitting Theorem, all of which involve some notion of dimension, appear in Coquand–Lombardi–Quitté [4, Corollary 3.2], Heitmann [10, Corollary 2.6], and Stafford [13, Proposition 5.5]. These results are sharp to the extent that the coordinate ring of the real d -sphere admits an indecomposable rank- d projective module for every positive even integer d [14, Theorem 3]. So, the existence of an epimorphism between two fixed modules,

Date: May 16, 2022.

2020 Mathematics Subject Classification. Primary 13E05; Secondary 13A02, 13B30, 13C05.

Key words and phrases. Basic element, cogeneration, cogenerator, general position, graded, injective.

though not a local property, does follow from, and generally requires, certain dimensional restrictions, given that some finiteness conditions are in place.

This fact naturally affects module generation, the subject of the second type of result that concerns us here. For modules M and N over a ring R , we say that u copies of M generate N if $\text{Hom}_R(M^{\oplus u}, N)$ contains an onto map. The local existence of such integers u does not ensure global existence: If $R = M$ is a direct product of infinitely many fields and N is the direct sum of the local rings of R , then N is finitely generated locally but not globally. In order to avoid this sort of situation, theorists traditionally assume global existence when applying local data to the issue of module generation. Exploiting this concession, the first author certifies in a forthcoming paper that, if M and N are finitely generated modules over a d -dimensional commutative Noetherian ring R such that $t - d$ copies of M generate N locally, then t copies of M generate N globally [2]. In the case that $M = R$, the preceding assertion is a theorem of Forster [9, Satz 2]. A refinement of Forster achieved by Swan [15, Theorem 2] is the celebrated theorem now bearing the names of both men. Further elaborations on Forster's Theorem are due to Coutinho [5, Corollary 8.5] [6, Theorem 5.4], Eisenbud–Evans [8, Theorem B], Lyubeznik [11, Theorem 2], and Warfield [16, Theorem 2]. Despite the many ways that Forster's Theorem has been extended, Forster's local condition cannot be improved in general: For all nonnegative integers d and t with $d < t$, there is a d -dimensional commutative ring R admitting a projective module of rank $t - d$ that is not a factor of $R^{\oplus(t-1)}$ [14, Theorem 4]. Thus, once again, we are confronted with a question on epimorphisms that resists reduction to the local realm and exhibits strong ties to dimension, even with the provision of several finiteness hypotheses.

In contrast, our solution to the problem of embeddability boils down to the local case and obviates dimensional considerations. Moreover, the Noetherian condition on the source of our maps, which constitutes our only finiteness assumption, cannot be removed: Adapting a prior example, we observe that, if $R = N$ is a direct product of infinitely many fields and M is the direct sum of the local rings of R , then N embeds into M locally but not globally. Assuming, on the other hand, that M and N are modules over a commutative ring R with N Noetherian, we may marshal the fact (Lemma 4.2 below) that a map $h \in \text{Hom}_R(N, M)$ is monic if and only if $h_{\mathfrak{p}}$, when restricted to the socle of $N_{\mathfrak{p}}$, is monic for every \mathfrak{p} in the finite set $\text{Ass}_R(N)$. This characterization of injectivity allows us to use a “general position” argument in the style of [1, Sections 6 and 7], which would suffice to prove our main theorem. To complement the existing literature on direct summands of factor modules and on factor modules of direct sums, we could then replace $\text{Hom}_R(N, M)$ in our main theorem with $\text{Hom}_R(N^{\oplus t}, M)$ or $\text{Hom}_R(N, M^{\oplus t})$ for every nonnegative integer t .

Instead, we take the following route: We begin by laying down the technical foundations for our argument in Section 2, introducing two invariants of the triple (M, N, R) : the *injective capacity of M with respect to N over R* and the *number of cogenerators of N with respect to M over R* . In Section 3, we establish results on roots of polynomials and ranks of matrices to which we appeal later in the paper. Section 4 offers enhancements of our main theorem using injective capacities and numbers of cogenerators. Our main theorem follows as a corollary of these statements. Finally, in Section 5, we initiate an investigation on embeddings of graded modules.

2. CONVENTIONS AND DEFINITIONS

In this section, we flesh out our conventions for the rest of the paper and cover definitions that will streamline our discussion.

Throughout this paper, the letter R refers to a commutative ring with unity. The set of all maximal ideals of R , called the *maximal spectrum of R* , is denoted $\text{Max}(R)$. The set of all prime ideals of R , referred to as the *prime spectrum of R* , is written as $\text{Spec}(R)$. For every $\mathfrak{p} \in \text{Spec}(R)$, the symbol $\kappa(\mathfrak{p})$ represents the *residue field of R at \mathfrak{p}* or, in other words, the ring $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$.

Every R -module in this paper is unital. The letter M refers to an arbitrary R -module, and N stands for a Noetherian R -module. The *annihilator of N in R* , written as $\text{Ann}_R(N)$, is the largest ideal I of R satisfying $IN = 0$. We signify with $\text{Supp}_R(N)$ the set of all $\mathfrak{p} \in \text{Spec}(R)$ such that $N_{\mathfrak{p}} \neq 0$, and we call this set the *support of N over R* . For every $\mathfrak{p} \in \text{Spec}(R)$, the *socle of $N_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$* , denoted $\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$, is the $\kappa(\mathfrak{p})$ -module consisting of all elements of $N_{\mathfrak{p}}$ annihilated by $\mathfrak{p}_{\mathfrak{p}}$. An *associated prime of N in R* is a prime \mathfrak{p} of R such that $\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \neq 0$. The set $\text{Ass}_R(N)$ is the finite set of all associated primes of N in R . For every R -module L and nonnegative integer t , the notation $L^{\oplus t}$ serves as shorthand for the direct sum of t copies of L , with $L^{\oplus 0}$ signifying the zero R -module. $\text{Hom}_R(N, M)$ refers to the R -module of all R -linear maps from N to M . For every nonnegative integer t , we view a member of $\text{Hom}_R(N^{\oplus t}, M)$ and a member of $\text{Hom}_R(N, M^{\oplus t})$ as

$$(h_1, \dots, h_t) \in \text{Hom}_R(N^{\oplus t}, M) \quad \text{and} \quad (h_1, \dots, h_t)^{\top} := \begin{pmatrix} h_1 \\ \vdots \\ h_t \end{pmatrix} \in \text{Hom}_R(N, M^{\oplus t}),$$

respectively, where $h_1, \dots, h_t \in \text{Hom}_R(N, M)$.

For an R -submodule F of $\text{Hom}_R(N, M)$, context will determine the meaning of the symbol $F^{\oplus t}$: If we write $F^{\oplus t} \subseteq \text{Hom}_R(N^{\oplus t}, M)$, then $F^{\oplus t}$ refers to the R -module of all rows (f_1, \dots, f_t) , where $f_1, \dots, f_t \in F$; if we write $F^{\oplus t} \subseteq \text{Hom}_R(N, M^{\oplus t})$, then $F^{\oplus t}$ represents the R -module of all columns of length t with entries in F .

We now expand on two terms first mentioned in the introduction.

Definition 2.1. Let M and N signify modules over a commutative ring R with N Noetherian, and let F stand for an R -submodule of $\text{Hom}_R(N, M)$. We define $\text{inj}_R^F(M, N)$ as the supremum of the nonnegative integers t for which an injection exists in $F^{\oplus t} \subseteq \text{Hom}_R(N^{\oplus t}, M)$. We call $\text{inj}_R^F(M, N)$ the *global injective capacity of M with respect to N over R when restricted to F* . For every $\mathfrak{p} \in \text{Spec}(R)$, we call $\text{inj}_{R_{\mathfrak{p}}}^{F_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ the *local injective capacity of M with respect to N over R when restricted to F at \mathfrak{p}* . We omit the superscript F and the phrase *when restricted to F* in the case that $F = \text{Hom}_R(N, M)$. We set $\text{sup}(\emptyset) = 0$ and $\text{sup}(\{0, 1, 2, \dots\}) = \infty$.

Definition 2.2. Once again, let F be an R -submodule of $\text{Hom}_R(N, M)$, where M and N are modules over a commutative ring R with N Noetherian. The *global number of cogenerators of N with respect to M over R when restricted to F* , denoted $\text{cog}_R^F(N, M)$, refers to the infimum of the nonnegative integers t such that there is an injection in $F^{\oplus t} \subseteq \text{Hom}_R(N, M^{\oplus t})$. For every $\mathfrak{p} \in \text{Spec}(R)$, the *local number of cogenerators of N with respect to M over R when restricted to F at \mathfrak{p}* is the number $\text{cog}_{R_{\mathfrak{p}}}^{F_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$. If $F = \text{Hom}_R(N, M)$, we withhold the superscript F and all references to restriction. We set $\text{inf}(\emptyset) = \infty$.

3. ROOTS AND RANKS

In preparation for our “general position” arguments, we prove three lemmas below that may attract independent interest. Lemma 3.1 describes a way to ensure, for example, that a Cartesian product of finite subsets of a field is not contained in (a fixed embedding of) a given affine algebraic set over that field. Building on this result, Lemma 3.2 specifies an instance in which two matrices, one of which has large rank, can be combined to produce a matrix that still has large rank. Lemma 3.3 is a variant of Lemma 3.2 that we use in the last section of our paper when discussing graded modules.

Lemma 3.1. *Let k_1, \dots, k_s be domains and x_1, \dots, x_t indeterminates. For every $i \in \{1, \dots, s\}$, let f_i be a nonzero polynomial in $k_i[x_1, \dots, x_t]$ such that every monomial in its support divides $x_1^{n_{i1}} \cdots x_t^{n_{it}}$. For every $j \in \{1, \dots, t\}$, let C_j be a set with $|C_j| > \sum_{i=1}^s n_{ij}$, and suppose that there are embeddings $\phi_{j1}, \dots, \phi_{js}$ of C_j into k_1, \dots, k_s , respectively. Then there exists $(c_1, \dots, c_t) \in C_1 \times \cdots \times C_t$ such that $f_i(\phi_{1i}(c_1), \dots, \phi_{ti}(c_t)) \neq 0$ for every $i \in \{1, \dots, s\}$.*

Proof. Since $k[x_1, \dots, x_t] = k[x_1, \dots, x_{t-1}][x_t]$ for every ring k , induction on t suffices. The case $t = 1$ follows from the fact that every $f_i(x_1) \in k_i[x_1]$ has at most n_{i1} roots in k_i . For every integer $t > 1$, we treat f_i as a polynomial in $k_i[x_1, \dots, x_{t-1}][x_t]$ and let $g_i \in k_i[x_1, \dots, x_{t-1}]$ be the leading coefficient of f_i . All monomials in the support of g_i divide $x_1^{n_{i1}} \cdots x_{t-1}^{n_{i(t-1)}}$. By the induction hypothesis, there exists $(c_1, \dots, c_{t-1}) \in C_1 \times \cdots \times C_{t-1}$ such that $g_i(\phi_{1i}(c_1), \dots, \phi_{(t-1)i}(c_{t-1})) \neq 0$ for every $i \in \{1, \dots, s\}$. Hence, for every $i \in \{1, \dots, s\}$, the polynomial $f_i(\phi_{1i}(c_1), \dots, \phi_{(t-1)i}(c_{t-1}), x_t) \in k_i[x_t]$ is nonzero of degree at most n_{it} . By the induction hypothesis again, there exists $c_t \in C_t$ such that $f_i(\phi_{1i}(c_1), \dots, \phi_{ti}(c_t)) \neq 0$ for every $i \in \{1, \dots, s\}$. \square

Lemma 3.2. *For all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, t\}$, let A_i and $B_i := (B_{i1} \mid \cdots \mid B_{it})$ be $m_i \times n_i$ matrices with entries in a domain k_i ; suppose that $\text{rank}(B_i) \geq r_i$ and that B_{ij} has size $m_i \times n_{ij}$; and let C_j be a set of size exceeding $\sum_{i=1}^s \min\{r_i, n_{ij}\}$ that can be embedded into k_1, \dots, k_s via maps $\phi_{j1}, \dots, \phi_{js}$, respectively. Then there exists $(c_1, \dots, c_t) \in C_1 \times \cdots \times C_t$ such that $\text{rank}(A_i + (\phi_{1i}(c_1)B_{i1} \mid \cdots \mid \phi_{ti}(c_t)B_{it})) \geq r_i$ for every $i \in \{1, \dots, s\}$. An analogous statement holds for the transposes of $A_1, B_1, \dots, A_s, B_s$.*

Proof. For every $i \in \{1, \dots, s\}$, there exists a nonzero $r_i \times r_i$ minor of B_i . To complete the proof, it suffices to restrict A_i , B_i and hence B_{ij} to the positions defining the submatrix that gives the nonzero minor of B_i . Thus we may assume, without loss of generality, that A_i and B_i are both $r_i \times r_i$ matrices and that $\det(B_i) \neq 0$. (So, now, $r_i = n_{i1} + \cdots + n_{it}$, with the possibility of $n_{ij} = 0$ for some j .) For every $i \in \{1, \dots, s\}$, let $f_i := \det(A_i + (x_1 B_{i1} \mid \cdots \mid x_t B_{it})) = \det(A_i) + \cdots + \det(B_i)(x_1^{n_{i1}} \cdots x_t^{n_{it}})$ so that f_i is a nonzero polynomial in $k_i[x_1, \dots, x_t]$ with every monomial in its support a factor of $x_1^{n_{i1}} \cdots x_t^{n_{it}}$. Lemma 3.1 now finishes the proof of the first claim of the lemma. The second claim can be verified in a similar manner. \square

Lemma 3.3. *For all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, t\}$, let A_i and B_{ij} be $m_i \times n_i$ matrices with entries in a domain k_i . Suppose that, for every $i \in \{1, \dots, s\}$, there exists $j_i \in \{1, \dots, t\}$ such that $\text{rank}(B_{i,j_i}) \geq r_i$. For every $j \in \{1, \dots, t\}$, let C_j be a set containing at least $1 + \sum_{i=1}^s r_i$ elements but at most $\min\{|k_1|, \dots, |k_s|\}$ elements so that there exist injections $\phi_{j1}, \dots, \phi_{js}$ from C_j to k_1, \dots, k_s , respectively. Then there exists $(c_1, \dots, c_t) \in C_1 \times \cdots \times C_t$ such that $\text{rank}(A_i + \phi_{1i}(c_1)B_{i1} + \cdots + \phi_{ti}(c_t)B_{it}) \geq r_i$ for every $i \in \{1, \dots, s\}$.*

Proof. For every $i \in \{1, \dots, s\}$, there exists a nonzero $r_i \times r_i$ minor of B_{i,j_i} . To complete the proof, it suffices to restrict A_i and all B_{ij} to the positions defining the submatrix that gives the nonzero minor of B_{i,j_i} . For this reason, we reduce to the case in which $m_i = n_i = r_i$ and $\det(B_{i,j_i}) \neq 0$ and define $f_i := \det(A_i + x_1 B_{i1} + \dots + x_t B_{it}) = \det(A_i) + \dots + \det(B_{i,j_i}) x_{j_i}^{r_i} + \dots$, a nonzero polynomial in $k_i[x_1, \dots, x_t]$ with every monomial in its support dividing $x_1^{r_i} \dots x_t^{r_i}$. Applying Lemma 3.1 is all that remains to be done. \square

4. MAIN RESULTS

Our purpose in this section is to achieve several stronger versions of our main theorem. Our primary message is that neither all of $\text{Hom}_R(N, M)$ nor the entirety of $\text{Supp}_R(N)$ necessarily needs to be studied when estimating $\text{inj}_R(M, N)$ or $\text{cog}_R(N, M)$. Indeed, it suffices to work with an R -submodule of $\text{Hom}_R(N, M)$ admitting large local injective capacities or small local numbers of cogenerators on the finite set $\text{Ass}_R(N)$.

Each observation in this section also contributes something unique to our discussion. Theorem 4.1 produces maps h_1, \dots, h_v in $\text{Hom}_R(N, M)$ such that, for every associated prime \mathfrak{p} of N and for every integer $t \leq v$, if $\text{inj}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \geq t$, then $(h_1, \dots, h_t)_{\mathfrak{p}}$ is injective. Theorem 4.4 serves as the dual of Theorem 4.1 for local numbers of cogenerators. Corollary 4.3 delineates a local criterion for an R -submodule L of M to intersect trivially with an isomorphic copy of $N^{\oplus t}$ in M , and Corollary 4.5 replaces L, M , and $N^{\oplus t}$ in the last statement with $L^{\oplus t}, M^{\oplus t}$, and N , respectively. This section ends with Theorem 4.6, which summarizes our findings: A global injective capacity is the infimum of its local analogues on $\text{Ass}_R(N)$; a global number of cogenerators is the supremum of the corresponding local invariants on $\text{Ass}_R(N)$; and an R -submodule of $\text{Hom}_R(N, M)$ contains an injection if and only if its localizations at the associated primes of N contain injections.

Theorem 4.1. *Let M and N be modules over a commutative ring R with N nonzero Noetherian, and let F be an R -submodule of $\text{Hom}_R(N, M)$. For every $\mathfrak{p} \in \text{Ass}_R(N)$, let $t(\mathfrak{p})$ be a positive integer, and suppose that $t(\mathfrak{p}) \leq \text{inj}_{R_{\mathfrak{p}}}^{F_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$. Next, for every multiplicatively closed subset S of R avoiding $\text{Ann}_R(N)$, define $u(S) := \min\{t(\mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R(N), \mathfrak{p} \cap S = \emptyset\}$. Also, let $v := \max\{t(\mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R(N)\}$. Then there exist $h_1, \dots, h_v \in F$ such that the map $S^{-1}(h_1, \dots, h_{u(S)})$ is injective for every multiplicatively closed subset S of R avoiding $\text{Ann}_R(N)$. Hence, letting $u := u(\{1\})$, we find that (h_1, \dots, h_u) is injective.*

To prove this theorem, it suffices to compute $h_1, \dots, h_v \in F$ such that $(h_1, \dots, h_{t(\mathfrak{p})})_{\mathfrak{p}}$, when restricted to $\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}^{\oplus t(\mathfrak{p})})$, is injective for every $\mathfrak{p} \in \text{Ass}_R(N)$, as Lemma 4.2 indicates below. To find such maps h_1, \dots, h_v , we use Lemma 3.2 as part of a “general position” argument somewhat reminiscent of [1, Sections 6 and 7].

Lemma 4.2. *Let M and N be modules over a commutative ring R with N Noetherian. Let $h \in \text{Hom}_R(N, M)$. Then h is injective if and only if $h_{\mathfrak{p}}$, when restricted to $\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$, is injective for every $\mathfrak{p} \in \text{Ass}_R(N)$.*

Proof of Lemma 4.2. The forward direction is obvious. For the reverse direction, let $K := \ker(h)$. Then $\text{Ass}_R(K) \subseteq \text{Ass}_R(N)$. Since $\text{Soc}_{R_{\mathfrak{p}}}(K_{\mathfrak{p}}) = K_{\mathfrak{p}} \cap \text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = 0$ for every $\mathfrak{p} \in \text{Ass}_R(N)$, we see that $\text{Ass}_R(K) = \emptyset$. Hence $K = 0$, and so h is injective. \square

Proof of Theorem 4.1. The main portion of our proof works only for $\mathfrak{p} \in \text{Ass}_R(N)$ such that $|\kappa(\mathfrak{p})| > \text{rank}_{\kappa(\mathfrak{p})}(\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}))$. This condition always holds for the members of $\text{Ass}_R(N)$ not

in $\text{Max}(R)$. To account for the remaining members of $\text{Ass}_R(N)$, we take inspiration from the “general position” argument in [1, Section 6]: For every $\mathfrak{m} \in \text{Ass}_R(N) \cap \text{Max}(R)$, let $s(\mathfrak{m})$ be an element of R that avoids \mathfrak{m} but belongs to every other member of $\text{Ass}_R(N) \cap \text{Max}(R)$, and let $d(\mathfrak{m})$ be a member of $F^{\oplus v} \subseteq \text{Hom}_R(N^{\oplus v}, M)$ whose first $t(\mathfrak{m})$ components form a map that becomes injective after localizing at \mathfrak{m} and whose last $v - t(\mathfrak{m})$ components are zero. Let

$$(e_1, \dots, e_v) := \sum_{\mathfrak{m} \in \text{Ass}_R(N) \cap \text{Max}(R)} s(\mathfrak{m})d(\mathfrak{m}).$$

Then, for every $\mathfrak{m} \in \text{Ass}_R(N) \cap \text{Max}(R)$, the map $(e_1, \dots, e_{t(\mathfrak{m})})_{\mathfrak{m}}$ becomes injective when restricted to $\text{Soc}_{R_{\mathfrak{m}}}(N_{\mathfrak{m}}^{\oplus t(\mathfrak{m})})$. Hence, if $\text{Ass}_R(N) \subseteq \text{Max}(R)$, then we may take $(h_1, \dots, h_v) := (e_1, \dots, e_v)$ and apply Lemma 4.2 to close out the proof.

Otherwise, we may continue by revising the “general position” argument from [1, Section 7] in the following way: Let $\mathfrak{q}_1, \dots, \mathfrak{q}_m$ be the distinct members of $\text{Ass}_R(N) \setminus \text{Max}(R)$, and suppose that, for every $\ell \in \{1, \dots, m\}$, the ideal $\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{\ell-1}$ is not contained in \mathfrak{q}_ℓ . Fix an $\ell \in \{1, \dots, m\}$, and choose a map $(f_1, \dots, f_v) \in F^{\oplus v}$ as follows: If $\ell = 1$, then take $(f_1, \dots, f_v) := (e_1, \dots, e_v)$; if $\ell \geq 2$, then suppose inductively that, for every $\mathfrak{p} \in \text{Ass}_R(N) \setminus \{\mathfrak{q}_\ell, \dots, \mathfrak{q}_m\}$, the map $(f_1, \dots, f_{t(\mathfrak{p})})_{\mathfrak{p}}$ becomes injective when restricted to $\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}^{\oplus t(\mathfrak{p})})$. Let J denote the intersection of the members of $\text{Ass}_R(N) \setminus \{\mathfrak{q}_\ell, \dots, \mathfrak{q}_m\}$, and let $(g_1, \dots, g_{t(\mathfrak{q})}) \in F^{\oplus t(\mathfrak{q})}$ be a map that becomes injective after localizing at $\mathfrak{q} := \mathfrak{q}_\ell$. We will now prepare to apply Lemma 3.2 over the field $\kappa := \kappa(\mathfrak{q})$ with $s := 1$ and $t := t(\mathfrak{q})$. For this purpose, we define the following objects:

$$\begin{cases} V := \text{Soc}_{R_{\mathfrak{q}}}(N_{\mathfrak{q}}); & W := \sum_{j=1}^t (f_j)_{\mathfrak{q}}(V) + (g_j)_{\mathfrak{q}}(V); \\ m := \text{rank}_{\kappa}(W); & n := \text{rank}_{\kappa}(V). \end{cases}$$

Next, we fix ordered bases for V and W over κ . With respect to these ordered bases, we represent the restrictions of the maps $(f_1, \dots, f_t)_{\mathfrak{q}}$ and $(g_1, \dots, g_t)_{\mathfrak{q}}$ to $V^{\oplus t}$ with $m \times nt$ matrices A and $B := (B_1 | \dots | B_t)$ with entries in κ and with B_1, \dots, B_t being $m \times n$ submatrices of B . Let C be a system of distinct representatives for $J/(J \cap \mathfrak{q})$ in J . Using the matrices A and B , we appeal to Lemma 3.2 to conclude that there exist $c_1, \dots, c_t \in C$ such that $(f_1, \dots, f_t) + (c_1 g_1, \dots, c_t g_t)$ becomes injective after localizing at \mathfrak{q} and restricting to $V^{\oplus t}$. We may thus take $(f_1 + c_1 g_1, \dots, f_t + c_t g_t, f_{t+1}, \dots, f_v)$ to complete our inductive step, and we can conclude the proof with an application of Lemma 4.2. \square

Corollary 4.3. *Let $A \leq C$ and $B \leq D$ be modules over a commutative ring R with A nonzero Noetherian, and let G be an R -submodule of $\text{Hom}_R(C, D)$. For every $\mathfrak{p} \in \text{Ass}_R(A)$, let $t(\mathfrak{p})$ be a positive integer, and suppose that there exists a map in $G_{\mathfrak{p}}^{\oplus t(\mathfrak{p})} \subseteq \text{Hom}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}^{\oplus t(\mathfrak{p})}, D_{\mathfrak{p}})$ under which the preimage of $B_{\mathfrak{p}}$ has trivial intersection with $A_{\mathfrak{p}}^{\oplus t(\mathfrak{p})}$. Next, for every multiplicatively closed subset S of R avoiding $\text{Ann}_R(A)$, define $u(S) := \min\{t(\mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R(A), \mathfrak{p} \cap S = \emptyset\}$. Also, let $v := \max\{t(\mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R(A)\}$. Then there exist $h_1, \dots, h_v \in G$ such that the preimage of $S^{-1}B$ under $S^{-1}(h_1, \dots, h_{u(S)})$ has trivial intersection with $S^{-1}A^{\oplus u(S)}$ for every multiplicatively closed subset S of R avoiding $\text{Ann}_R(A)$. Hence, if $u := u(\{1\})$, then the preimage of B under (h_1, \dots, h_u) has trivial intersection with $A^{\oplus u}$.*

Proof. This is an application of Theorem 4.1 to $\text{inj}_R^F(M, N)$ with $N = A$, $M = D/B$, and F being the image of G under the natural map $\text{Hom}_R(C, D) \rightarrow \text{Hom}_R(A, D/B)$. \square

Theorem 4.4. *Let M and N be modules over a commutative ring R with N nonzero Noetherian, and let F be an R -submodule of $\text{Hom}_R(N, M)$. For every $\mathfrak{p} \in \text{Ass}_R(N)$, suppose that $\text{cog}_{R_{\mathfrak{p}}}^{F_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$ is finite, and let $t(\mathfrak{p})$ be an integer $\geq \text{cog}_{R_{\mathfrak{p}}}^{F_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$. For every multiplicatively closed subset S of R avoiding $\text{Ann}_R(N)$, let $v(S) := \max\{t(\mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R(N), \mathfrak{p} \cap S = \emptyset\}$. Also, let $v := v(\{1\})$. Then there exist $h_1, \dots, h_v \in F$ such that $S^{-1}(h_1, \dots, h_{v(S)})^{\top}$ is injective for every multiplicatively closed subset S of R avoiding $\text{Ann}_R(N)$. Hence $(h_1, \dots, h_v)^{\top}$ is injective.*

Proof. We may proceed as in the proof of Theorem 4.1 except that we must take the transpose of every matrix involved and consider $N_{\mathfrak{p}}$ and $M_{\mathfrak{p}}^{\oplus t(\mathfrak{p})}$ instead of $N_{\mathfrak{p}}^{\oplus t(\mathfrak{p})}$ and $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Ass}_R(N)$. Note that this proof uses the second claim of Lemma 3.2 rather than the first. \square

Corollary 4.5. *Let $A \leq C$ and $B \leq D$ be modules over a commutative ring R with A nonzero Noetherian, and let G be an R -submodule of $\text{Hom}_R(C, D)$. For every $\mathfrak{p} \in \text{Ass}_R(A)$, let $t(\mathfrak{p})$ be an integer, and suppose that there is a map in $G_{\mathfrak{p}}^{\oplus t(\mathfrak{p})} \subseteq \text{Hom}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}, D_{\mathfrak{p}}^{\oplus t(\mathfrak{p})})$ under which the preimage of $B_{\mathfrak{p}}^{\oplus t(\mathfrak{p})}$ has trivial intersection with $A_{\mathfrak{p}}$. Next, for every multiplicatively closed subset S of R avoiding $\text{Ann}_R(A)$, define $v(S) := \max\{t(\mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R(A), \mathfrak{p} \cap S = \emptyset\}$. Also, let $v := v(\{1\})$. Then there exist $h_1, \dots, h_v \in G$ such that the preimage of $S^{-1}B^{\oplus v(S)}$ under $S^{-1}(h_1, \dots, h_{v(S)})^{\top}$ has trivial intersection with $S^{-1}A$ for every multiplicatively closed subset S of R avoiding $\text{Ann}_R(A)$. Hence, if $v := v(\{1\})$, then the preimage of $B^{\oplus v}$ under $(h_1, \dots, h_v)^{\top}$ has trivial intersection with A .*

Proof. This is an application of Theorem 4.4 to $\text{cog}_R^F(N, M)$ with $N = A$, $M = D/B$, and F being the image of G under the natural map $\text{Hom}_R(C, D) \rightarrow \text{Hom}_R(A, D/B)$. \square

Theorem 4.6. *Let M and N be modules over a commutative ring R with N Noetherian, and let F be an R -submodule of $\text{Hom}_R(N, M)$. Then the following statements hold:*

- (1) $\text{inj}_R^F(M, N) = \inf\{\text{inj}_{R_{\mathfrak{p}}}^{F_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) : \mathfrak{p} \in \text{Ass}_R(N)\}$.
- (2) $\text{cog}_R^F(N, M) = \sup\{\text{cog}_{R_{\mathfrak{p}}}^{F_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) : \mathfrak{p} \in \text{Ass}_R(N)\}$.
- (3) F contains an injection if and only if $F_{\mathfrak{p}}$ contains an injection for every $\mathfrak{p} \in \text{Ass}_R(N)$.

In fact, in each of the statements above, we can replace $\text{Ass}_R(N)$ with the set consisting of just the maximal members of $\text{Ass}_R(N)$.

Proof. (1) Let t and u denote the left and right sides of the asserted equation, respectively. Since an injective map remains injective upon localization, $t \leq u$. To prove the reverse inequality, we note that either $u = 0$, in which case $u \leq t$ automatically, or $u > 0$, in which case Theorem 4.1 shows that $u \leq t$, including the case in which $u = \infty$.

(2) Label the left and right sides of the proposed equality as v and w , respectively. If w is infinite, then $v \leq w$ at once; if w is finite, then Theorem 4.4 suffices. We may also consider two cases to prove that $w \leq v$: If v is infinite, then $w \leq v$ immediately; if v is finite, then we may appeal to the exactness of localization.

(3) We may apply either Part (1) or Part (2) of the present theorem since F contains an injection if and only if $\text{inj}_R^F(M, N) \geq 1$ if and only if $\text{cog}_R^F(N, M) \leq 1$.

Having proved Parts (1), (2), and (3), we yield the last claim of the theorem by recalling once again that localization preserves injectivity. \square

5. A GRADED EMBEDDING THEOREM

Throughout this section, let $(G, +)$ denote an abelian group, and let N and M signify G -graded modules over a G -graded commutative ring R with N Noetherian. In this setting, $\text{Hom}_R(N, M)$ is a G -graded R -module. Also, for every $\mathfrak{p} \in \text{Ass}_R(N)$, every G -graded module over the domain $k(\mathfrak{p}) := R_{(\mathfrak{p})}/\mathfrak{p}_{(\mathfrak{p})}$ is free; in particular, the G -graded $k(\mathfrak{p})$ -module $\text{Soc}_{R_{(\mathfrak{p})}}(N_{(\mathfrak{p})}) := \{v \in N_{(\mathfrak{p})} : \mathfrak{p}_{(\mathfrak{p})}v = 0\}$ is free. If, moreover, $G = (\mathbb{Z}^{\oplus r}, +)$ for some nonnegative integer r , then every associated prime of N is G -graded. However, if G is an arbitrary abelian group, then N may admit an associated prime that is not G -graded. For example, if k is a field, $G := (\mathbb{Z}/2\mathbb{Z}, +)$, and $R = N$ is the G -graded ring $k[X]/(X^2 - 1)k[X] = k \oplus kx$ with $\deg(k) = \bar{0} \in G$ and $\deg(kx) = \bar{1} \in G$, then the associated primes of N are $(x + 1)R$ and $(x - 1)R$, and neither of these is G -graded.

We close this paper with a G -graded version of Theorem 4.6 in the case that R_0/\mathfrak{p}_0 is sufficiently large for every $\mathfrak{p} \in \text{Ass}_R(N)$. The cardinality condition on the rings R_0/\mathfrak{p}_0 automatically holds if, for example, R is the coordinate ring of a projective variety over an infinite field. We are unaware of whether the cardinality condition in Theorem 5.1 can be removed.

Theorem 5.1. *Let M and N be G -graded modules over a G -graded commutative ring R with N Noetherian. For every $\mathfrak{p} \in \text{Ass}_R(N)$, let $r(\mathfrak{p}) := \text{rank}_{\kappa(\mathfrak{p})}(\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}))$; let $P(\mathfrak{p}) := \{\mathfrak{q} \in \text{Ass}_R(N) : \mathfrak{q}_0 = \mathfrak{p}_0\}$; and assume that $|R_0/\mathfrak{p}_0| > \sum_{\mathfrak{q} \in P(\mathfrak{p})} r(\mathfrak{q})$. Let F be a G -graded R -submodule of $\text{Hom}_R(N, M)$, and let $d \in G$ be a fixed degree.*

If, for every $\mathfrak{p} \in \text{Ass}_R(N)$, the d th homogeneous component F_d of F contains a map that becomes injective after localization at \mathfrak{p} , then F_d contains an injection.

In particular, if every $\mathfrak{p} \in \text{Ass}_R(N)$ is G -graded and, for every $\mathfrak{p} \in \text{Ass}_R(N)$, the d th homogeneous component F_d of F contains a map that becomes injective after homogeneous localization at \mathfrak{p} , then F_d contains an injection.

The following example shows that we cannot remove the uniformity condition on degree in Theorem 5.1: Let G be an abelian group with a nonzero element d ; let $R := R_0$ be a G -graded commutative ring with two incomparable prime ideals \mathfrak{p} and \mathfrak{q} ; let $N := (R/\mathfrak{p})[0] \oplus (R/\mathfrak{q})[0]$; and let $M := (R/\mathfrak{p})[-d] \oplus (R/\mathfrak{q})[0]$. Then $\text{Hom}_R(N, M)$ contains homogeneous maps f and g such that $f_{(\mathfrak{p})}$ and $g_{(\mathfrak{q})}$ are injective, but $\text{Hom}_R(N, M)$ does not itself contain a homogeneous injection.

Since the uniformity condition in Theorem 5.1 cannot be removed, we would like to mention one way to guarantee that this condition is satisfied: First assume that, for every $\mathfrak{p} \in \text{Ass}_R(N)$, there exists a homogeneous map $f(\mathfrak{p})$ in $\text{Hom}_R(N, M)$ of degree $d(\mathfrak{p})$ that becomes injective after localization (or homogeneous localization) at \mathfrak{p} . Next, suppose that there exists $d \in G$ such that, for every $\mathfrak{p} \in \text{Ass}_R(N)$, we have $R_{d-d(\mathfrak{p})} \not\subseteq \mathfrak{p}$. Then, for every $\mathfrak{p} \in \text{Ass}_R(N)$, we can choose $s(\mathfrak{p}) \in R_{d-d(\mathfrak{p})} \setminus \mathfrak{p}$ and then replace $f(\mathfrak{p})$ with $s(\mathfrak{p})f(\mathfrak{p}) \in F_d$.

Proof of Theorem 5.1. In the case that a prime $\mathfrak{p} \in \text{Ass}_R(N)$ is graded, if a map in F_d becomes injective after homogeneous localization at \mathfrak{p} , then via further localization the same map becomes injective after localization at \mathfrak{p} . Thus it suffices to prove the first claim concerning localization at the associated primes of N . List the members of the set $\Pi := \{\mathfrak{p}_0 : \mathfrak{p} \in \text{Ass}_R(N)\}$ so that no member contains any of its predecessors. Let $\pi \in \Pi$, and suppose inductively that there is $f \in F_d$ such that, for every $\mathfrak{p} \in \text{Ass}_R(N)$ with \mathfrak{p}_0 a predecessor of π , the map $f_{\mathfrak{p}}$ becomes injective when restricted to $\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$. Let J denote

the intersection of the predecessors of π , and let $P := \{\mathfrak{p} \in \text{Ass}_R(N) : \mathfrak{p}_0 = \pi\}$. Note that $|J/(J \cap \pi)| = |(J + \pi)/\pi| = |R_0/\pi| > \sum_{\mathfrak{p} \in P} r(\mathfrak{p})$.

We will now prepare to apply Lemma 3.3. In this application, the numbers s and t from Lemma 3.3 will be $|P|$. However, since numerical subscripts refer to degrees in the present section of the paper, we will replace the subscript notation from Lemma 3.3 with in-line parenthetical indices corresponding to primes in P . For example, for every $\mathfrak{p} \in P$, we will let $g(\mathfrak{p})$ be a map in F_d that becomes injective following localization at \mathfrak{p} .

Next, we will temporarily fix $\mathfrak{p} \in P$ to define a collection of objects (indexed by \mathfrak{p}) with greater ease. We begin with the following objects:

$$\begin{cases} V(\mathfrak{p}) := \text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}); & W(\mathfrak{p}) := f_{\mathfrak{p}}(V(\mathfrak{p})) + \sum_{\mathfrak{q} \in P} g(\mathfrak{q})_{\mathfrak{p}}(V(\mathfrak{p})); \\ m(\mathfrak{p}) := \text{rank}_{\kappa(\mathfrak{p})}(W(\mathfrak{p})); & n(\mathfrak{p}) := r(\mathfrak{p}). \end{cases}$$

We also fix ordered bases for $V(\mathfrak{p})$ and $W(\mathfrak{p})$ over $\kappa(\mathfrak{p})$. With respect to these ordered bases, we represent, for every $\mathfrak{q} \in P$, the restrictions of the maps $f_{\mathfrak{p}}$ and $g(\mathfrak{q})_{\mathfrak{p}}$ to $V(\mathfrak{p})$ with $m(\mathfrak{p}) \times n(\mathfrak{p})$ matrices $A(\mathfrak{p})$ and $B(\mathfrak{p}, \mathfrak{q})$ with entries in $\kappa(\mathfrak{p})$.

We now let \mathfrak{p} vary across P once again. Let C be a system of distinct representatives for $J/(J \cap \pi)$ in J . Using all of the matrices $A(\mathfrak{p})$ and $B(\mathfrak{p}, \mathfrak{q})$, we appeal to Lemma 3.3 to conclude that there is a function $c : P \rightarrow C$ such that, for every $\mathfrak{p} \in P$, the map $f + \sum_{\mathfrak{q} \in P} c(\mathfrak{q})g(\mathfrak{q})$ becomes injective after localization at \mathfrak{p} and restriction to $\text{Soc}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$. Induction, followed by an application of Lemma 4.2, brings the proof to a close. \square

As a final remark, we observe that, if $(H, +)$ is a submonoid of G , then an H -graded ring is also a G -graded ring, and so the preceding discussion covers rings and modules graded over $(\mathbb{N}_0^{\oplus r}, +)$ for every nonnegative integer r .

ACKNOWLEDGEMENTS

The authors would like to thank Ela and Olgur Celikbas, Guantao Chen, Florian Enescu, Neil Epstein, Karl Schwede, Alexandra Smirnova, and Janet Vassilev for providing opportunities to speak about the results of this paper. The authors would also like to thank Craig Huneke and the anonymous referee for offering helpful feedback on the content of the paper.

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