GLOBAL FROBENIUS BETTI NUMBERS AND FROBENIUS EULER CHARACTERISTICS

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Abstract. We extend the notion of Frobenius Betti numbers to large classes of finitely generated modules over rings of prime characteristic, which are not assumed to be local. To do so, we introduce new invariants, that we call Frobenius Euler characteristics. We prove uniform convergence and upper semi-continuity results for Frobenius Betti numbers and Euler characteristics. These invariants detect the singularities of a ring, extending two results from the local to the global setting.

1. Introduction

Throughout this article, R will denote a commutative Noetherian ring with unity, and of prime characteristic $p > 0$. For $e \in \mathbb{Z}_{>0}$, let $F^e : R \to R$ be the e-th iterate of the Frobenius endomorphism, that is, the p^e-th power map on R. Moreover, let F^e_*R denote R as a module over itself, under restriction of scalars via F^e. Unless otherwise stated, we assume that R is F-finite, that is, the Frobenius is a finite map.

A clear methodology for globalizing measurements of singularities associated with the Frobenius endomorphism is established in [DSPY19]. In [DSPY19], these authors extended the definitions of Hilbert-Kunz multiplicity and F-signature to rings that are not necessarily local by combining strong convergence results unique to prime characteristic rings with techniques of basic element theory. We follow closely the methodology/philosophy of [DSPY19] in order to globalize Frobenius Betti numbers (see [DSHNB17]). However, the explicit differences in the numerical invariants considered in this paper, as compared to those of [DSPY19], requires unique attention to verify that they can indeed be defined and provide useful measurements of singularities for rings which are not assumed to be local.

When $(R, \mathfrak{m}, k)$ is a local ring, Frobenius Betti numbers provide an asymptotic measure of the Betti numbers of the modules F^e_*R. More explicitly, the i-th Frobenius Betti number of R is defined as

$$\beta^F_i(R) = \lim_{e\to\infty} \frac{\lambda_R(\operatorname{Tor}^R_i(k, F^e_*R))}{\operatorname{rank}(F^e_*R)}.$$

Here, $\lambda_R(-)$ denotes the length of an R-module. To prove our results about Frobenius Betti numbers, we introduce certain auxiliary invariants, that we call *Frobenius Euler characteristics*, and we denote by $\chi^F_i(R)$. If $(R, \mathfrak{m}, k)$ is local, these invariants are defined as

$$\chi^F_i(R) = \lim_{e\to\infty} \left[\sum_{j=0}^{i} (-1)^{i-j} \frac{\lambda_R(\operatorname{Tor}^R_j(k, F^e_*R))}{\operatorname{rank}(F^e_*R)}\right].$$

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Our first main result consists in proving upper semi-continuity results and uniform convergence results for the sequences converging to these two invariants (see Proposition 3.1 and Theorem 3.4). As a consequence, we obtain upper semi-continuity of such invariants.

Theorem A (See Corollary 3.5)**.** *Let R be an F-finite domain of prime characteristic $p > 0$. The functions $P \mapsto \beta_i^F(R_P)$ and $P \mapsto \chi_i^F(R_P)$ are upper semi-continuous functions as maps from* $\operatorname{Spec}(R)$ *to* $\mathbb{R}$*.*

Next, we define Frobenius Betti numbers and Frobenius Euler characteristic for rings that are not necessarily local, giving an appropriate interpretation of the numerators in the limits above as minimal number of generators of syzygies in appropriate resolutions, which we call minimal.

Theorem B (See Theorem 4.10)**.** *Let R be an F-finite domain of prime characteristic $p > 0$, not necessarily local. Then*

(1) The limits $\beta_i^F(R)$ and $\chi_i^F(R)$ exist.
(2) We have equality $\chi_i^F(R) = \max\{\chi_i^F(R_P) \mid P \in \operatorname{Spec}(R)\}$.

We are stating Theorem B only for global Frobenius Betti numbers and Frobenius Euler characteristic of the ring R. In Section 4, however, we obtain more general results for global Frobenius Betti numbers of any finitely generated R-module.

We also point out that part (2) of Theorem B shows how global Frobenius Euler characteristics are related to local Frobenius Euler characteristics. An analogous relation does not hold in full generality for global Frobenius Betti numbers (see Example 4.12).

Our third result is a way to detect the singularities of a ring using global Frobenius Betti numbers and Euler characteristics. This extends analogous statements, that were previously known only for local rings [AL08, PS19].

Theorem C (See Theorems 4.17 and 4.18)**.** *Let R be an F-finite domain of prime characteristic $p > 0$. The following conditions are equivalent:*

(1) R is regular.
(2) $\beta_i^F(R) = 0$ for some (equivalently, for all) $i > 0$.
(3) R is strongly F-regular and $\chi_i^F(R) = (-1)^i$ for some (equivalently, for all) $i \geqslant 0$.

Finally, we also show an associativity formula for Frobenius Betti numbers, that generalizes that for Hilbert-Kunz multiplicities (see Corollary 4.16), and we show that if R is a positively graded algebra over a local ring $(R_0, \mathfrak{m}_0)$, then $\beta_i^F(R) = \beta_i^F(R_\mathfrak{m})$ and $\chi_i^F(R) = \chi_i^F(R_\mathfrak{m})$, where $\mathfrak{m} = \mathfrak{m}_0 + R_{>0}$ (see Proposition 4.19).

2. Background on Frobenius Betti numbers of local rings

Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of dimension d. For an R-module M, we denote by $F_*^e M$ the module structure on M induced by restriction of scalars via $F^e : R \to R$, the e-th iterate of the Frobenius endomorphism on R. For any R-module L of finite length, and any finitely generated R-module M, the R-module $\operatorname{Tor}_i^R(L, F_*^e M)$ has finite length, for all $i \geqslant 0$ and all $e \in \mathbb{Z}_{>0}$. For a prime ideal P of R, let $\kappa(P)$ denote the residue field $R_P/PR_P \cong (R/P)_P$ of the local ring R_P. We set $\gamma(R) = \max\{\alpha(P) \mid P \in \operatorname{Spec}(R)\}$, where $\alpha(P) = \log_p[F_*\kappa(P) : \kappa(P)]$. Moreover, given a finitely generated R-module M, we set

$\gamma(M) = \gamma(R/\operatorname{ann}(M))$. In [Sei89], Seibert proves that the limit

$$\beta_i^F(L, M, \gamma) = \lim_{e\to\infty} \frac{\lambda_R\left(\operatorname{Tor}_i^R(L, F_*^e M)\right)}{p^{e\gamma}}$$

exists for every integer $\gamma \geqslant \gamma(M)$. When $L = R/\mathfrak{m}$, we simply denote $\beta_i^F(R/\mathfrak{m}, M, \gamma)$ by $\beta_i^F(M, \gamma)$. In addition, if $\gamma = \gamma(M)$, we only write $\beta_i^F(M)$, omitting γ from the notation, and we call this invariant the *i-th Frobenius Betti number of M*. We warn the reader that, in [DSHNB17], the normalization factor in the denominator of the i-th Frobenius Betti number is chosen to be $p^{e\gamma(R)}$, rather than $p^{e\gamma(M)}$.

Observe that, for all e, the length $\lambda_R(\operatorname{Tor}_i^R(R/\mathfrak{m}, F_*^e M))$ in the numerator of $\beta_i^F(M)$ is the i-th Betti number of the R-module $F_*^e M$. Moreover

$$\beta_0^F(M, \gamma(R)) = \lim_{e\to\infty} \frac{\lambda_R(R/\mathfrak{m} \otimes_R F_*^e M)}{p^{e\gamma(R)}} = \mathrm{e}_{\mathrm{HK}}(M)$$

is the Hilbert-Kunz multiplicity of M, with respect to the maximal ideal $\mathfrak{m}$.

To ease up the notation, we often write $\beta_i(e, \mathfrak{m}, M)$ for $\lambda_R(\operatorname{Tor}_i^R(R/\mathfrak{m}, F_*^e M))$. More generally, for $P \in \operatorname{Spec}(R)$ and integers $e, i \geqslant 0$, we define

$$\beta_i(e, P, M) = \lambda_{R_P}(\operatorname{Tor}_i^{R_P}(\kappa(P), F_*^e(M_P))).$$

With this notation, the i-th Frobenius Betti number of M_P as an R_P-module is

$$\beta_i^F(M_P) = \lim_{e\to\infty} \frac{\beta_i(e, P, M)}{p^{e\gamma(M_P)}}.$$

Remark 2.1. We warn the reader about a potential source of confusion with our notation. If we view M_P as an R-module, $\beta_i^F(M_P)$ is equal to $\lim\limits_{e\to\infty} \beta_i(e, \mathfrak{m}, M_P)/p^{e\gamma(M_P)}$. On the other hand, viewing M_P as an R_P-module, $\beta_i^F(M_P)$ is equal to $\lim\limits_{e\to\infty} \beta_i(e, P, M_P)/p^{e\gamma(M_P)}$. We could fix the problem by specifying the underlying ring; however, when writing $\beta_i^F(M_P)$ for a finitely generated R-module M, we will always view M_P as an R_P-module, to guarantee that the module stays finitely generated. Therefore, we will not specify the underlying ring, to avoid making the notation heavier.

3. Uniform convergence and upper semi-continuity results

A key ingredient that is used in [DSPY19] for developing a global theory of Hilbert-Kunz multiplicity and F-signature are certain semicontinuity results. In particular, to relate the global Hilbert-Kunz multiplicity to the invariants in the localization, the upper semi-continuity of the functions

$$\lambda_e : \operatorname{Spec}(R) \longrightarrow \mathbb{R}$$

$$P \longrightarrow \frac{\lambda_{R_P}(M_P/P^{[p^e]}M_P)}{p^{e\,\mathrm{ht}(P)}}$$

for locally equidimensional excellent rings, and the uniform convergence to their limit, play a crucual role. The upper semi-continuity was first established in [Kun76] (Kunz claimed that this result was true for an equidimensional ring, but Shepherd-Barron noted in [SB79] that the locally equidimensional assumption is needed). The uniform convergence was established in [Pol18, Theorem 5.1]. An immediate consequence of these two facts is that the Hilbert-Kunz function is upper semi-continuous on the spectrum of a locally equidimensional ring [Smi16, Pol18].

As for the Hilbert-Kunz multiplicity, $\beta_i^F(-)$ is additive on short exact sequences, therefore for several arguments we can reduce to the case $M = R/\mathfrak{q}$, where $\mathfrak{q}$ is a prime. We warn the reader that this does not allow us to reduce to the case when R is a domain, as for the Hilbert-Kunz multiplicity. In fact, we still have to compute the lengths modules $\operatorname{Tor}_i^{R_P}(\kappa(P), F_*^e(R/\mathfrak{q})_P)$ over the rings R_P. This complicates the arguments for the Frobenius Betti numbers, and adds some technical work to the proofs of uniform convergence and upper semi-continuity in this section.

We now recall some general notions regarding semi-continuity. Let X be a topological space. A function $f : X \to \mathbb{R}$ is *upper semi-continuous* if for all $x \in X$, for all $\varepsilon > 0$, there exists an open set $U \subseteq X$ containing x such that $f(y) < f(x) + \varepsilon$ for all $y \in U$. We say that f is *dense upper semi-continuous* if, for all $x \in X$, there exists an open set $U \subseteq X$ containing x such that $f(y) \leqslant f(x)$ for all $y \in U$.

In what follows, it will be helpful to consider an Euler characteristic version of the Frobenius Betti numbers, in part inspired by Dutta multiplicities [Dut83]. For integers $i, e \geqslant 0$, a finitely generated R-module M, and a prime $P \in \operatorname{Spec}(R)$, recall that we have defined $\beta_i(e, P, M) = \lambda_{R_P}(\operatorname{Tor}_i^{R_P}(\kappa(P), F_*^e(M_P)))$. In the same setup, we let

$$\chi_i(e, P, M) = \sum_{j=0}^{i} (-1)^{i-j} \beta_j(e, P, M).$$

Proposition 3.1. *Let R be an F-finite ring, and M be a finitely generated R-module. For $i \in \mathbb{Z}_{\geqslant 0}$, and $e \in \mathbb{Z}_{>0}$, the functions*

$$P \in \operatorname{Spec}(R) \mapsto \beta_i(e, P, M) \qquad \text{and} \qquad P \in \operatorname{Spec}(R) \mapsto \chi_i(e, P, M)$$

are dense upper semi-continuous. In particular, they are upper semi-continuous.

Proof. Let $P \in \operatorname{Spec}(R)$, and let $e > 0$. Consider a minimal free resolution of $F_*^e(M_P)$:

$$\ldots \longrightarrow R_P^{\beta_i(e)} \xrightarrow{\varphi_i} R_P^{\beta_{i-1}(e)} \xrightarrow{\varphi_{i-1}} \ldots\ldots \longrightarrow R_P^{\beta_0(e)} \xrightarrow{\varphi_0} F_*^e(M_P) \longrightarrow 0,$$

where $\beta_j(e) = \beta_j(e, P, M)$ is the j-th Betti number of $F_*^e(M_P)$ as an R_P-module. Since the rank of each free R_P-module in the resolution is finite, for all j we can find lifts $\psi_j \in \operatorname{Hom}_R(R^{\beta_j(e)}, R^{\beta_{j-1}(e)})$ of φ_j from R_P to R, giving maps

$$\ldots \longrightarrow R^{\beta_i(e)} \xrightarrow{\psi_i} R^{\beta_{i-1}(e)} \xrightarrow{\psi_{i-1}} \ldots\ldots \longrightarrow R^{\beta_0(e)} \xrightarrow{\psi_0} F_*^e M \longrightarrow 0.$$

Note that this is not even necessarily a complex. Fix an integer $i \geqslant 0$. Since R is Noetherian and all the free modules appearing above have finite rank, by inverting an element $s \in R \smallsetminus P$ we can assume that $\ker((\psi_j)_s) = \operatorname{im}((\psi_{j+1})_s)$ for all $j = 0, \ldots, i$, and that $\operatorname{im}((\psi_0)_s) = F_*^e(M_s)$. In other words,

$$R_s^{\beta_{i+1}(e)} \xrightarrow{(\psi_{i+1})_s} R_s^{\beta_i(e)} \xrightarrow{(\psi_i)_s} R_s^{\beta_{i-1}(e)} \xrightarrow{(\psi_{i-1})_s} \ldots\ldots \longrightarrow R_s^{\beta_0(e)} \xrightarrow{(\psi_0)_s} F_*^e(M_s) \longrightarrow 0.$$

is the start of a free resolution of $F_*^e(M_s)$ over the ring R_s. In particular, by localizing at any prime $Q \in \operatorname{Spec}(R)$ not containing s, the complex is still exact, and it becomes a free resolution of $F_*^e(M_Q)$ over R_Q. However, it may not be minimal. That is, $\beta_i(e, Q, M) \leqslant \beta_i$

for all $i \geqslant 0$. If we consider the Zariski open set $D(s) = \{Q \in \operatorname{Spec}(R) \mid s \notin Q\}$, we therefore have that $\beta_i(e, Q, M) \leqslant \beta_i(e, P, M)$ for all $Q \in D(s)$. This shows dense upper semicontinuity of the function $P \mapsto \beta_i(e, P, M)$. We now focus on the function $P \mapsto \chi_i(e, P, M)$. Let P, φ_j, ψ_j and $s \in R \smallsetminus P$ be as above. Let $Q \in D(s)$, and denote $\Omega_j = \ker((\psi_{j-1})_Q)$, for all $j = 0, \ldots, i$. This gives short exact sequences of R_Q-modules:

$$0 \longrightarrow \Omega_j \longrightarrow R_Q^{\beta_{j-1}(e)} \longrightarrow \Omega_{j-1} \longrightarrow 0$$

for all $j = 1, \ldots, i$. Tensoring with $\kappa(Q)$, we obtain long exact sequences

$$0 \longrightarrow \operatorname{Tor}_1^{R_Q}(\kappa(Q), \Omega_{j-1}) \longrightarrow \Omega_j/Q\Omega_j \longrightarrow \kappa(Q)^{\beta_{j-1}(e)} \longrightarrow \Omega_{j-1}/Q\Omega_{j-1} \longrightarrow 0.$$

Let $\mu_{R_Q}(-)$ denote the minimal number of generators of an R_Q-module. For $j = 1, \ldots, i$ the exact sequence above gives

$$\mu_{R_Q}(\Omega_j) = \lambda_{R_Q}(\Omega_j/Q\Omega_j) = \lambda_{R_Q}(\operatorname{Tor}_1^{R_Q}(\kappa(Q), \Omega_{j-1})) + \beta_{j-1}(e) - \mu_{R_Q}(\Omega_{j-1}).$$

Because $\operatorname{Tor}_1^{R_Q}(\kappa(Q), \Omega_{j-1}) \cong \operatorname{Tor}_j^{R_Q}(\kappa(Q), F_*^e(M_Q))$, by repeatedly using the relation above we obtain that

$$\mu_{R_Q}(\Omega_i) + \sum_{j=0}^{i-1}(-1)^{i-j}\beta_j(e) = \sum_{j=0}^{i}(-1)^{i-j}\lambda_{R_Q}(\operatorname{Tor}_j^{R_Q}(\kappa(Q), F_*^e(M_Q))) = \chi_i(e, Q, M).$$

Since $\mu_{R_Q}(\Omega_i) \leqslant \beta_i(e)$, we get the desired conclusion

$$\chi_i(e, Q, M) \leqslant \beta_i(e) + \sum_{j=0}^{i-1}(-1)^{i-j}\beta_j(e) = \chi_i(e, P, M). \qquad \square$$

We recall the following global version of a result due to Dutta [Dut83].

Lemma 3.2 ([Pol18, Lemma 2.2])**.** *Let R be an F-finite domain. There exists a finite set of nonzero primes $\mathcal{S}(R)$, and a constant C, such that for every $e \in \mathbb{Z}_{>0}$*

(1) there is a containment of R-modules $R^{\oplus p^{e\gamma(R)}} \subseteq F_*^e R$,
(2) which has a prime filtration with prime factors isomorphic to R/Q, where $Q \in \mathcal{S}(R)$,
(3) and for each $Q \in \mathcal{S}(R)$, the prime factor R/Q appears no more than $Cp^{e\gamma(R)}$ times in the chosen prime filtration of $R^{\oplus p^{e\gamma(R)}} \subseteq F_*^e R$.

Lemma 3.2 is used by the second author in [Pol18] to establish the presence of strong uniform length bounds for all F-finite rings, and in [DSPY19] to show the existence of global Hilbert-Kunz multiplicity and F-signature. Moreover, a weaker version of the uniform length bounds of [Pol18] were provided by Smirnov in [Smi16] and were central to proving the Hilbert-Kunz multiplicity function is upper semicontinuous. The reader is encouraged to compare the following lemma and its proof techniques with [Pol18, Corollary 3.4] and [Smi16, Key Lemma].

Lemma 3.3. *Let R be an F-finite ring, $\mathfrak{q} \in \operatorname{Spec}(R)$, i and γ be non-negative integers, with $\gamma \geqslant \gamma(R/\mathfrak{q})$. There exists a constant A, that only depends on i and $\mathfrak{q}$, such that*

$$\left|\frac{\beta_i(e_1+e_2, P, R/\mathfrak{q})}{(q_1q_2)^\gamma} - \frac{\beta_i(e_2, P, R/\mathfrak{q})}{q_2^\gamma}\right| \leqslant \frac{A}{q_2}$$

for all $q_1 = p^{e_1}$, $q_2 = p^{e_2}$ *and* $P \in \operatorname{Spec}(R)$. *In particular, the sequence* $\left\{ \dfrac{\beta_i(e, \bullet, R/\mathfrak{q})}{p^{e\gamma}} \right\}_{e \in \mathbb{Z}_{>0}}$ *converges uniformly on* $\operatorname{Spec}(R)$.

Proof. Let $\gamma' := \gamma(R/\mathfrak{q})$. Note that, for all $P \in \operatorname{Spec}(R)$, the limit

$$\lim_{e \to \infty} \frac{\beta_i(e, P, R/\mathfrak{q})}{p^{e\gamma'}}$$

exists and it is finite by [Sei89]. We will first show that $\frac{\beta_i(e,\bullet,R/\mathfrak{q})}{p^{e\gamma'}}$ converges uniformly to this limit. Let $q_1 = p^{e_1}$. Consider a set of primes $\mathcal{S}(R/\mathfrak{q})$ as in Lemma 3.2 for the inclusion $(R/\mathfrak{q})^{q_1^{\gamma'}} \subseteq F_*^{e_1}(R/\mathfrak{q})$, and let C be the constant given by the Lemma. For each $\mathfrak{p} \in \mathcal{S}(R/\mathfrak{q})$, the ring $R/\mathfrak{p}$ is an F-finite domain with $\gamma(R/\mathfrak{p}) \leqslant \gamma' - 1$. Applying Lemma 3.2 to each $R/\mathfrak{p}$, we obtain lists $\mathcal{S}(R/\mathfrak{p})$ and constants $D_\mathfrak{p}$. Let $\mathcal{S} = \bigcup_{\mathfrak{p} \in \mathcal{S}(R/\mathfrak{q})} \mathcal{S}(R/\mathfrak{p})$, and let $D = \max\{D_\mathfrak{p} \mid \mathfrak{p} \in \mathcal{S}\}$. Note that, for all $\mathfrak{p} \in \mathcal{S}(R/\mathfrak{q})$ and all $q_2 = p^{e_2}$, the inclusion $(R/\mathfrak{p})^{q_2^{\gamma(R/\mathfrak{p})}} \subseteq F_*^{e_2}(R/\mathfrak{p})$ has a filtration by cyclic modules of the form $R/\mathfrak{a}$, with $\mathfrak{a}$ inside $\mathcal{S}$. Furthermore, each ideal in such a filtration appears at most $Dq_2^{\gamma(R/\mathfrak{p})} \leqslant Dq_2^{\gamma'-1}$ times. For an integer $j \geqslant 0$ and a prime $\mathfrak{p}' \in \operatorname{Spec}(R)$, let $E_{j,\mathfrak{p}'}$ be the minimal number of generators of a j-th syzygy of $R/\mathfrak{p}'$ over R. Let $E_j := \max\{E_{j,\mathfrak{p}'} \mid \mathfrak{p}' \in \mathcal{S}\}$. Note that $\lambda_{R_P}(\operatorname{Tor}_j^{R_P}(\kappa(P), (R/\mathfrak{p}')_P)) \leqslant E_j$ for all $\mathfrak{p}' \in \mathcal{S}$ and all primes $P \in \operatorname{Spec}(R)$, since after localizing a resolution of $R/\mathfrak{p}'$ over R at P it stays exact, but may not be minimal.

Now let $P \in \operatorname{Spec}(R)$. After localizing everything at P we still have filtrations as before, possibly with fewer factors, since some of them may have collapsed. We remove from the lists $\mathcal{S}(R/\mathfrak{q})$ and $\mathcal{S}$ prime ideals $\mathfrak{p}$ such that $\mathfrak{p}R_P = R_P$; we still call the new lists $\mathcal{S}(R/\mathfrak{q})$ and $\mathcal{S}$. If $(R/\mathfrak{q})_P = 0$ there is nothing to show, so let us assume that $(R/\mathfrak{q})_P \neq 0$. Consider the short exact sequence

$$0 \longrightarrow (R/\mathfrak{q})_P^{q_1^{\gamma'}} \longrightarrow F_*^{e_1}(R/\mathfrak{q})_P \longrightarrow T(q_1)_P \longrightarrow 0$$

where $T(q_1)_P$ are $(R/\mathfrak{q})_P$-modules of dimension at most $\dim(R/\mathfrak{q}) - 1$. The functor $F_*^{e_2}$ is exact, and yields a short exact sequence

$$0 \longrightarrow (F_*^{e_2}(R/\mathfrak{q})_P)^{q_1^{\gamma'}} \longrightarrow F_*^{e_1+e_2}(R/\mathfrak{q})_P \longrightarrow F_*^{e_2}T(q_1)_P \longrightarrow 0$$

Applying $\operatorname{Tor}^{R_P}(\kappa(P), -)$ and counting lengths we obtain

$$\left|\lambda\left(\operatorname{Tor}_i^{R_P}(\kappa(P), F_*^{e_1+e_2}(R/\mathfrak{q})_P)\right) - p^{e_1\gamma'}\lambda\left(\operatorname{Tor}_i^{R_P}(\kappa(P), F_*^{e_2}(R/\mathfrak{q})_P)\right)\right| \leqslant$$
$$\leqslant \max\left\{\lambda\left(\operatorname{Tor}_i^{R_P}(\kappa(P), F_*^{e_2}T(q_1)_P)\right), \lambda\left(\operatorname{Tor}_{i+1}^{R_P}(\kappa(P), F_*^{e_2}T(q_1)_P)\right)\right\}.$$

Equivalently, we obtain that

$$\left|\beta_i(e_1 + e_2, P, R/\mathfrak{q}) - p^{e_1\gamma'}\beta_i(e_2, P, R/\mathfrak{q})\right| \leqslant \max\left\{\beta_i(e_2, P, T(q_1)), \beta_{i+1}(e_2, P, T(q_1))\right\}.$$

The modules $T(q_1)_P$ have filtrations $0 \subseteq T_1 \subseteq \ldots \subseteq T_{i(q_1)} = T(q_1)_P$ as in Lemma 3.2, and by exactness of $F_*^{e_2}$ we then have filtrations $0 \subseteq F_*^{e_2}T_1 \subseteq \ldots \subseteq F_*^{e_2}T_{i(q_1)} = F_*^{e_2}T(q_1)_P$. The relative quotients are isomorphic to $F_*^{e_2}(R/\mathfrak{p})_P$, for some $\mathfrak{p} \in \mathcal{S}$ appearing at most $Cq_1^{\gamma'}$ times in the filtration. Applying $\operatorname{Tor}_\bullet^{R_P}(\kappa(P), -)$ to the resulting short exact sequences, for all j we then have that

$$\beta_j(e_2, P, T(q_1)) \leqslant C\,|\mathcal{S}(R/\mathfrak{q})|\,q_1^{\gamma'}\max\left\{\beta_j(e_2, P, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{S}(R/\mathfrak{q})\right\}.$$

For $\mathfrak{p} \in \mathcal{S}(R/\mathfrak{q})$, the inclusion $(R/\mathfrak{p})_P^{q_2^{\gamma(R/\mathfrak{p})}} \subseteq F_*^{e_2}(R/\mathfrak{p})_P$ has a filtration by prime ideals in $\mathcal{S}$ appearing at most $Dq_2^{\gamma'-1}$ times. Applying $\operatorname{Tor}_\bullet^{R_P}(\kappa(P), -)$ to the resulting short exact sequences, we obtain that for all $\mathfrak{p} \in \mathcal{S}(R/\mathfrak{q})$ and all $j \geqslant 0$

$$\beta_j(e_2, P, R/\mathfrak{p}) \leqslant D\,|\mathcal{S}|\,q_2^{\gamma'-1}\max\{\beta_j(0, P, R/\mathfrak{p}') \mid \mathfrak{p}' \in \mathcal{S}\} \leqslant D\,|\mathcal{S}|\,E_j q_2^{\gamma'-1},$$

where $\beta_j(0, P, R/\mathfrak{p}')$ is just the j-th Betti number of the R_P-module $F_*^0((R/\mathfrak{p}')_P) = (R/\mathfrak{p}')_P$, that is, $\lambda_{R_P}(\operatorname{Tor}_j^{R_P}(\kappa(P), (R/\mathfrak{p}')_P))$. Recall that the constants C, D, E_j are completely independent of q, q', and the prime P. Set $A := CD\,|\mathcal{S}|\,|\mathcal{S}'|\max\{E_i, E_{i+1}\}$ and divide by $(q_1q_2)^{\gamma'}$, to obtain

$$\left|\frac{\beta_i(e_1+e_2, P, R/\mathfrak{q})}{(q_1q_2)^{\gamma'}} - \frac{\beta_i(e_2, P, R/\mathfrak{q})}{q_2^{\gamma'}}\right| \leqslant \frac{A}{q_2}$$

for all q_1, q_2, for all $P \in \operatorname{Spec}(R)$. This shows that $\frac{\beta_i(e,\bullet,R/\mathfrak{q})}{p^{e\gamma'}}$ converges uniformly.

If $\gamma = \gamma'$ then there is nothing else to prove. If $\gamma > \gamma'$, then the sequence

$$\frac{\beta_i(e, P, R/\mathfrak{q})}{p^{e\gamma}} = \frac{\beta_i(e, P, R/\mathfrak{q})}{p^{e\gamma'}} \cdot \frac{1}{p^{e(\gamma-\gamma')}}$$

converges to zero. Furthermore, the convergence is still uniform, and to see this it is enough to show that the limit function $\lim\limits_{e\to\infty} \frac{\beta_i(e,\bullet,R/\mathfrak{q})}{p^{e\gamma'}}$ is bounded on $\operatorname{Spec}(R)$. To see that, observe that, combining uniform convergence of the sequence $\frac{\beta_i(e,\bullet,R/\mathfrak{q})}{p^{e\gamma'}}$ with Proposition 3.1, we obtain that $P \mapsto \lim\limits_{e\to\infty} \frac{\beta_i(e,P,R/\mathfrak{q})}{p^{e\gamma'}}$ is upper semi-continuous. Finally, by quasi-compactness of $\operatorname{Spec}(R)$, we conclude that

$$\sup\left\{\lim_{e\to\infty}\frac{\beta_i(e, P, R/\mathfrak{q})}{p^{e\gamma'}} \mid P \in \operatorname{Spec}(R)\right\} = \max\left\{\lim_{e\to\infty}\frac{\beta_i(e, P, R/\mathfrak{q})}{p^{e\gamma'}} \mid P \in \operatorname{Spec}(R)\right\} < \infty.$$

□

Theorem 3.4. *Let R be an F-finite ring, and let M be a finitely generated R-module. Let γ be an integer satisfying $\gamma \geqslant \gamma(M)$. For any fixed $i \in \mathbb{Z}_{\geqslant 0}$, the sequence of functions*

$$\begin{aligned} \operatorname{Spec}(R) &\longrightarrow \mathbb{R} \\ P &\longrightarrow \frac{\beta_i(e, P, M)}{p^{e\gamma}} \end{aligned}$$

is uniformly bounded over $\operatorname{Spec}(R)$*, and converges uniformly to its limits as $e \to \infty$.*

Proof. We proceed by induction on $\gamma(M)$. If $\gamma(M) = 0$, then $\operatorname{Supp}(M)$ consists of finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. In addition, there exists e_0, depending on M, such that $\operatorname{ann}(F_*^e M) = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_t$ for all $e \geqslant e_0$. By the Chinese Remainder Theorem, we have that $F_*^e M \cong \oplus_{j=1}^r F_*^e(R/\mathfrak{m}_j)^{\lambda_{R_{\mathfrak{m}_j}}(M_{\mathfrak{m}_j})}$ for all $e \geqslant e_0$. Thus, for $e \geqslant e_0$, we have

$$\frac{\beta_i(e, \bullet, M)}{p^{e\gamma}} = \sum_{j=1}^r \lambda_{R_{\mathfrak{m}_j}}(M_{\mathfrak{m}_j})\frac{\beta_i(e, \bullet, R/\mathfrak{m}_j)}{p^{e\gamma}}.$$

Uniform convergence of $\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}$ then follows from Lemma 3.3, since the first $e_0 - 1$ terms do not affect this kind of considerations. Furthermore, by Proposition 3.1, the function $\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}$ is bounded on $\operatorname{Spec}(R)$ for every fixed $e \in \mathbb{Z}_{>0}$, as a consequence of its upper semi-continuity and of quasi-compactness of $\operatorname{Spec}(R)$. It then follows that $\left\{\frac{\beta_i(e,\bullet,R/\mathfrak{m}_j)}{p^{e\gamma}}\right\}_{e\in\mathbb{Z}_{>0}}$ is uniformly bounded on $\operatorname{Spec}(R)$ for each j, and thus so is $\left\{\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}\right\}_{e\in\mathbb{Z}_{>0}}$.

Now assume that $\gamma(M)>0$, and suppose that $\operatorname{ann}(M)$ is radical first. Consider a prime filtration of M:

$$0=M_0\subseteq M_1\subseteq M_2\subseteq\ldots\subseteq M_t=M,$$

where $M_j/M_{j-1}\cong R/P_j$ for some $P_j\in\operatorname{Spec}(R)$, for $j=1,\ldots,t$. Consider the R-module $N:=\bigoplus_{j=1}^t R/P_j$, and let $W=R\smallsetminus\bigcup\{\mathfrak{p}\mid\mathfrak{p}\in\operatorname{Min}(M)\}$. Since $\operatorname{ann}(M)$ is radical, we have an isomorphism $M_W\cong\prod_{\mathfrak{p}\in\operatorname{Min}(M)}\kappa(\mathfrak{p})^{\lambda_{R_\mathfrak{p}}(M_\mathfrak{p})}\cong N_W$ of R_W-modules. Because M and N are finitely generated over R, we can find an R-linear map $\varphi:M\to N$ that, after localizing at W, becomes an isomorphism. In addition, if we write

$$0\longrightarrow K\longrightarrow M\xrightarrow{\varphi}N\longrightarrow C\longrightarrow 0,$$

we have that by an observation of Kunz, [Kun76, Proposition 2.3], $\gamma(K)$ and $\gamma(C)$ are at most $\gamma(M)-1$. Denote by T the image of φ. After applying the functors $F_*^e(-)$ and $\operatorname{Tor}^{R_\bullet}(\kappa(\bullet),-_\bullet)$, and comparing lengths, we obtain that

$$|\beta_i(e,\bullet,M)-\beta_i(e,\bullet,T)|\leqslant\max\{\beta_i(e,\bullet,K),\beta_{i-1}(e,\bullet,K)\}$$

and that

$$|\beta_i(e,\bullet,T)-\beta_i(e,\bullet,N)|\leqslant\max\{\beta_i(e,\bullet,C),\beta_{i+1}(e,\bullet,C)\}.$$

By the triangle inequality, we get that

$$|\beta_i(e,\bullet,M)-\beta_i(e,\bullet,N)|\leqslant n_i(\bullet),$$

where $n_i(\bullet)=\max\{\beta_i(e,\bullet,K),\beta_{i-1}(e,\bullet,K)\}+\max\{\beta_i(e,\bullet,C),\beta_{i+1}(e,\bullet,C)\}$. By the inductive hypothesis, we have that the sequences

$$\frac{\beta_i(e,\bullet,K)}{p^{e\gamma}},\quad\frac{\beta_{i-1}i(e,\bullet,K)}{p^{e\gamma}},\quad\frac{\beta_i(e,\bullet,C)}{p^{e\gamma}},\quad\frac{\beta_{i+1}(e,\bullet,C)}{p^{e\gamma}}$$

are uniformly bounded, and converge uniformly on $\operatorname{Spec}(R)$ as $e\to\infty$. Therefore, the sequence of functions $\frac{n_i(\bullet)}{p^{e\gamma}}$ given by the sum of the maxima as defined above satisfies the same properties. In addition, since $\gamma(K)$ and $\gamma(C)$ are at most $\gamma(M)-1<\gamma$, we have that $\frac{n_i(\bullet)}{p^{e\gamma}}$ converges to zero uniformly. In particular, we have that

$$\beta_i^F(M_\bullet,\gamma)=\lim_{e\to\infty}\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}=\lim_{e\to\infty}\frac{\beta_i(e,\bullet,N)}{p^{e\gamma}}.$$

Since $F_*^eN\cong\oplus_{j=1}^tF_*^e(R/P_j)$, the sequence $\left\{\frac{\beta_i(e,\bullet,N)}{p^{e\gamma}}\right\}$ converges uniformly by Lemma 3.3. Therefore, for all $\varepsilon>0$, there exists $e_1>0$ such that for all $e>e_1$ we have

- $\left|\dfrac{\beta_i(e,\bullet,N)}{p^{e\gamma}}-\beta_i^F(M_\bullet)\right|<\dfrac{\varepsilon}{2}$.
- $\dfrac{n_i(\bullet)}{p^{e\gamma}}<\dfrac{\varepsilon}{2}$.

By the triangle inequality, we obtain

$$\left|\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}-\beta_i^F(M_\bullet)\right|\leqslant\left|\frac{\beta_i(e,\bullet,N)}{p^{e\gamma}}-\beta_i^F(M_\bullet)\right|+\frac{n_i(\bullet)}{p^{e\gamma}}<\varepsilon,$$

that is, $\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}$ converges uniformly. Finally, the fact that $\left\{\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}\right\}_{e\in\mathbb{Z}_{>0}}$ is uniformly bounded on $\operatorname{Spec}(R)$ follows from Proposition 3.1, as for the case $\gamma(M)=0$.

If $R/\operatorname{ann}(M)$ is not reduced, we can find $e_0>0$ such that $\operatorname{ann}(F^e_*M)=\sqrt{\operatorname{ann}(M)}$ for all $e\geqslant e_0$. Consider $M':=F^{e_0}_*M$, and note that $F^e_*M'\cong F^{e+e_0}_*M$ for all $e\geqslant 0$. In addition, $R/\operatorname{ann}(M')$ is reduced. We replace the sequence $\dfrac{\beta_i(e,\bullet,M)}{p^{e\gamma}}$ with the sequence $\dfrac{\beta_i(e,\bullet,M')}{p^{e\gamma}}$. Since they only differ by finitely many terms, and by a correction term of $p^{e_0\gamma}$, uniform convergence and uniform boundedness of the former would follow those for the latter. Given that $\operatorname{ann}(M')$ is radical, this has been proved above. □

Let $i\geqslant 0$ be an integer, M be a finitely generated R-module, and $\gamma\geqslant\gamma(M)$ be an integer. Using the notation introduced in Section 2, we let $\beta^F_i(M_\bullet,\gamma)$ be the limit function of the sequence considered in Theorem 3.4. Recall that, for $P\in\operatorname{Spec}(R)$, the i-th Frobenius Betti number of the R_P-module M_P is

$$\beta^F_i(M_P)=\lim_{e\to\infty}\frac{\beta_i(e,P,M)}{p^{e\gamma(M_P)}}.$$

The difference between $\beta^F_i(M_P,\gamma)$ and $\beta^F_i(M_P)$ is a possibly different normalization in the denominator. More specifically, let $Z_{M,\gamma}=\{P\in\operatorname{Spec}(R)\mid\gamma(M_P)=\gamma\}$. The set $Z_{M,\gamma}$ in the case $M=R$ and $\gamma=\gamma(R)$ has been introduced in [DSPY19] to study relations between local and global invariants for general F-finite rings. Clearly one has $\beta^F_i(M_P,\gamma)=\beta^F_i(M_P)$ whenever $P\in Z_{M,\gamma}$. On the other hand, one has $\beta^F_i(M_P,\gamma)=0$ if $P\notin Z_{M,\gamma}$. Similar considerations hold for the sequence of functions

$$\begin{aligned}\operatorname{Spec}(R)&\longrightarrow\mathbb{R}\\ P&\longrightarrow\frac{\chi_i(e,P,M)}{p^{e\gamma(R)}}\end{aligned}$$

and its limit as $e\to\infty$, that we denote by $\chi^F_i(M_\bullet,\gamma):\operatorname{Spec}(R)\to\mathbb{R}$.

Corollary 3.5. *Let R be an F-finite ring, M be a finitely generated R-module, and γ be an integer satisfying $\gamma\geqslant\gamma(M)$. For any fixed $i\in\mathbb{Z}_{\geqslant 0}$, the functions*

$$\begin{aligned}\operatorname{Spec}(R)&\longrightarrow\mathbb{R}\\ P&\longrightarrow\beta^F_i(M_P,\gamma)\end{aligned}\qquad\textit{and}\qquad\begin{aligned}\operatorname{Spec}(R)&\longrightarrow\mathbb{R}\\ P&\longrightarrow\chi^F_i(M_P,\gamma)\end{aligned}$$

are upper semi-continuous.

Proof. Since dividing by $p^{e\gamma}$ does not affect semi-continuity of the functions $P\in\operatorname{Spec}(R)\mapsto\beta_i(e,P,M)$ and $P\in\operatorname{Spec}(R)\mapsto\chi_i(e,P,M)$, Proposition 3.1 gives that $\frac{\beta_i(e,\bullet,M)}{p^{e\gamma}}$ and $\frac{\chi_i(e,\bullet,M)}{p^{e\gamma}}$ are upper semi-continuous for all $e\geqslant 0$. Because the second sequence is built as a finite alternating sum of elements from the first, Theorem 3.4 gives uniform convergence over $\operatorname{Spec}(R)$ as $e\to\infty$ for both sequences. It then follows that $P\in\operatorname{Spec}(R)\mapsto\beta^F_i(M_P,\gamma)$ and $P\in\operatorname{Spec}(R)\mapsto\chi^F_i(M_P,\gamma)$ are upper semi-continuous, as they are the uniform limit of upper semi-continuous functions. □

4. Minimal free resolutions and existence of global limits

In this section, we introduce and justify the notion of global Frobenius Betti numbers and Frobenius Euler characteristic. In what follows, $\mu_R(-)$ will denote the minimal number of generators of an R-module.

We start with an easy consequence of Schanuel's lemma.

Lemma 4.1. *Let R be a Noetherian ring of Krull dimension d, $i \geqslant 1$ an integer, and let M be a finitely generated R-module. Let*

$$0 \longrightarrow \Omega \longrightarrow R^{b_{i-1}} \longrightarrow \cdots\cdots \longrightarrow R^{b_0} \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow \Omega' \longrightarrow R^{b'_{i-1}} \longrightarrow \cdots\cdots \longrightarrow R^{b'_0} \longrightarrow M \longrightarrow 0$$

be exact sequences. Then $\left|\left(\mu_R(\Omega) + \sum_{j=1}^{i}(-1)^j b_{i-j}\right) - \left(\mu_R(\Omega') + \sum_{j=1}^{i}(-1)^j b'_{i-j}\right)\right| \leqslant d$.

Proof. By Schanuel's lemma, we have that

$$\Omega \oplus R^{\sum_{j \text{ odd}} b'_{i-j} + \sum_{j \text{ even}} b_{i-j}} \cong \Omega' \oplus R^{\sum_{j \text{ odd}} b_{i-j} + \sum_{j \text{ even}} b'_{i-j}}.$$

By the Forster-Swan Theorem [For64, Swa67], we may choose $\mathfrak{m} \in \operatorname{Max}\operatorname{Spec}(R)$ such that $\mu_R(\Omega') \leqslant \mu_{R_\mathfrak{m}}(\Omega'_\mathfrak{m}) + d$. Consequently we see

$$\begin{aligned}
\mu_R(\Omega) + \sum_{j \text{ odd}} b'_{i-j} + \sum_{j \text{ even}} b_{i-j} &\geqslant \mu_R(\Omega \oplus R^{\sum_{j \text{ odd}} b'_{i-j} + \sum_{j \text{ even}} b_{i-j}}) \\
&= \mu_R(\Omega' \oplus R^{\sum_{j \text{ odd}} b_{i-j} + \sum_{j \text{ even}} b'_{i-j}}) \\
&\geqslant \mu_{R_\mathfrak{m}}((\Omega' \oplus R^{\sum_{j \text{ odd}} b_{i-j} + \sum_{j \text{ even}} b'_{i-j}})_\mathfrak{m}) \\
&= \mu_{R_\mathfrak{m}}(\Omega'_\mathfrak{m}) + \sum_{j \text{ odd}} b_{i-j} + \sum_{j \text{ even}} b'_{i-j} \\
&\geqslant \mu_R(\Omega') - d + \sum_{j \text{ odd}} b_{i-j} + \sum_{j \text{ even}} b'_{i-j}.
\end{aligned}$$

Therefore $\left(\mu_R(\Omega') + \sum_{j=1}^{i}(-1)^j b'_{i-j}\right) - \left(\mu_R(\Omega) + \sum_{j=1}^{i}(-1)^j b_{i-j}\right) \leqslant d$. Using a symmetric argument we establish the Lemma. ☐

Remark 4.2. In the notation of Lemma 4.1, let $\gamma \geqslant \min\{1, d\}$ be an integer. For every $e \in \mathbb{Z}_{>0}$, fix a free resolution

$$\cdots \longrightarrow R^{b_i(e)} \xrightarrow{\varphi_i(e)} R^{b_{i-1}(e)} \xrightarrow{\varphi_{i-1}(e)} \cdots\cdots \longrightarrow R^{b_0(e)} \xrightarrow{\varphi_0(e)} F^e_* M \longrightarrow 0$$

of $F^e_* M$. It follows from Lemma 4.1 that

$$\chi_i^F(M, \gamma) = \lim_{e \to \infty} \frac{\mu_R(\operatorname{im}(\varphi_i(e))) + \sum_{j=1}^{i}(-1)^j b_{i-j}(e)}{p^{e\gamma}}$$

is independent of the choices of resolutions for $F^e_* M$. When $\gamma = \gamma(M)$, we omit γ from the notation, and call $\chi_i^F(M)$ the *i-th (global) Frobenius Euler characteristic of M*. At the moment, we are not claiming that the limit exists.

To study global Frobenius Betti numbers, we need a version of Lemma 4.1 that compares minimal number of generators of the modules $\operatorname{im}(\varphi_i(e))$ in different resolutions. First, we record the following special case of Lemma 4.1.

Lemma 4.3. *Let R be a Noetherian ring of Krull dimension d, and let M be a finitely generated R-module. Let $0 \to \Omega \to R^n \to M \to 0$ and $0 \to \Omega' \to R^n \to M \to 0$ be short exact sequences. Then $|\mu_R(\Omega) - \mu_R(\Omega')| \leqslant d$.*

Definition 4.4. Let M be a finitely generated R-module, and let $e > 0$ be an integer. Consider a free resolution of $F^e_* M$

$$\cdots \longrightarrow R^{b_i(e)} \xrightarrow{\varphi_i(e)} R^{b_{i-1}(e)} \xrightarrow{\varphi_{i-1}(e)} \cdots\cdots \longrightarrow R^{b_0(e)} \xrightarrow{\varphi_0(e)} F^e_* M \longrightarrow 0.$$

We say that the resolution is *minimal* if, setting $\Omega_i(e) := \operatorname{im}(\varphi_i(e))$, we have $\mu_R(\Omega_i(e)) = b_i(e)$ for all $i \geqslant 0$.

Lemma 4.5. *Let R be a Noetherian ring of Krull dimension d, and M a finitely generated R-module. Let*

$$\cdots \longrightarrow R^{b_i(e)} \xrightarrow{\varphi_i(e)} R^{b_{i-1}(e)} \xrightarrow{\varphi_{i-1}(e)} \cdots\cdots \longrightarrow R^{b_0(e)} \xrightarrow{\varphi_0(e)} F^e_* M \longrightarrow 0.$$

and

$$\cdots \longrightarrow R^{b'_i(e)} \xrightarrow{\psi_i(e)} R^{b'_{i-1}(e)} \xrightarrow{\psi_{i-1}(e)} \cdots\cdots \longrightarrow R^{b'_0(e)} \xrightarrow{\psi_0(e)} F^e_* M \longrightarrow 0$$

be minimal free resolutions of $F^e_* M$. Then $|b_i(e) - b'_i(e)| \leqslant d2^{i-1}$.

Proof. This follows immediately from a repeated application of Lemma 4.3. □

Remark 4.6. In the notation of Lemma 4.5, let $\gamma \geqslant \min\{1, d\}$ be an integer. We have that

$$\beta^F_i(M,\gamma) = \lim_{e\to\infty} \frac{\mu_R(\Omega_i(e))}{p^{e\gamma}} = \lim_{e\to\infty} \frac{\mu_R(\Omega'_i(e))}{p^{e\gamma}},$$

and is therefore independent of the choice of a minimal free resolution for $F^e_* M$. When $\gamma = \gamma(M)$, we simply write $\beta^F_i(M)$, and we call it the *i-th (global) Frobenius Betti number of M*. As in Remark 4.2, we are not yet claiming that the limits exist. We are only stating that one limit exists if and only if the other one does and, in this case, they coincide. Observe that $\beta^F_0(M, \gamma(R)) = \mathrm{e}_{\mathrm{HK}}(M)$ is the global Hilbert-Kunz multiplicity of M, therefore we know the limit exists in this case [DSPY19].

Remark 4.7. For a finitely generated R-module M and integers $i \geqslant 0$ and γ between $\gamma(M)$ and $\gamma(R)$, recall the notation $Z_{M,\gamma} = \{P \in \operatorname{Spec}(R) \mid \gamma(M_P) = \gamma\}$ introduced at the end of Section 3. We have already observed that $\chi^F_i(M_P, \gamma) = \chi^F_i(M_P)$ if $P \in Z_{M,\gamma}$, while $\chi^F_i(M_P, \gamma) = 0$ if $P \notin Z_{M,\gamma}$.

Proposition 4.8. *Let R be an F-finite ring, M be a finitely generated R-module, $i \geqslant 0$ and γ be integers, with $\gamma \geqslant \gamma(M)$. For all integers $e \geqslant 0$, let $P_e \in \operatorname{Spec}(R)$ be such that $\chi_i(e, P_e, M) = \max\{\chi_i(e, P, M) \mid P \in \operatorname{Spec}(R)\}$. Then*

$$\lim_{e\to\infty} \frac{\chi_i(e, P_e, M)}{p^{e\gamma}} = \lim_{e\to\infty} \chi^F_i(M_{P_e}, \gamma) = \max\{\chi^F_i(M_P, \gamma) \mid P \in \operatorname{Spec}(R)\}.$$

Let χ be the common value of the equation above. If either $Z_{M,\gamma} = \operatorname{Spec}(R)$ or $\chi \neq 0$, we also have

$$\lim_{e\to\infty} \frac{\chi_i(e, P_e, M)}{p^{e\gamma}} = \max\{\chi^F_i(M_P) \mid P \in Z_{M,\gamma}\}.$$

Proof. Let $Q \in \operatorname{Spec}(R)$ be such that $\chi_i^F(M_Q,\gamma) = \max\{\chi_i^F(M_P,\gamma) \mid P \in \operatorname{Spec}(R)\}$, and let $\varepsilon > 0$. By Theorem 3.4, the sequence $\chi_i(e,\bullet,M)/p^{e\gamma}$ converges uniformly to its limit $\chi_i^F(M_\bullet,\gamma)$ on $\operatorname{Spec}(R)$. Therefore, there exists e_0 such that for all $e > e_0$

$$\left|\frac{\chi_i(e,P,M)}{p^{e\gamma}} - \chi_i^F(M_P,\gamma)\right| < \frac{\varepsilon}{2}$$

holds for all $P \in \operatorname{Spec}(R)$. Then, for all $e > e_0$ we obtain

$$\chi_i^F(M_Q,\gamma) \geqslant \chi_i^F(M_{P_e},\gamma) > \frac{\chi_i(e,P_e,M)}{p^{e\gamma}} - \frac{\varepsilon}{2} \geqslant \frac{\chi_i(e,Q,M)}{p^{e\gamma}} - \frac{\varepsilon}{2} > \chi_i^F(M_Q,\gamma) - \varepsilon.$$

Since ε is arbitrary, this completes the proof of the first part of the Proposition. The second claim is now clear if $Z_{M,\gamma} = \operatorname{Spec}(R)$, since in this case $\chi_i^F(M_P,\gamma) = \chi_i^F(M_P)$ for all $P \in \operatorname{Spec}(R)$. On the other hand, if $\chi \neq 0$, by the first part there exists $P \in \operatorname{Spec}(R)$ such that $\chi_i^F(M_P,\gamma) = \chi \neq 0$. It then follows from Remark 4.7 that $\max\{\chi_i^F(M_P,\gamma) \mid P \in \operatorname{Spec}(R)\} = \max\{\chi_i^F(M_P,\gamma) \mid P \in Z_{M,\gamma}\}$. Using that $\chi_i^F(M_P,\gamma) = \chi_i^F(M_P)$ for $P \in Z_{M,\gamma}$, we finally conclude that

$$\begin{aligned}\lim_{e\to\infty} \frac{\chi_i^F(e,P_e,M)}{p^{e\gamma}} &= \max\{\chi_i^F(M_P,\gamma) \mid P \in \operatorname{Spec}(R)\} \\ &= \max\{\chi_i^F(M_P,\gamma) \mid P \in Z_{M,\gamma}\} \\ &= \max\{\chi_i^F(M_P) \mid P \in Z_{M,\gamma}\}.\end{aligned}$$

□

The assumptions for the second claim in the Proposition are needed, as the following example shows.

Example 4.9. Let $S = \mathbb{F}_p$, and $T = \mathbb{F}_p(t)$, and consider the ring $R = S \times T$. Since $\chi_1(e, S \times 0, R) = -p^e$ and $\chi_1(e, 0 \times T, R) = -1$, using the notation of Proposition 4.8 we have $P_e = 0 \times T$ for all e. Using $\gamma = \gamma(R) = 1$, it then follows that $\lim_{e\to\infty} \frac{\chi_1(e,P_e,R)}{p^e} = 0$. However, one has $\max\{\chi_1^F(R_P) \mid P \in Z_{R,1}\} = \chi_1^F(R_{S\times 0}) = \chi_1^F(\mathbb{F}_p(t)) = -1$. Observe that there is no contradiction with the first part of the Proposition, since $\max\{\chi_1^F(R_P,1) \mid P \in \operatorname{Spec}(R)\} = \chi_1^F(R_{0\times T},1) = \chi_1^F(\mathbb{F}_p,1) = 0$.

Theorem 4.10. *Let R be an F-finite ring of prime characteristic $p > 0$, M be a finitely generated R-module, and $\gamma \geqslant \gamma(M)$ be an integer such that $\gamma \geqslant \min\{1,\dim(R)\}$. For every $e \in \mathbb{Z}_{>0}$, fix any free resolution $(G_\bullet(e),\varphi_\bullet(e))$ of the module $F_*^e M$:*

$$\cdots \longrightarrow R^{b_i(e)} \xrightarrow{\varphi_i(e)} R^{b_{i-1}(e)} \xrightarrow{\varphi_{i-1}(e)} \cdots \longrightarrow R^{b_0(e)} \xrightarrow{\varphi_0(e)} F_*^e M \longrightarrow 0.$$

For $i \in \mathbb{Z}_{\geqslant 0}$, let $\Omega_i(e) = \operatorname{im}(\varphi_i(e))$. Then:

(1) The limit $\chi_i^F(M,\gamma) = \lim_{e\to\infty} \dfrac{\mu_R(\Omega_i(e)) + \sum_{j=1}^{i}(-1)^j b_{i-j}(e)}{p^{e\gamma}}$ exists.

(2) $\chi_i^F(M,\gamma) = \max\{\chi_i^F(M_P,\gamma) \mid P \in \operatorname{Spec}(R)\}$. Moreover, if either this value is non-zero or $Z_{M,\gamma} = \operatorname{Spec}(R)$, then it is also equal to $\max\{\chi_i^F(M_P) \mid P \in Z_{M,\gamma}\}$.

(3) Assume further that, for all $e \in \mathbb{Z}_{>0}$, the free resolution $G_\bullet(e)$ is chosen to be minimal. Then the limit $\beta_i^F(M,\gamma) = \lim_{e\to\infty} \dfrac{b_i(e)}{p^{e\gamma}}$ exists.

Proof. For all $P \in \operatorname{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$, localizing the resolution $(G_\bullet(e), \varphi_\bullet(e))$ at P gives an exact sequence:

$$0 \longrightarrow \Omega_i(e)_P \longrightarrow R_P^{b_{i-1}(e)} \longrightarrow \cdots\cdots \longrightarrow R_P^{b_0(e)} \longrightarrow F_*^e(M_P) \longrightarrow 0,$$

which gives

$$\mu_{R_P}(\Omega_i(e)_P) + \sum_{j=1}^{i} (-1)^j b_{i-j}(e) = \chi_i(e, P, M).$$

In particular, this shows that

$$\max\{\mu_{R_P}(\Omega_i(e)_P) \mid P \in \operatorname{Spec}(R)\} + \sum_{j=1}^{i} (-1)^j b_{i-j}(e) = \max\{\chi_i(e, P, M) \mid P \in \operatorname{Spec}(R)\}.$$

For all $e \in \mathbb{Z}_{>0}$, let P_e be a prime that achieves such maximum. Then, by the Forster-Swan Theorem [For64, Swa67], we have that $\mu_{R_{P_e}}(\Omega_i(e)_{P_e}) \leqslant \mu_R(\Omega_i(e)) \leqslant \mu_{R_{P_e}}(\Omega_i(e)_{P_e}) + \dim(R)$. Therefore, for all $e > 0$, we have

$$\frac{\chi_i(e, P_e, M)}{p^{e\gamma}} \leqslant \frac{\mu_R(\Omega_i(e)) + \sum_{j=0}^{i} (-1)^j b_{i-j}(e)}{p^{e\gamma}} \leqslant \frac{\chi_i(e, P_e, M) + \dim(R)}{p^{e\gamma}}.$$

Part (1) and (2) now follow from Proposition 4.8, since the difference between the two terms on the sides of the inequality goes to zero because of our assumptions on γ. Given that $\chi_i^F(M, \gamma)$ exists as a limit, for part (3) it is enough to observe that, if $G_\bullet$ is minimal, then we have

$$\beta_i^F(M, \gamma) = \chi_i^F(M, \gamma) + \chi_{i-1}^F(M, \gamma).$$

□

Remark 4.11. As a consequence of Theorem 4.10 (2), we have that $\chi_i^F(M, \gamma) = 0$ for all $i \in \mathbb{Z}_{\geqslant 0}$ whenever $\gamma > \gamma(M)$. Therefore, $\beta_i^F(M, \gamma) = 0$ for $\gamma > \gamma(M)$ as well.

Unlike the case of $\chi_i^F(M, \gamma)$, $\beta_i^F(M, \gamma)$ does not coincide with the maximal value of the local invariants achieved on $\operatorname{Spec}(R)$.

Example 4.12. Consider the ring $R = \mathbb{F}_p \times \mathbb{F}_p$ and the R-module $M = \mathbb{F}_p \times 0$, so $\gamma(M) = 0$ and $F_*^e M \cong M$ for all $e \geqslant 0$. Since M is projective over R hence locally free, we see $\max\{\beta_i^F(M_P, 0) \mid P \in \operatorname{Spec}(R)\} = 0$ for all $i \geqslant 1$. On the other hand, for each $e \geqslant 0$, it is easy to see a minimal free resolution of $F_*^e M$ (cf. Definition 4.4)

$$\cdots \longrightarrow R \longrightarrow R \longrightarrow \cdots \longrightarrow R \longrightarrow F_*^e M \longrightarrow 0,$$

which yields $\beta_i^F(M, 0) = 1$ for all $i \geqslant 0$. More generally, if $R = R_1 \times R_2$ with each R_i a regular F-finite local ring such that $\gamma(R_1) \geqslant \gamma(R_2)$ and if $M = R_1 \times 0$, then $\max\{\beta_i^F(M_P, \gamma(R)) \mid P \in \operatorname{Spec}(R)\} = 0$ for all $i \geqslant 1$ while $\beta_i^F(M, \gamma(R)) = 1$ for all $i \geqslant 0$.

Or let $R = R_1 \times R_2$ with each R_i F-finite and regular local such that $\gamma(R_1) \neq \gamma(R_2)$. Then $\max\{\beta_i^F(R_P, \gamma(R)) \mid P \in \operatorname{Spec}(R)\} = 0$ for all $i \geqslant 1$ while $\beta_i^F(R, \gamma(R)) = 1$ for all $i \geqslant 0$.

We ask the following question.

Question 4.13. Under what condition does the following equality hold:

$$\beta_i^F(M, \gamma) = \max\{\beta_i^F(M_P, \gamma) \mid P \in \operatorname{Spec}(R)\}?$$

The following example provides evidence that studying Question 4.13 may lead to some interesting consequences.

Example 4.14. Let Q be an F-finite regular ring, and let f be an non-unit element of Q. Let $R = Q/(f)$, and assume that $Z_{R,\gamma(R)} = \operatorname{Spec}(R)$. By [DSHNB17, Example 3.2], for all $P \in \operatorname{Spec}(R)$, we have

$$\beta_i^F(R_P) = \begin{cases} \mathrm{e}_{\mathrm{HK}}(R_P) & i = 0 \\ \mathrm{e}_{\mathrm{HK}}(R_P) - \mathrm{s}(R_P) & i > 0 \end{cases}$$

where $\mathrm{s}(R_P)$ is the F-signature of the local ring R_P. Therefore

$$\chi_i^F(R_P) = \begin{cases} \mathrm{e}_{\mathrm{HK}}(R_P) & i \text{ even} \\ -\mathrm{s}(R_P) & i \text{ odd} \end{cases}$$

By Theorem 4.10 we have that

$$\chi_1^F(R) = \max\{\chi_1^F(R_P) \mid P \in \operatorname{Spec}(R)\} = -\min\{\mathrm{s}(R_P) \mid P \in \operatorname{Spec}(R)\},$$

and

$$\chi_0^F(R) = \max\{\chi_0^F(R_P) \mid \in \operatorname{Spec}(R)\} = \max\{\mathrm{e}_{\mathrm{HK}}(R_P) \mid P \in \operatorname{Spec}(R)\}.$$

Since $\beta_1^F(R) = \chi_1^F(R) + \chi_0^F(R)$, it follows that $\beta_1^F(R) = \max\{\beta_1^F(R_P) \mid P \in \operatorname{Spec}(R)\} = \max\{\mathrm{e}_{\mathrm{HK}}(R_P) - \mathrm{s}(R_P) \mid P \in \operatorname{Spec}(R)\}$ if and only if

$$(*) \qquad \{P \in \operatorname{Spec}(R) \mid \mathrm{e}_{\mathrm{HK}}(R_P) \text{ is maximal}\} \cap \{P \in \operatorname{Spec}(R) \mid \mathrm{s}(R_P) \text{ is minimal}\} \neq \emptyset.$$

If we can find F-finite local hypersurface rings R_i with $\gamma(R_1) = \gamma(R_2)$ such that $\mathrm{e}_{\mathrm{HK}}(R_1) > \mathrm{e}_{\mathrm{HK}}(R_2)$ and $\mathrm{s}(R_1) > \mathrm{s}(R_2)$, then $R = R_1 \times R_2$ is a counterexample to $(*)$. We are also interested in whether $(*)$ holds when $\operatorname{Spec}(R)$ is connected (e.g., R is a domain).

We now prove an associativity-type formula for global Frobenius Betti numbers and Frobenius Euler characteristics. Given a finitely generated R-module M, we let $\operatorname{Assh}(M, \gamma)$ denote the set of associated primes P of M such that $\gamma(R/P) = \gamma$. We first establish the behavior of the invariants $\chi_i^F(-, \gamma)$ under short exact sequences.

Proposition 4.15. *Let R be an F-finite ring, and $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated R-modules, and $\gamma \geqslant \gamma(B)$ be an integer such that $\gamma \geqslant \min\{1, \dim(R)\}$. For $i \in \mathbb{Z}_{\geqslant 0}$, we have*

(1) $\chi_i^F(B, \gamma) = \chi_i^F(A \oplus C, \gamma)$.
(2) $\chi_i^F(B, \gamma) \leqslant \chi_i^F(A, \gamma) + \chi_i^F(C, \gamma)$.
(3) $\chi_i^F(B, \gamma) = \chi_i^F\left(\bigoplus_{P \in \operatorname{Assh}(B,\gamma)} \bigoplus_{i=1}^{\lambda(B_P)} R/P, \gamma \right)$.

Proof. We prove (1). It follows from [Sei89, Proposition 1 (b)] that, for all $P \in \operatorname{Spec}(R)$, we have equalities $\chi_i^F(B_P, \gamma) = \chi_i^F(A_P, \gamma) + \chi_i^F(C_P, \gamma) = \chi_i^F((A \oplus C)_P, \gamma)$. Using Theorem 4.10 (2), we conclude that

$$\begin{aligned}\chi_i^F(B, \gamma) &= \max\{\chi_i^F(B_P, \gamma) \mid P \in \operatorname{Spec}(R)\} \\ &= \max\{\chi_i^F((A \oplus C)_P, \gamma) \mid P \in \operatorname{Spec}(R)\} = \chi_i^F(A \oplus C, \gamma).\end{aligned}$$

For (2), let $P \in \operatorname{Spec}(R)$ be such that $\chi_i^F(M, \gamma) = \chi_i^F(M_P, \gamma)$, which exists by Theorem 4.10 (2). Using the result of Seibert mentioned above, we get

$$\chi_i^F(B, \gamma) = \chi_i^F(B_P, \gamma) = \chi_i^F(A_P, \gamma) + \chi_i^F(C_P, \gamma) \leqslant \chi_i^F(A, \gamma) + \chi_i^F(C, \gamma).$$

Part (3) follows from a repeated application of (1), using a prime filtration of B. ☐

As a consequence of Proposition 4.15, we extend a version of associativity formula from the global Hilbert-Kunz multiplicity [DSPY19, Corollary 3.10] to all global Frobenius Betti numbers.

Corollary 4.16. *Let R be an F-finite ring, and $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated R-modules, and $\gamma \geqslant \gamma(B)$ be an integer such that $\gamma \geqslant \min\{1, \dim(R)\}$. For $i \in \mathbb{Z}_{\geqslant 0}$, we have*

(1) $\beta_i^F(B,\gamma) = \beta_i^F(A \oplus C, \gamma)$.
(2) $\beta_i^F(B,\gamma) \leqslant \beta_i^F(A,\gamma) + \beta_i^F(C,\gamma)$.
(3) $\beta_i^F(B,\gamma) = \beta_i^F\left(\bigoplus_{P \in \operatorname{Assh}(B,\gamma)} \bigoplus_{i=1}^{\lambda(B_P)} R/P, \gamma\right)$.

Proof. The proof follows at once from Proposition 4.15 and the relation $\beta_i^F(-,\gamma) = \chi_i^F(-,\gamma) + \chi_{i-1}^F(-,\gamma)$. □

We now establish connections between the new global invariants and the singularities of the ring, extending results that were previously applicable only to the local setting.

The first result concerns Frobenius Betti numbers, and extends [AL08, Corollary 3.2].

Theorem 4.17. *Let R be an F-finite ring such that $Z_{R,\gamma(R)} = \operatorname{Spec}(R)$. Then $\beta_i^F(R) = 0$ for some (equivalently, for all) $i > 0$ if and only if R is regular.*

Proof. Assume that $\beta_i^F(R) = 0$ for some $i > 0$. If $\Omega_i(e)$ is the i-th syzygy module of any minimal free resolution of $F_*^e R$, we always have $\beta_i(e,P,R) \leqslant \mu_{R_P}(\Omega_i(e)_P) \leqslant \mu_R(\Omega_i(e))$. Since $\beta_i^F(R) = \lim_{e\to\infty} \frac{\mu_R(\Omega_i(e))}{p^{e\gamma(R)}} = 0$, we have $\beta_i^F(R_P) = 0$ for all $P \in \operatorname{Spec}(R)$. By [AL08], we conclude that R_P is regular for all primes P, hence, R is regular. Conversely, if R is regular, for all $P \in \operatorname{Spec}(R)$ we have that $\beta_i^F(R_P) = 0$ for all $i > 0$, and $\mathrm{e_{HK}}(R_P) = \beta_0^F(R_P) = 1$. In particular, $\chi_i^F(R_P) = (-1)^i$ for all $P \in \operatorname{Spec}(R)$. By Theorem 4.10, we have $\chi_i^F(R) = (-1)^i$ for all i, and it follows that $\beta_i^F(R) = \chi_i^F(R) + \chi_{i-1}^F(R) = 0$. □

We recall that an F-finite ring R is said to be strongly F-regular if, for every $c \in R$ that does not belong to any minimal prime, the map $R \to F_*^e R$ sending 1 to $F_*^e c$ splits (as an R-module map) for $e \gg 0$.

The second author and Smirnov proved a result analogous to [AL08, Corollary 3.2] for Frobenius Euler characteristics [PS19, Theorem B], under the additional assumption that the local ring is strongly F-regular. We extend this result to the global setting.

Theorem 4.18. *Let R be an F-finite and strongly F-regular ring such that $Z_{R,\gamma(R)} = \operatorname{Spec}(R)$. Then $\chi_i^F(R) = (-1)^i$ for some (equivalently, for all) $i \geqslant 0$ if and only if R is regular.*

Proof. Assume that $\chi_i^F(R) = (-1)^i$ for some $i \geqslant 0$. By Proposition 4.8 we have that $\chi_i^F(R_P) \leqslant (-1)^i$ for each $P \in \operatorname{Spec}(R)$ and therefore R is regular by [PS19, Lemma 3.17 and Theorem B]. Conversely if R is regular then $\chi_i^F(R) = (-1)^i$ for all $i \geqslant 0$ as we have seen in the proof of Theorem 4.17. □

We end this section by showing that, for positively graded algebras over a local ring $(R_0, \mathfrak{m}_0)$, the global Frobenius Betti numbers coincide with the ones in the localization at the irrelevant maximal ideal $\mathfrak{m} = \mathfrak{m}_0 + R_{>0}$.

Proposition 4.19. *Let $(R_0, \mathfrak{m}_0, k)$ be an F-finite local ring and let R be a positively graded algebra of finite type over R_0. Let $R_{>0}$ be the ideal of R generated by elements of positive degree, $\mathfrak{m} = \mathfrak{m}_0 + R_{>0}$, and M be a finitely generated graded R-module. Let $\gamma \geqslant \gamma(M)$ be an integer such that $\gamma \geqslant \min\{1, \dim(R)\}$. For all $i \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\beta_i^F(M,\gamma) = \beta_i^F(M_{\mathfrak{m}},\gamma) \quad \text{and} \quad \chi_i^F(M,\gamma) = \chi_i^F(M_{\mathfrak{m}},\gamma).$$

Proof. It is sufficient to show the equality $\beta_i^F(M,\gamma) = \beta_i^F(M_{\mathfrak{m}},\gamma)$. Observe that $F_*^e M$ is a $\mathbb{Q}$-graded R-module. Given any finitely generated graded R-module N, the minimal number of generators of N is the length of $N/\mathfrak{m}N$, by the graded version of Nakayama's Lemma. Applying this observation to the $\mathbb{Q}$-graded syzygies of $F_*^e M$ for $e \in \mathbb{Z}_{>0}$, one can construct a graded exact sequence

$$(1) \qquad 0 \longrightarrow \Omega_i(e) \longrightarrow \bigoplus_{j=1}^{b_{i-1}(e)} R[n_{i-1,j}] \xrightarrow{\varphi_{i-1}(e)} \cdots\cdots \longrightarrow \bigoplus_{j=1}^{b_0(e)} R[n_{0,j}] \xrightarrow{\varphi_0(e)} F_*^e M \longrightarrow 0,$$

where each syzygy $\Omega_j(e) = \operatorname{im}(\varphi_j(e))$ is graded, and $b_j(e) = \mu_R(\Omega_j(e)) = \lambda_R(\Omega_j(e)/\mathfrak{m}\Omega_j(e))$ for all j. In the resolution, $R[n_{\ell,j}]$ denotes the cyclic $\mathbb{Q}$-graded free module with generator in degree $-n_{\ell,j} \in \mathbb{Q}$. In particular, this is a minimal free resolution of $F_*^e M$, and it follows from Theorem 4.10 (3) that $\beta_i^F(M,\gamma) = \lim\limits_{e\to\infty} \frac{b_i(e)}{p^{e\gamma}}$. On the other hand, since all the maps and all the modules in (1) are graded minimal, after localizing at $\mathfrak{m}$ we obtain a minimal free resolution of $F_*^e(M_{\mathfrak{m}})$:

$$0 \longrightarrow \Omega_i(e)_{\mathfrak{m}} \longrightarrow R_{\mathfrak{m}}^{b_{i-1}(e)} \longrightarrow \cdots \longrightarrow R_{\mathfrak{m}}^{b_0(e)} \longrightarrow F_*^e(M_{\mathfrak{m}}) \longrightarrow 0.$$

In particular, since $\lambda_{R_{\mathfrak{m}}}(\operatorname{Tor}_i^R(k, F_*^e(M_{\mathfrak{m}}))) = \lambda_R(\Omega_i(e)/\mathfrak{m}\Omega_i(e)) = b_i(e)$, we have

$$\beta_i^F(M_{\mathfrak{m}},\gamma) = \lim_{e\to\infty} \frac{\lambda_{R_{\mathfrak{m}}}(\operatorname{Tor}_i(k, F_*^e(M_{\mathfrak{m}})))}{p^{e\gamma}} = \lim_{e\to\infty} \frac{b_i(e)}{p^{e\gamma}} = \beta_i^F(M,\gamma). \qquad \square$$

Corollary 4.20. *Let R and $\mathfrak{m}$ be as in Proposition 4.19. For all finitely generated R-modules M, we have $\mathrm{e}_{\mathrm{HK}}(M) = \mathrm{e}_{\mathrm{HK}}(M_{\mathfrak{m}})$.*

Proof. By Proposition 4.19, we have $\mathrm{e}_{\mathrm{HK}}(M) = \beta_0^F(M,\gamma(R)) = \beta_0^F(M_{\mathfrak{m}},\gamma(R))$. Since $\gamma(R) = \gamma(R_{\mathfrak{m}})$, we get $\mathrm{e}_{\mathrm{HK}}(M) = \beta_0^F(M_{\mathfrak{m}},\gamma(R_{\mathfrak{m}})) = \mathrm{e}_{\mathrm{HK}}(M_{\mathfrak{m}})$. $\square$

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