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GENERIC LOCAL DUALITY AND PURITY EXPONENTS

MELVIN HOCHSTER AND YONGWEI YAO

Abstract. We prove a form of generic local duality that generalizes a result of Karen E. Smith. Specifically, let R be a Noetherian ring, let P be a prime ideal of R of height h, let $A := R/P$, and W be a subset of R that maps onto $A \setminus \{0\}$. Suppose that R_P is Cohen-Macaulay, and that ω is a finitely generated R-module such that ω_P is a canonical module for R_P. Let $E := H^h_P(\omega)$. We show that for every finitely generated R-module M there exists $g \in W$ such that for all $j \geqslant 0$, $H^j_P(M)_g \cong \mathrm{Hom}_R(\mathrm{Ext}^{h-j}_R(M, \omega), E)_g$, and that, moreover, every $H^j_P(M)_g$ has an ascending filtration by a countable sequence of finitely generated submodules such that the factors are finitely generated free A_g-modules. In fact, this sequence may be taken to be $\{\mathrm{Ann}_{H^j_P(M)_g} P^n\}_n$. We use this result to study the purity exponent for a nonzerodivisor c in a reduced excellent Noetherian ring R of prime characteristic p, which is the least $e \in \mathbb{N}$ such that the map $R \to R^{1/p^e}$ with $1 \mapsto c^{1/p^e}$ is pure. In particular, in the case where R is a homomorphic image of an excellent Cohen-Macaulay ring and is S_2, we establish an upper semicontinuity result for the function $\mathfrak{e}_c : \mathrm{Spec}(R) \to \mathbb{N}$, where $\mathfrak{e}_c(P)$ is the purity exponent for the image of c in R_P. This result enables us to prove that excellent strongly F-regular rings are very strongly F-regular (also called F-pure regular). Another consequence is that the F-pure locus is open in an S_2 ring that is a homomorphic image of an excellent Cohen-Macaulay ring.

1. Introduction

We prove a form of generic local duality that generalizes a result of Karen E. Smith [Sm18]. Our result may be described as follows. Let R be a Noetherian ring, let P be a prime ideal of R of height h, let $A := R/P$, and W be a subset of R that maps onto $A \setminus \{0\}$ under the natural homomorphism $R \to R/P$. Suppose that R_P is Cohen-Macaulay, and that ω is a finitely generated R-module such that ω_P is a canonical module for R_P. Let $E := H^h_P(\omega)$. We show that for every finitely generated R-module

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M there exists $g \in W$ such that for all $j \in \mathbb{N}$, [1] $H^j_P(M)_g \cong \mathrm{Hom}_R(\mathrm{Ext}^{h-j}_R(M,\, \omega),\, E)_g$, and $H := H^j_P(M)_g$ satisfies the following condition:

(†) The module H has an ascending filtration by a countable sequence of finitely generated R_g-submodules such that the factors are free A_g-modules and the union of the submodules in the sequence is the whole module. More precisely, we show that $\{\mathrm{Ann}_{H^j_P(M)_g} P^n\}_n$ gives such a filtration, and the factors are A_g-free of finite rank.

We call a module satisfying (†) *free filterable relative to* P_g. The formal definition of this notion is given in Definition 2.2.

In the situation studied by Smith in [Sm18], A is a subring of R such that A is isomorphic to R/P under the natural map $R \to R/P$, and $W = A \setminus \{0\}$. We compare the results of this manuscript and those of [Sm18] further in §3. In the applications here, there is usually no copy of R/P in R (i.e., $R \twoheadrightarrow R/P$ does not split as a map of rings). Most frequently, W is simply $R \setminus P$. A key property of R-modules M that are free filterable relative to P is that every sequence of elements in R that maps to a possibly improper regular sequence in A is a possibly improper regular sequence[2] on M; moreover, this property is preserved if we make a flat base change $R \to R'$. In fact, this property also holds under substantially weaker assumptions than in the definition of free filterable relative to P. We want to study this property even when P is replaced by an ideal $\mathfrak{P}$ that is not necessarily prime.

We are therefore led to make the following definition:

Definition 1.1. If $\mathfrak{P} \subsetneq R$ is any proper ideal, we define an R-module M to be $(R \mathbin{\backslash\!\backslash} \mathfrak{P})$-*pseudoflat* if, for every flat R-algebra R', a sequence in R' that maps to a possibly improper regular sequence in $R'/\mathfrak{P}R'$ is a possibly improper regular sequence on $R' \otimes_R M$.

If M is killed by P and $A := R/P$ is regular, then M is $(R \mathbin{\backslash\!\backslash} P)$-pseudoflat if and only if M is A-flat (see Proposition 4.1(i)). In this paper we are primarily interested in the case where M is an R-module, not necessarily finitely generated, P is a prime ideal in R such that $A := R/P$ is regular, and such that every element of M is killed by a power of P. We briefly develop some facts about $(R \mathbin{\backslash\!\backslash} \mathfrak{P})$-pseudoflat modules in §4.

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[1] Note that in the statement above, if either R, or $R_{g'}$ for some $g' \in R \setminus P$, is excellent and has finite Krull dimension, then $g \in W$ can be chosen so that the isomorphism $H^j_P(M)_g \cong \mathrm{Hom}_R(\mathrm{Ext}^{h-j}_R(M,\, \omega),\, E)_g$ holds *for all* $j \in \mathbb{Z}$. See Theorem 2.19, part (e).

[2] A sequence $f_1, \ldots, f_s$ of elements of a ring R is called a *possibly improper regular sequence* on an R-module M (which is often the ring) if for all $0 \leqslant i \leqslant s-1$, the element f_{i+1} is a nonzerodivisor on $M/(f_1, \ldots, f_i)M$. Note that we do not require $(f_1, \ldots, f_s)M \neq M$.

In §7, we use our results to study the purity exponent for a nonzerodivisor c in a ring R that is a homomorphic image of an excellent Cohen-Macaulay ring and is S_2. The purity exponent for c is the least integer $e \in \mathbb{N}$ such that the map $R \to R^{1/p^e}$ with $1 \mapsto c^{1/p^e}$ is pure. In particular, we establish an upper semicontinuity result for the function $\mathfrak{e}_c : \mathrm{Spec}(R) \to \mathbb{N}$, where $\mathfrak{e}_c(P)$ is the purity exponent for the image of c in R_P. This result enables us to prove that excellent strongly F-regular rings are F-pure regular (also called very strongly F-regular). The terminology is explained in §7.4. This result was previously known when R is F-finite, when R is essentially of finite type over an excellent semilocal ring, and in a handful of other cases. We refer the reader to [Hash10], [DaMuSm20], [DaSm16], [HoY23], and [DET23] for these earlier results. The semicontinuity result also implies that the the F-pure locus is open in a ring that is an S_2 image of an excellent Cohen-Macaulay ring.

A technical problem that arises is that an excellent Cohen-Macaulay local ring R_P may fail to have a canonical module. This difficulty can be overcome because there is always a local étale extension of R_P with the same residue class field that has a canonical module. See §6.

There are no restrictions on the characteristic of the ring in §§1–6. The results in §7 typically need the assumption that the ring has prime characteristic $p > 0$.

For background results in commutative algebra, see [BH93, Mat70, Mat87, Na62].

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2. Generic local duality

In this section, in Theorem 2.19, we state and prove our main results on generic local duality. We begin with a brief recap of the basic results that we are generalizing.

Discussion 2.1 (Local duality). We give a brief treatment of the basic facts about local duality over a Cohen-Macaulay local ring R. We refer the reader to [Gro67] and [BH93, Ch. 3] for a detailed discussion. Throughout this discussion, (R, P, κ) is a Cohen-Macaulay local ring of Krull dimension h, and ω is a canonical module for R. The following statements are part of the standard theory. Let M be a finitely generated R-module (although some statements hold without finite generation).

(1) The module ω is a small (i.e., finitely generated) Cohen-Macaulay module over R of Krull dimension h such that $H^h_P(\omega)$ is an injective hull for κ. This statement characterizes ω up to noncanonical isomorphism.
(2) Hence, $\mathrm{Ext}^i_R((M, H^h_P(\omega)\big) = 0$ for all $i \geqslant 1$, and $\mathrm{Hom}_R\big(\kappa, H^h_P(\omega)\big) \cong \kappa$.
(3) If $\mathfrak{p}$ is any prime ideal of R, $\omega_{\mathfrak{p}}$ is a canonical module for $R_{\mathfrak{p}}$.
(4) If $x_1, \ldots, x_k$ is part of a system of parameters for R, $\omega/(x_1, \ldots, x_k)\omega$ is a canonical module for $R/(x_1, \ldots, x_k)$.

(5) The homothety map $R \to \operatorname{Hom}_R(\omega, \omega)$, determined by $r \mapsto (u \mapsto ru)$, is an isomorphism.
(6) (Local duality) For every i, there is a natural isomorphism of covariant functors $H^i_P(\_)$ and $\operatorname{Hom}_R\big((\operatorname{Ext}^{h-i}_R(\_, \omega), H^h_P(\omega)\big)$, so that $H^i_P(M)$ is the Matlis dual of $\operatorname{Ext}^{h-i}_R(M, \omega)$. In consequence, for all i, the modules $H^i_P(M)$ have DCC.
(7) If M has finite length, i.e., if M is killed by a power of P, then $\operatorname{Ext}^h_R(M, \omega)$ has a natural identification with the Matlis dual of M, which by (1) above may be thought of as $\operatorname{Hom}_R\big(M, H^h_P(\omega)\big)$.
(8) For all $i \in \mathbb{N}$, $\dim\big(\operatorname{Ext}^i_R(M, \omega)\big) \leqslant h - i$.
(9) If $M \neq 0$ is a finitely generated Cohen-Macaulay module of dimension δ, then $\operatorname{Ext}^i_R(M, \omega) = 0$ for all i except when $i = h - \delta$, which is also the height of the annihilator of M. Moreover, $\operatorname{Ext}^{h-\delta}_R(M, \omega)$ is always nonzero. In consequence, $\operatorname{Ext}^{h-\delta}_R(\_, \omega)$ is exact on finitely generated Cohen-Macaulay R-modules of dimension δ. In particular, $\operatorname{Hom}_R(\_, \omega)$ is an exact contravariant functor from finitely generated Cohen-Macaulay modules of dimension h to finitely generated Cohen-Macaulay modules of dimension h. In addition, since ω is Cohen-Macaulay of dimension h, $\operatorname{Ext}^i_R(\omega, \omega) = 0$ for $i > 0$.

All are well known, but we give short proofs of (7), (8) and (9), assuming the preceding items (1)–(6). To prove (7), we note that if M is killed by a power of P, then we have, using (6), that $M = H^0_P(M) \cong \operatorname{Hom}_R\big(\operatorname{Ext}^h_R(M, \omega), H^h_P(\omega)\big)$, and then each of M and $\operatorname{Ext}^h_R(M, \omega)$ is the Matlis dual of the other $\square$

For (8), let $N := \operatorname{Ext}^i_R(M, \omega)$. We may assume $0 \leqslant i \leqslant h$. Observe that N is not supported at any prime $\mathfrak{p}$ with $\operatorname{height}(\mathfrak{p}) < i$, since $N_{\mathfrak{p}} \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, \omega_{\mathfrak{p}}) = 0$. Thus, $\dim(N) \leqslant h - i$. $\square$

For (9), we also remark that for any finitely generated module M over a Noetherian ring R and any ideal I of R, the modules $H^i_I(M)$ all vanish if $IM = M$, while if $IM \neq M$ the first nonvanishing local cohomology module occurs when $i = \operatorname{depth}_I M$, and all local cohomology modules vanish if i exceeds either $\dim(M)$ or the number of generators of an ideal with the same radical as I. It follows that when M is a finitely generated Cohen-Macaulay module over a local ring (R, P), $H^i_P(M) = 0$ for all i except when $i = \dim(M)$. By locality duality, we establish (9). $\square$

Before concluding the discussion, we note also that these results imply that a canonical module ω has injective dimension precisely h, that R is a canonical module for R if and only if R is a Gorenstein local ring, and that R has a canonical module if and only if it is a homomorphic image of a Gorenstein local ring S, in which case, if $R \cong S/I$, we may take $\omega = \operatorname{Ext}^c_S(R, S)$, where $c := \dim(S) - \dim(R) = \operatorname{height} I$.

2.1. **Modules free filterable with respect to a prime.** We first define modules that are free filterable with respect to an ideal (especially, with respect to a prime ideal).

Definition 2.2 (Compare with (†))**.** Let R be a ring, $\mathfrak{P}$ an ideal of R, and M an R-module. We shall say that an R-module M is *free filterable relative to* $\mathfrak{P}$ if it has an ascending filtration by a sequence of finitely generated R-submodules $\{M_n\}_{n\in\mathbb{N}}$ such that all of the factors M_{n+1}/M_n are free over $R/\mathfrak{P}$ and such that the union $\bigcup_{n\in\mathbb{N}} M_n$ of the submodules is equal to M.

In particular, we are interested in modules that are free filterable with respect to a prime ideal. Next, we begin the development of several auxiliary results that will be needed in the proof of the main result, Theorem 2.19.

Notation 2.3. From this point on in this section, let R be a Noetherian ring, let $P \subseteq R$ be a prime ideal of R, let $A := R/P$, and let M denote an R-module that is not necessarily finitely generated until Notation 2.17, where the restriction that M be finitely generated will be imposed.

The following fact, although obvious, is very useful if there happens to be a "copy" of A in R.

Proposition 2.4. *Suppose that $A \hookrightarrow R$ and that P is a prime ideal of R such that the composite map $A \hookrightarrow R \twoheadrightarrow R/P$ is an isomorphism. Then an R-module M is free filterable relative to P if and only if it is free of finite or countably infinite rank.*

Proposition 2.5. *Let R, M, P, and A be as in Notation 2.3.*

(a) *If M is free filterable relative to P, then $\operatorname{Ass}_R(M) \subseteq \{P\}$, and every finitely generated submodule of M is killed by a power of P.*
(b) *If M is any finitely generated module killed by a power of P, there exists $g \in R \setminus P$ such that M_g is free filterable relative to PR_g over R_g.*
(c) *If M is free filterable relative to P and N is a finitely generated submodule, then there exists $g \in R\setminus P$ such that $(M/N)_g$ and N_g are free filterable relative to PR_g over R_g.*

Proof. Part (a) follows from the fact that every finitely generated submodule N of M is contained in a submodule N' with a finite filtration by finitely many factors that are direct sums of A, with the additional property that M/N' is free filterable relative to P. To prove part (b), take a finite prime cyclic filtration of M and localize at g that is not in P but is in all other primes Q_i such that R/Q_i occurs in the filtration. For part (c), choose N' as in the proof of part (a) and then, by part (b), there exists $g \notin P$ such that $(N'/N)_g$ and N_g become free filterable with respect to PR_g over R_g. It follows that $(M/N)_g$ is free filterable relative to PR_g over R_g. □

Remark 2.6. Clearly, if M has an ascending filtration by a sequence of finitely generated modules such that each factor is free filterable relative to P, then we may refine that filtration to get one in which the factors are all A-free (simply inserting finitely many modules between each pair of consecutive terms of the original filtration to make all factors A-free). Another useful fact is recorded in Lemma 2.8. We need a preliminary discussion.

Discussion 2.7 (**Double filtrations and compatible submodules**). Suppose that we are given an ascending sequence filtration (these may be finite or infinite) of a module M, say

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots \subseteq M,$$

where $\bigcup_{n\in\mathbb{N}} M_n = M$, and for each $n \in \mathbb{N}$ a filtration

$$M_n = M_{n,0} \subseteq M_{n,1} \subseteq \cdots \subseteq M_{n,s} \subseteq \cdots \subseteq M_{n+1}$$

such that $\bigcup_s M_{n,s} = M_{n+1}$ and every $M_{n,s+1}/M_{n,s}$ is finitely generated. This, equivalently, gives a filtration of each factor M_{n+1}/M_n using the modules $M_{n,s}/M_n$. We refer to the $M_{n,s}$ as a *double filtration* of M. We refer to the modules M_{n+1}/M_n as the *factors* of the double filtration and the modules $M_{n,s+1}/M_{n,s}$ as the *double factors* of the double filtration.

We say that a submodule module $N \subseteq M$ is *compatible* with this filtration if for all n, the submodule $(M_{n+1} \cap N) + M_n$ (which is the same as $M_{n+1} \cap (N + M_n)$) equals $M_{n,s(n,N)}$ for some $s(n,N) \in \mathbb{N} \cup \{\infty\}$ with the understanding that $M_{n,\infty} = M_{n+1}$.

(1) Given a compatible submodule N of M, we have an induced filtration $\{M_n \cap N\}_n$ of N, with the factors $\dfrac{M_{n+1} \cap N}{M_n \cap N}$.

(2) Denote $N_n := M_n \cap N$. We have a double filtration of N as follows. By the compatibility hypothesis on N, we have that $(M_{n+1} \cap N) + M_n = M_{n,s(n,N)}$ for some $s(n,N) \in \mathbb{N} \cup \{\infty\}$. Thus, the factors of N are

$$\frac{N_{n+1}}{N_n} \cong \frac{M_{n+1} \cap N}{M_n \cap N} \cong \frac{(M_{n+1} \cap N) + M_n}{M_n} = \frac{M_{n+1} \cap (N + M_n)}{M_n} = \frac{M_{n,s(n,N)}}{M_n}.$$

Now we filter N_{n+1}/N_n using the images of the modules $N_{n,s} := M_{n,s} \cap N$. It is straightforward to see that

$$\frac{N_{n,s+1}}{N_{n,s}} = \frac{M_{n,s+1} \cap N}{M_{n,s} \cap N} \cong \frac{(M_{n,s+1} \cap N) + M_{n,s}}{M_{n,s}} = \begin{cases} M_{n,s+1}/M_{n,s} & \text{if } s < s(n,N) \\ 0 & \text{if } s \geqslant s(n,N) \end{cases}.$$

This $\{N_{n,s}\}_s$ is the induced double filtration of N.

(3) The module M/N has the induced filtration $\{(M_n + N)/N\}_n$ with factors

$$\frac{(M_{n+1} + N)/N}{(M_n + N)/N} \cong \frac{M_{n+1} + N}{M_n + N} \cong \frac{M_{n+1}}{M_{n+1} \cap (N + M_n)} \cong \frac{M_{n+1}}{M_{n,s(n,N)}}.$$

The induced double filtration, by $\{(M_{n,s}+N)/N\}_s$, has double factors

$$\frac{(M_{n,s+1}+N)/N}{(M_{n,s}+N)/N} \cong \frac{M_{n,s+1}}{M_{n,s+1}\cap(N+M_{n,s})} \cong \begin{cases} 0 & \text{if } s < s(n,N) \\ M_{n,s+1}/M_{n,s} & \text{if } s \geqslant s(n,N) \end{cases}.$$

(4) Suppose that Q is another compatible submodule of M such that $N \subseteq Q$. We claim that N is a compatible submodule of Q. The induced filtration on Q from part (1) has submodules $Q_n = M_n \cap Q$. The induced filtration from these on N is the same as the one induced by the original filtration on M, since $(M_n \cap Q) \cap N = M_n \cap N$. It is also true that the double filtrations on N induced by M and by Q, as described in part (2), are the same, since the double filtration that Q induces on N consists of the modules $Q_{n,s} \cap N = (M_{n,s} \cap Q) \cap N = M_{n,s} \cap N$.

(5) Hence, if $N \subseteq Q \subseteq M$ where N and Q are both compatible with double filtration on M, then Q/N has a filtration whose double factors are all double factors of the filtration of M.

(6) We also observe the following: if G is any finitely generated submodule of the doubly filtered module M, then there is a finitely generated compatible submodule N of M such that $G \subseteq N$. It follows that M is the directed union of its finitely generated compatible submodules. To find N, note that there exists $n \in \mathbb{N}$ such that $G \subseteq M_{n+1}$. Then there exists $s \in \mathbb{N}$ such that $G + M_n \subseteq M_{n,s}$. We shall prove by induction on n that we can find a finitely generated submodule N with $G \subseteq N \subseteq M_{n+1}$ such that N is a compatible with the double filtration on M. (Note that the modules M_t in the filtration of M with $t > n+1$ will play no further role in the argument.) If $n = 0$, then $N := M_{0,s}$ can be used as the choice of N. Assume $n \geqslant 1$. As $M_{n,s}/M_n$ and G are both finitely generated, there exists a finitely generated R-submodule H such that $G \subseteq H \subseteq M_{n,s}$ and $H + M_n = M_{n,s}$. Clearly, $M_n \cap H$ is finitely generated and contained in M_n. By the induction hypothesis, we can choose a finitely generated submodule L with $M_n \cap H \subseteq L \subseteq M_n$ such that L is compatible with the double filtration on M. Let $N := L + H$, so that $N \subseteq M_{n,s} \subseteq M_{n+1}$ and N is finitely generated. As $L \subseteq M_n$, we have $M_{n+1} \cap (N + M_n) = M_{n+1} \cap (H + M_n) = M_{n,s}$. For all $t \geqslant n+1$, it is clear that $M_{t+1} \cap (N + M_t) = M_t = M_{t,0}$. Also note that $M_n \cap N = M_n \cap (L + H) = L + (M_n \cap H) = L$, since $M_n \cap H \subseteq L \subseteq M_n$ by the choice of L. Thus, for all $t = 0, \ldots, n-1$, we have $M_{t+1} \cap N = M_{t+1} \cap L$ and, hence,

$$(M_{t+1} \cap N) + M_t = (M_{t+1} \cap L) + M_t = M_{t,s(t,L)} \text{ for some } s(t,L) \in \mathbb{N} \cup \{\infty\},$$

since L is a compatible submodule of M. This proves that N is compatible with the double filtration on M.

Lemma 2.8. *If M has a filtration by an ascending sequence of submodules whose union is M and whose factors are free filterable relative to P, then M is free filterable relative to P.*

Proof. The hypothesis says that M has a filtration by submodules M_n such that every M_{n+1}/M_n is free filterable relative to P. This gives rise to a double filtration as in Discussion 2.7 such that every $M_{n,s+1}/M_{n,s}$ is finitely generated and free over A. Choose a finite set of elements $S_{n,t} \subseteq M_{n,t+1}$ whose images in $M_{n,t+1}/M_{n,t}$ generate the double factor $M_{n,t+1}/M_{n,t}$. It is easy to see that the union of the sets $S_{n,t}$, namely $\bigcup_{n\geqslant 0,\, t\geqslant 0} S_{n,t}$, generates M. Take $N_0 := 0$ and recursively construct a sequence of finitely generated compatible submodules N_n of M such that for each n, N_{n+1} contains N_n and $\bigcup_{i\leqslant n,\, t\leqslant n} S_{i,t}$, which is possible by part (6) of Discussion 2.7. By part (5) of Discussion 2.7, the modules N_{n+1}/N_n have finite filtrations with factors that are finitely generated and A-free, by Remark 2.6. □

Remark 2.9. In this paper, we shall be interested exclusively in the case where the module is countably generated over R. When we do not assume countable generation, a more suitable alternative definition, of an R-module H being free filterable relative to P, may be that H be a directed union of finitely generated R-submodules such that if $G_i \hookrightarrow G_j$ is in the system, then G_j/G_i has a filtration such that all factors are A-free. This second definition is weaker than our original definition, but agrees with the one we have chosen for this manuscript if H is countably generated by elements $u_1, \ldots, u_n, \ldots$: let $G_0 = 0$, and, recursively, pick an element G_n of the limit system that contains the $u_1, \ldots, u_n$ and G_{n-1}. Now fill in finitely many modules between G_{n-1} and G_n so that all factors are A-free.

2.2. **Graded generic freeness.** In the sequel, we need a slightly strengthened version of generic freeness in graded situations. We give a quite short proof.

Lemma 2.10 (Graded generic freeness)**.** *Let $R = \bigoplus_{i\in\mathbb{N}}[R]_i$ be an $\mathbb{N}$-graded ring that is finitely generated over a Noetherian domain $A \hookrightarrow [R]_0$, and let M be a finitely generated $\mathbb{Z}$-graded R-module. Then there exists $g \in A \setminus \{0\}$ such that $[M_g]_d$ is free over A_g for all d. When $A \hookrightarrow [R]_0$ is module-finite, the modules $[M_g]_d$ are free of finite rank over A_g for all d.*

Proof. Since M has a finite filtration by graded cyclic modules, we can reduce at once to the case where $M = R/I$, where I is homogeneous, and we change notation and write R for R/I. That is, we only need to do the case where the module is a finitely generated graded A-algebra. We use induction on the number n of homogeneous generators of R over A (the case where $n = 0$ is obvious). So we may write $R = B[u]$ where B needs strictly fewer than $n \geqslant 1$ generators and u is homogeneous. We have a filtration of R by B-submodules $C_t := B + Bu + \cdots + Bu^t$, and a typical factor is $D_t := C_t/C_{t-1} \cong B/J_t$, where $J_t := \{b \in B : bu^t \in C_{t-1}\}$. The ideals J_t form an ascending sequence, and so stabilize. The filtration therefore has only finitely many distinct factors, each of which is an $\mathbb{N}$-graded quotient of B, and so has strictly fewer than n generators. Thus, we can localize at one element of $A \setminus \{0\}$ so that every graded component of every factor D_t is A-free. It follows that $[R]_d$ has a countable ascending filtration by A-modules such that every factor is A-free. □

Note that Lemma 2.10 does not follow from the assertion that one can localize at $g \in A \setminus \{0\}$ so that M_g becomes A_g-free: that only implies that the $[M_g]_d$ are projective modules over A_g.

Remark 2.11. Let R be a Noetherian algebra over a Noetherian ring A and suppose that I is an ideal of R such that I is contained in the Jacobson radical of R, which holds if R is I-adically complete. Let M be a finitely generated R-module such that M/IM is module-finite over A. Suppose that $\mathrm{gr}_I M$ is A-flat. Then M/I^tM is A-flat for all $t \in \mathbb{N}$, and M is A-flat. The first statement follows because M/I^tM has a finite filtration in which the factors $I^jM/I^{j+1}M$ are all A-flat. For the second statement, if $B \hookrightarrow C$ are finitely generated A-modules, if $u \in \mathrm{Ker}(B \otimes_A M \to C \otimes_A M)$, then for all $t \in \mathbb{N}$ the image of u is in $\mathrm{Ker}\big(B \otimes_A (M/I^tM) \to C \otimes_A (M/I^tM)\big)$, and so $u \in \bigcap_{t\in\mathbb{N}} I^t(B \otimes_A M) = 0$, since I is in the Jacobson radical of R.

2.3. **The main result on generic local duality.**

For a general reference on local cohomology, see [Gro67].

Remark 2.12. We shall be using freely throughout that Ext and, in particular, Hom, commutes with flat base change when the ring is Noetherian and the first (left) input module is finitely generated. This will most frequently be applied in the case of localization, but will also be needed to pass to the P-adic completion of R.

Remark 2.13. We recall that, for *any* three R-modules G, ω, $\mathcal{C}$, we have the following natural maps

(1) $\phi : \mathrm{Hom}_R(G, \omega) \otimes_R \mathcal{C} \to \mathrm{Hom}_R(G, \omega \otimes_R \mathcal{C})$.
(2) $\psi : G \otimes_R \mathrm{Hom}_R(\omega, \mathcal{C}) \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(G, \omega), \mathcal{C}\big)$.

Both commute with finite direct sums of choices of G. The map ϕ is induced by the bilinear map $(h, c) \mapsto (u \mapsto h(u) \otimes c)$ and the map ψ is induced by the bilinear map $(u, f) \mapsto (h \mapsto f(h(u)))$. Both ϕ and ψ provide very useful natural isomorphisms when $G = R$ and, hence, when G is any finitely generated projective module. Moreover, when G is finitely presented, the map ϕ is an isomorphism when $\mathcal{C}$ is flat and the map ψ is an isomorphism when $\mathcal{C}$ is injective.[3]

[3] Take a presentation $G_1 \to G_0 \to G \to 0$ with G_1, G_0 free of finite rank. Then ϕ (resp., ψ) gives a natural map from the row $0 \to \mathrm{Hom}_R(G, \omega) \otimes_R \mathcal{C} \to \mathrm{Hom}_R(G_0, \omega) \otimes_R \mathcal{C} \to \mathrm{Hom}_R(G_1, \omega) \otimes_R \mathcal{C}$ to the row $0 \to \mathrm{Hom}_R(G, \omega \otimes_R \mathcal{C}) \to \mathrm{Hom}_R(G_0, \omega \otimes_R \mathcal{C}) \to \mathrm{Hom}_R(G_1, \omega \otimes_R \mathcal{C})$ (resp., from the row $G_1 \otimes_R \mathrm{Hom}_R(\omega, \mathcal{C}) \to G_0 \otimes_R \mathrm{Hom}_R(\omega, \mathcal{C}) \to G \otimes_R \mathrm{Hom}_R(\omega, \mathcal{C}) \to 0$ to the row $\mathrm{Hom}_R\big(\mathrm{Hom}_R(G_1, \omega), \mathcal{C}\big) \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(G_0, \omega), \mathcal{C}\big) \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(G, \omega), \mathcal{C}\big) \to 0$) where exactness holds in the first (resp., second) row because $\mathcal{C}$ is flat (resp., injective). The two maps for the G_0, G_1 terms are isomorphisms and then so is the map for the G term by the five lemma.

Remark 2.14. We shall also be using freely throughout that the regular locus, the Cohen-Macaulay locus, and the Gorenstein locus in an excellent ring are Zariski open sets. We refer the reader to [EGAIV65, §§6.11–6.13, §§7.6–7.8] for a detailed treatment of openness of loci. There is also a particularly readable discussion of some of this material in [Mat70, Ch. 13]. Note that when R is excellent, if ω is a finitely generated R-module and P a prime such that ω_P is a canonical module for R_P, there exists $g \in R \setminus P$ such that R_g is Cohen-Macaulay and ω_g is a global canonical module. To see this, first note that since ω_P is faithful over R_P, the same holds for ω and R after localizing at one element of $R \setminus P$. Next, consider the Nagata idealization $S := R \oplus \omega$ (see, for example, the proof of [BH93, Thm. (3.36)], where S is denoted $R * M$) of ω, where $\omega^2 = 0$. Note that S is finitely generated as an R-module, and the algebra maps $R \hookrightarrow R \oplus \omega \twoheadrightarrow R$, where the second map is the quotient map that kills the ideal ω, induce isomorphisms of the spectra. In fact, the composite map is the identity on $\mathrm{Spec}(R)$. Under this identification of spectra, P corresponds to $P \oplus \omega$. Then $(R \oplus \omega)_{P \oplus \omega} \cong R_P \oplus \omega_P = S_P$, and since ω_P is a canonical module for R_P, we have that S_P is Gorenstein, as in [BH93, Thm. (3.36)]. Since S is also excellent, we can choose $g \in R \setminus P$ such that $S_g \cong R_g \oplus \omega_g$ is Gorenstein. Thus, if $Q \in D(g)$, $S_Q \cong R_Q \oplus \omega_Q$ is Gorenstein, and $\mathrm{Hom}_{S_Q}(S_Q/\omega_Q, S_Q) \cong \mathrm{Ann}_{S_Q}\omega_Q = \omega_Q$ (since $\mathrm{Ann}_{R_Q}\omega_Q = 0$) is a canonical module for $S_Q/\omega_Q = R_Q$, as claimed.

Definition 2.15. Let W be a subset of R. We shall say that a statement about R-modules, ideals of R, and/or maps of R-modules holds *W-generically* if there exists an element $g \in W$ such that the statement holds after we make a base change from R to R_g.

Remark 2.16. Assume that P is a prime ideal of R, M is an R-module in which every element is killed by a power of P, and $g,\, g' \in R \setminus P$ satisfy $g' \equiv g \mod P$. Then $M_g \cong M_{g'}$. That is, inverting one element with a given residue mod P inverts every element with the same residue. To see this, note that $g' = g - (g - g') = g[1 - (g - g')/g] \in R_g$ and $(g - g')/g$ acts nilpotently on every cyclic submodule of M_g, which shows that g' acts bijectively on every cyclic submodule of M_g. Consequently, g' acts bijectively on M_g. By symmetry, g acts bijectively on $M_{g'}$. Thus $M_g \cong M_{g'}$.[4]

Suppose that $H(\_)$ is a functor that commutes with localization and whose values are modules in which every element is killed by a power of P, for example $H^i_P(\_)$, $\mathrm{Ext}^i_R(L,\, H^i_P(\_))$ or $\mathrm{Ext}^i_R(N,\, \_)$, where L and N are finitely generated module over a Noetherian ring R such that N killed by a power of P. Suppose that Q is any module, *not necessarily a module such that every element is killed by a power of P*. Then we again have $H(Q_g) \cong H(Q_{g'})$ for all $g,\, g' \in R \setminus P$ such that $g' \equiv g \mod P$, since we may apply the paragraph above to $M = H(Q)$. We shall frequently make use of these observations without comment.

[4]Note also that g' is invertible in the PR_g-adic completion S of R_g, and so acts invertibly on every S-module: in this case, $g - g'$ is in the Jacobson radical of S.

Notation 2.17. Let R be a Noetherian ring, P a prime ideal of R, $A := R/P$, $W \subseteq R$ a subset that maps onto $A \setminus \{0_A\}$ under the natural map $R \to R/P$, and let M be a finitely generated R-module. Assume also that R_P is Cohen-Macaulay of Krull dimension h and that ω is a finitely generated R-module such that ω_P is a canonical module for R_P. Let $E := H^h_P(\omega)$. For every prime $\mathfrak{p}$ of R let $\kappa_\mathfrak{p}$ denote $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ which is naturally isomorphic with $\mathrm{frac}(R/\mathfrak{p})$.

In the sequel, given an R-module M, we write $\mathrm{E}_R(M)$ for the injective hull of M over R, which is unique up to non-unique isomorphism.

Remark 2.18. With Notation 2.17 above, note that the map $R \to \mathrm{Hom}_R(\omega, \omega)$ via the homothety map $r \mapsto (u \mapsto ru)$ becomes an isomorphism when we localize at one $g \in R \setminus P$, since that is true once we localize at P, and the modules are finitely generated. Moreover, since ω_P is a canonical module for the Cohen-Macaulay ring R_P, $\mathrm{E}_{R_P}(R_P/PR_P)$, which is canonically isomorphic with $\mathrm{E}_R(R/P)$, is isomorphic with E_P. The next theorem, one of the main results of this paper, asserts that E behaves very much like $\mathrm{E}_R(R/P)$ on a Zariski neighborhood of P. The size of the neighborhood is typically adjusted, depending on a specified set of finitely many finitely generated R-modules. We want to emphasize parts (e) and (f) of Theorem 2.19.

Theorem 2.19 (Main theorem on generic local duality)**.** *Let notation be as in 2.17.*

(a) *For every $i \geqslant 1$, there exists $g \in W$, where g depends on M and i, such that $\mathrm{Ext}^i_R\left(M, E\right)_g = 0$. That is, for every $i \geqslant 1$, the module $\mathrm{Ext}^i_R\left(M, H^h_P(\omega)\right)$ is W-generically 0. Hence, if $0 \to M' \to M \to M'' \to 0$ is a sequence of finitely generated R-modules that becomes exact after localization at P, then the induced sequence*

$$0 \to \mathrm{Hom}_R(M'', E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M', E) \to 0$$

is W-generically exact. Therefore, the functor $\mathrm{Hom}_R(\_, E)$ is W-generically exact on any given finite set of short exact sequences of finitely generated R-modules, and, hence, on any given finite set of finite long exact sequences of finitely generated R-modules. The choice of $g \in W$, at which we localize, depends on which finite set of finite exact sequences one chooses.

(b) *If M_P is killed by a power of P then we have, W-generically, an isomorphism*

$$\mathrm{Hom}_R(M, E) \cong \mathrm{Ext}^h_R(M, \omega).$$

Consequently, W-generically, $\mathrm{Hom}_R(M, E)$ is a finitely generated R-module.

Moreover, the unique largest A-module contained in E, i.e., $\mathrm{Ann}_E P \cong \mathrm{Hom}_R(R/P, E)$, is W-generically isomorphic with $A = R/P$. That is, W-generically, we have $R/P \cong \mathrm{Ann}_E P$. Thus, after localization at $R \setminus P$, we obtain a map $\kappa_P \mapsto E_P$ such that the image of κ_P is the socle in $E_P \cong \mathrm{E}_R(\kappa_P)$.

(c) *W-generically, $H^h_P(R) \cong \mathrm{Hom}_R(\omega, E)$.*

(d) *Let* $N := \operatorname{Hom}_R(M, E)$*, where* M *is a finitely generated* R*-module. Then after localizing at one element of* W that is independent of $n \in \mathbb{N}$*, the filtration of* N *by the modules* $\operatorname{Ann}_N P^n$*, which ascends as* n *increases with* $\bigcup_{n\in\mathbb{N}} \operatorname{Ann}_N P^n = N$*, has factors that are finitely generated free modules over* A*. In particular, if* $\mathfrak{a}$ *is any ideal of* R*, the conclusion holds for* $N = \operatorname{Ann}_E \mathfrak{a} \cong \operatorname{Hom}_R(R/\mathfrak{a}, E)$*.*

(e) *Assuming the isomorphism in part (c), we have a natural transformation of functors from the category of finitely generated* R*-modules to the category of* R*-modules, namely*

$$\operatorname{Hom}_R\big(\operatorname{Ext}_R^{h-i}(\_, \omega),\, H_P^h(\omega)\big) \longrightarrow H_P^i(\_),$$

such that for every finitely generated R*-module* M *there exists* $g \in W$ *such that for all* $i \geqslant 0$*,*

$$(*) \qquad \operatorname{Hom}_R\big(\operatorname{Ext}_R^{h-i}(M, \omega),\, H_P^h(\omega)\big)_g \xrightarrow{\cong} H_P^i(M)_g$$

is an isomorphism. Note that g *depends on* M *but* not *on* $i \in \mathbb{N}$*. If* R *(or some localization of* R *at an element of* $R \setminus P$*) is excellent and has finite Krull dimension, then we can choose* $g \in W$ *that depends on* M *but* not *on* $i \in \mathbb{Z}$ *such that* $(*)$ *is an isomorphism for all* $i \in \mathbb{Z}$*.*

(f) *Hence, if* M *is any finitely generated* R*-module, after localizing at one element* $g \in W$*, all of the local cohomology modules* $H_P^i(M)$ *are free filterable relative to* P*. That is, the local cohomology modules* $H_P^i(M)$ *are* W*-generically free filterable relative to* P*. More precisely, after localization at one element of* W*, the* R*-modules* $\operatorname{Ann}_{H_P^i(M)} P^n$*,* $n \in \mathbb{N}$*, give an ascending filtration of* $H_P^i(M)$ *such that all of the factors are* A*-free of finite rank with* $\bigcup_{n\in\mathbb{N}} \operatorname{Ann}_{H_P^i(M)} P^n = H_P^i(M)$*.*

Proof. By Remark 2.16, it suffices to assume $W = R \setminus P$. In the course of the proof we may repeatedly, but finitely many times, localize at one element $g \in W$. Each time, we make a change of terminology, and continue to use R to denote the resulting ring R_g. Likewise, we use P for its extension to R_g, and for every module under consideration we use the same letter for the module after base change from R to R_g (finitely generated modules under consideration are replaced by their localizations at g: this is the same as base change from R to R_g).

There exist $\underline{x} := x_1, \ldots, x_h \in R$ whose images are a system of parameters for R_P, where they form a regular sequence on R_P and ω_P. After inverting an element of $W = R \setminus P$ (and using R to denote the resulting ring R_g, as agreed above), we may assume that $\operatorname{Rad}(\underline{x}) = P$ in R. Therefore, we have $E = H_P^h(\omega) = H_{(\underline{x})}^h(\omega)$. For the same reason, we may assume throughout that $x_1, \ldots, x_h$ is a regular sequence on R and on ω such that $\operatorname{Rad}(\underline{x}) = P$ in R, and that $E = H_P^h(\omega) = H_{(\underline{x})}^h(\omega)$.

The proofs of (a), (b), and (c) depend on some results about the spectral sequences of a double complex, which are collected for reference in (‡) below.

Discussion (‡). **The spectral sequences of a double complex.** We only need the four standard facts listed below from the theory of spectral sequences for a double complex with bounded diagonals that relate the iterated cohomology (or second page, $\mathbf{E}_2^{\bullet\bullet}$) of the double complex with associated graded modules of the cohomology $\mathcal{H}^\bullet$ of the total complex. Let k, $v \in \mathbb{Z}$ be fixed integers.

(1) The spectral sequence calculation commutes with flat base change, including localization.
(2) If the terms $\mathbf{E}_2^{ij} = 0$ for all $(i, j) \in \mathbb{Z}^2$ such that $i + j = k$, then $\mathcal{H}^k = 0$.
(3) If the terms $\mathbf{E}_2^{ij} = 0$ for all $(i, j) \in \mathbb{Z}^2$ such that $j \neq \nu$ (respectively, for all $(i, j) \in \mathbb{Z}^2$ such that $i \neq \nu$), then $\mathcal{H}^{i+\nu} \cong \mathbf{E}_2^{i,\nu}$ for all i (respectively, $\mathcal{H}^{\nu+j} \cong \mathbf{E}_2^{\nu,j}$ for all j).
(4) If $\mathbf{E}_2^{ij} = 0$ for all $(i, j) \in \mathbb{Z}^2$ such that $k - 1 \leqslant i + j \leqslant k + 1$ except when $i = \nu$ or $j = k - \nu$, then[5] $\mathcal{H}^k \cong \mathbf{E}_2^{\nu,\, k-\nu}$.

In the proofs for (a), (b) and (c) below, using one spectral sequence we show that, W-generically, the cohomology of the total complex in degree $i + h$ is $\mathrm{Ext}^i_R(M, E)$. Using the other spectral sequence we prove, in each of cases (a), (b) and (c), that W-generically there are zeros at a certain spots of the second page, enabling us to reach the conclusion we need from one of the facts just above.

Let $\mathcal{C}^\bullet$ denote the usual modified Čech complex used to calculate $H^\bullet_P(R) \cong H^\bullet_{(\underline{x})}(R)$, namely

$$0 \to R \to \cdots \to \bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant h} R_{x_{i_1} \cdots x_{i_j}} \to \cdots \to R_{x_1 \cdots x_h} \to 0,$$

where the central/typical term is $\mathcal{C}^j$, which is flat over R. Let $G_\bullet$ be a free resolution of M by free R-modules of finite rank. Consider the isomorphic (see Remark 2.13) double complexes

$$\mathrm{Hom}_R(G_\bullet, \omega \otimes_R \mathcal{C}^\bullet) \cong \mathrm{Hom}_R(G_\bullet, \omega) \otimes_R \mathcal{C}^\bullet.$$

We compute the cohomology of the total complex by using the two spectral sequences for iterated cohomology obtained from this double complex. The typical module in the double complex is $\mathrm{Hom}_R(G_i, \omega \otimes_R \mathcal{C}^j) \cong \mathrm{Hom}_R(G_i, \omega) \otimes \mathcal{C}^j$. If we fix i and let j vary, then because $\underline{x}$ is a regular sequence on any finite direct sum of copies of ω, the cohomology vanishes except when $j = h$. This shows that the spectral sequence degenerates, and that, W-generically, the cohomology of the total complex in degree $h + i$ is $\mathcal{H}^{h+i} \cong \mathrm{Ext}^i_R\big(M, H^h_P(\omega)\big) = \mathrm{Ext}^i_R(M, E)$. If we fix j and let i vary (noting

[5] We comment only on (4). We have for $r \geqslant 2$ that all of the spots on the diagonal where $i+j=k$ are stable as the page index r increases: at spots where $i \neq \nu$, this holds because $\mathbf{E}_r^{i,k-i} = 0$. At the $(\nu, k-\nu)$ spot this holds because for $r \geqslant 2$ the graded component of the differential d^r mapping to (resp., from) $\mathbf{E}_r^{\nu,\, k-\nu}$ has domain (resp., target) on the diagonal of degree $k-1$ (resp., degree $k+1$) at a spot that is *not* in the row or column of $(\nu, k-\nu)$, and so is 0. Hence, there is only one possibly nonzero factor, namely $\mathbf{E}_2^{\nu,k-\nu}$, in the filtration of $\mathcal{H}^k$ one gets from the $\mathbf{E}_\infty^{\bullet\bullet}$ page.

that $\mathcal{C}^j$ is flat), then we first get cohomology $\operatorname{Ext}^i_R(M,\,\omega) \otimes \mathcal{C}^\bullet$, while the iterated cohomology is $H^j_P\big(\operatorname{Ext}^i_R(M,\,\omega)\big)$ on the second page. We denote pages of this second spectral sequence by $\mathbf{E}^{\bullet\bullet}_r$, $r \geqslant 2$ throughout the rest of the proof of parts (a), (b) and (c). In particular, explicitly, $\mathbf{E}^{ij}_2 \cong H^j_P\big(\operatorname{Ext}^i_R(M,\,\omega)\big)$. Quite generally, this spectral sequence is 0 for all the spots with $j < 0$ or $j > h$ from the second page onward. We now prove the statements (a), (b) and (c), by proving that, in each case, certain of the modules $H^j_P\big(\operatorname{Ext}^i_R(M,\,\omega)\big)$ vanish generically for a relevant set of pairs $(i,\,j)$.

(a) Replacing M by a suitable module of syzygies,[6] we see that it suffices to prove the case $i = 1$. By Discussion (‡)(2), to show that $\operatorname{Ext}^1_R(M,\,E)$ vanishes after localization at one element $g \in W$, it suffices to prove that there exists $g \in W$ such that $H^j_P\big(\operatorname{Ext}^i_R(M,\omega)\big)_g = 0$ when $i + j = h + 1$ and $0 \leqslant j \leqslant h$. Thus, we want to show that $H^j_P\big(\operatorname{Ext}^{h+1-j}_R(M,\omega)\big)$ vanishes W-generically for $0 \leqslant j \leqslant h$, which proves that the spectral sequence stabilizes at 0 on those spots from the second page onward. When $j = 0$, this holds because $\operatorname{injdim}_{R_P}(\omega_P) = \dim(R_P) = h$ and so $\operatorname{Ext}^{h+1-0}_R(M,\,\omega)$ becomes 0 after localizing at one element of $W = R \setminus P$. Assume $1 \leqslant j \leqslant h$. Let $I = \operatorname{Ann}_R\big(\operatorname{Ext}^{h+1-j}_R(M,\omega)\big)$. By localizing at one element of W, we may assume that I is contained in P. Next note that $\operatorname{height}(P/I) \leqslant j-1$, because this statement is unaffected by localization at P, and, from Discussion 2.1(8), we see that $\operatorname{height}(P_P/I_P) = \dim(R_P/IR_P) = \dim\big(\operatorname{Ext}^{h+1-j}_{R_P}(M_P,\,\omega_P)\big) \leqslant j-1$. Thus, PR_P/IR_P has a system of parameters consisting of images of elements of R with at most $j-1$ elements. Consequently, after localizing at one element of W, we have that P/I is the radical of an ideal $(\underline{z}) = (z_1,\,\dots,\,z_{j-1})$, with at most $j-1$ generators. But then, for modules killed by I, $H^j_P(\_) \cong H^j_{P/I}(\_) \cong H^j_{\underline{z}}(\_) \cong 0$.

Thus, the $\mathbf{E}^{\bullet\bullet}_2$ page degenerates to 0 for all $(i,\,j)$ spots with $i+j = h+1$. It follows that at once from (‡)(2) that, $\operatorname{Ext}^1_R\big(M,E\big)_g = 0$ as required and, hence, that for fixed $i \geqslant 1$, $\operatorname{Ext}^i_R\big(M,E\big)_g = 0$ for suitable g. Now it is straightforward to prove the remaining claims on the W-generic exactness of $\operatorname{Hom}_R(\_,\,E)$ on short exact sequences of finitely generated R-modules. This concludes the proof of (a).

(b) First note that if $M_P = 0$, we may localize at one element of $W = R \setminus P$ and assume that $M = 0$, in which case the result is obvious. Otherwise, after we localize at one element of W, we may assume that the radical of the annihilator of M is P. Hence $H^0_P\big(\operatorname{Ext}^i_R(M,\,\omega)\big) \cong \operatorname{Ext}^i_R(M,\,\omega)$ and $H^j_P\big(\operatorname{Ext}^i_R(M,\,\omega)\big) = 0$ for all $i \in \mathbb{N}$ and all $j \neq 0$, since a power of P kills $\operatorname{Ext}^i_R(M,\,\omega)$.

In light of this, we see that $\mathbf{E}^{\bullet\bullet}_2$ degenerates: all the nonzero terms can only occur for $j = 0$ and are simply the modules $\operatorname{Ext}^i_R(M,\,\omega)$. By (‡)(3), the module $\operatorname{Ext}^i_R(M,\,\omega)$ is the cohomology $\mathcal{H}^i$ of the total complex in degree i. In particular, when $i = h$, we have that, W-generically, $\operatorname{Ext}^h_R(M,\,\omega) \cong \mathcal{H}^h \cong \operatorname{Hom}_R(M,\,E)$.

[6]This is not necessary, but makes the argument a bit easier to follow.

Next, let $N := \operatorname{Hom}_R(R/P,\, E) \cong \operatorname{Ann}_E P$. By the first statement in part (b), proved above, we know that, W-generically, $N \cong \operatorname{Ext}^h_R(R/P,\, \omega)$ is finitely generated. After localization at P, we have $N_P \cong \kappa_P = \operatorname{frac} A$. We may choose an element $u \in \operatorname{Hom}_R(R/P,\, E)$ that becomes a generator this module once we localize at P. Hence, the linear map $\theta : A \to \operatorname{Hom}_R(R/P,\, E)$ such that $1 \mapsto u$ becomes an isomorphism from κ_P to κ_P when we localize at P. Since the kernel and cokernel of θ are finitely generated R-modules, they both vanish after localization at some $g \in W$, which then makes θ an isomorphism. This concludes the proof of part (b).

(c) Note that, W-generically, $\operatorname{Ext}^0_R(\omega,\, \omega) = \operatorname{Hom}_R(\omega,\, \omega) \cong R$ (cf. Remark 2.18) and $\operatorname{Ext}^i_R(\omega,\, \omega) = 0$ for all $1 \leqslant i \leqslant h+1$ (cf. Discussion 2.1(9)). We apply (‡)(4) to $\mathbf{E}_2^{\bullet\bullet}$ with $M = \omega$, $k = h$ and $\nu = 0$ to conclude that W-generically we have $\operatorname{Hom}_R(\omega,\, E) \cong \mathcal{H}^h \cong H^{h-0}_P\big(\operatorname{Ext}^0(\omega,\, \omega)\big) \cong H^h_P(R)$, as the nonzero terms $\mathbf{E}_2^{ij}$ on the diagonals corresponding to total degrees $h-1$, h and $h+1$ can only occur when $i = 0$.

(d) First note that, for all $s \in \mathbb{N}$, we have following series of isomorphisms (in which the second isomorphism is by the adjointness of $\otimes$ and Hom)

$$\begin{aligned}\operatorname{Ann}_{\operatorname{Hom}_R(M,\, E)} P^s &\cong \operatorname{Hom}_R\big(R/P^s,\, \operatorname{Hom}_R(M,\, E)\big)\\ &\cong \operatorname{Hom}_R\big((R/P^s) \otimes_R M,\, E\big) \cong \operatorname{Hom}_R(M/P^sM,\, E).\end{aligned}$$

Let $N_s := \operatorname{Hom}_R(M/P^sM,\, E)$, for $s \in \mathbb{N}$. It suffices to show that there exists $g \in W$ such that for all $s \in \mathbb{N}$, $(N_{s+1}/N_s)_g$ is A_g-free of finite rank. Given what have been proved, we can localize at $g \in W$ so that

(i) $\operatorname{Hom}_R(A,\, E) \cong A$ and $\operatorname{Ext}^1_R(A,\, E) = 0$, by parts (a) and (b) above. Therefore, $\operatorname{Hom}_R(A^{\oplus k},\, E) \cong A^{\oplus k}$ and $\operatorname{Ext}^1_R(A^{\oplus k},\, E) = 0$ for all $k \in \mathbb{N}$.

(ii) For all $s \in \mathbb{N}$, the graded component $[\operatorname{gr}_P M]_s = P^sM/P^{s+1}M$ is A-free of finite rank, by Lemma 2.10. Consequently, for each $s \in \mathbb{N}$, M/P^sM admits a finite filtration whose factors are free A-modules of finite rank.

(iii) In light of (i) and (ii) above, we see that $\operatorname{Ext}^1_R(P^sM/P^{s+1}M,\, E) = 0$ for all $s \in \mathbb{N}$. Moreover, a straightforward induction on the length of the filtration, as well as the long exact sequence for Ext, shows that $\operatorname{Ext}^1_R(M/P^sM,\, E) = 0$ for all $s \in \mathbb{N}$.

(iv) Similarly, by (i) and (ii) above, we see that $\operatorname{Hom}_R(P^sM/P^{s+1}M,\, E)$ is A-free of finite rank for all $s \in \mathbb{N}$.

We show that, under the conditions (i)–(iv), all of the modules N_s/N_{s+1} are A-free of finite rank. For all $s \in \mathbb{N}$, we have an exact sequence

$$0 \to P^sM/P^{s+1}M \to M/P^{s+1}M \to M/P^sM \to 0.$$

Additionally, as $\operatorname{Ext}^1_R(M/P^sM,\, E) = 0$ by (iii) above, we apply $\operatorname{Hom}_R(\_,\, E)$ to obtain the following exact sequence:

$$0 \to N_s \to N_{s+1} \to \operatorname{Hom}_R(P^sM/P^{s+1}M, E) \to 0.$$

To complete the proof of (d), we observe that $\mathrm{Hom}_R(P^sM/P^{s+1}M,\, E)$ is A-free of finite rank, by (iv) above. Also note that $\bigcup_{s\in\mathbb{N}} \mathrm{Ann}_{\mathrm{Hom}_R(M,\,E)}P^s = \mathrm{Hom}_R(M,\, E)$, as every element of $\mathrm{Hom}_R(M,\, E)$ is annihilated by a power of P.

(e) Let $G_\bullet$ denote a projective resolution of M by finitely generated modules and let $\mathcal{C}^\bullet$ be the modified Čech complex for $\underline{x}$. Then $H^i\big(\mathrm{Hom}(G_\bullet,\omega)\big) \cong \mathrm{Ext}^i_R(M,\omega)$ and $H^i_P(M) \cong H^i_{(\underline{x})}(M) \cong H^i(M \otimes \mathcal{C}^\bullet)$. Since, W-generically, $x_1,\, \ldots,\, x_h$ is a regular sequence with radical P in R, we may take $\mathcal{C}^\bullet$ as a flat resolution of $H^h_{(\underline{x})}(R)$ and use it to calculate the Tor. Note, however, that $\mathcal{C}^\bullet$ is numbered for calculating cohomology. Therefore, for fixed M and any finite set of choices for $i \in \mathbb{Z}$, W-generically we have

$$\begin{aligned}
H^i_P(M) &\cong H^i_{(\underline{x})}(M) \cong H^i(M \otimes \mathcal{C}^\bullet) \\
&\cong \mathrm{Tor}^R_{h-i}\left(M,\, H^h_P(R)\right) \cong H_{h-i}\big(G_\bullet \otimes_R H^h_P(R)\big) \\
&\cong H_{h-i}\Big(G_\bullet \otimes_R \mathrm{Hom}_R\left(\omega,\, H^h_P(\omega)\right)\Big) && \text{(by part (d))} \\
&\cong H_{h-i}\Big(\mathrm{Hom}_R\left(\mathrm{Hom}_R(G_\bullet,\omega),\, H^h_P(\omega)\right)\Big) && \text{(by Remark 2.13)} \\
&\overset{\alpha}{\cong} \mathrm{Hom}_R\Big(H^{h-i}\big(\mathrm{Hom}_R(G_\bullet,\omega)\big),\, H^h_P(\omega)\Big) && \text{(by part (a))} \\
&\cong \mathrm{Hom}_R\left(\mathrm{Ext}^{h-i}_R(M,\omega),\, H^h_P(\omega)\right).
\end{aligned}$$

Here we would like to point out that, in order to see the isomorphism $\overset{\alpha}{\cong}$, we need $\mathrm{Hom}_R\left(\_,\, H^h_P(\omega)\right)$ to preserve the exactness of several (but finitely many) short exact sequences, which can be achieved by repeated application of part (a).

Next, we observe that the usual homotopy arguments show that the identification

$$\mathrm{Hom}_R\left(\mathrm{Ext}^i_R(M,\omega), H^h_P(\omega)\right) \cong \mathrm{Tor}^R_i\left(M,\, H^h_P(R)\right)$$

is independent of the choice of projective resolution $G_\bullet$ for M, which is needed to see that this is a natural isomorphism of functors on the variable module M.

It remains to explain the statements in which i is allowed to take on infinitely many values. We already know that we may allow $0 \leqslant i \leqslant h$ or any larger *finite* set of choices for i. But since we have, W-generically, that $P = \mathrm{Rad}(x_1,\, \ldots,\, x_h)R$, the local cohomology modules $H^i_P(M)$ vanish when $i > h$, as do the modules involving $\mathrm{Ext}^{h-i}_R(M,\, \omega)$ since $h - i < 0$. Thus, we have proved that for fixed M, the result holds W-generically for all integers $i \geqslant 0$.

For $i < 0$, the local cohomology modules $H^i_P(M)$ vanish. To complete the proof of (e), it remains only to prove that under the hypothesis that R or R_g, for some $g \in R \setminus P$, is excellent of finite Krull dimension, say k, we have for some $g' \in W$ and for all $j > k$ that

$$\mathrm{Hom}_{R_{g'}}\big(\mathrm{Ext}^j_{R_{g'}}(M_{g'}, \omega_{g'}),\, E_{g'}\big) = 0,$$

since there are only finitely many integers i such that $h+1 \leqslant h-i \leqslant k$. Consequently, it suffices to show that for some choice of $g' \in W$, we have $\mathrm{Ext}^j_{R_{g'}}(M_{g'}, \omega_{g'}) = 0$ for all $j > k$. For this it is enough to show that the injective dimension of $\omega_{g'}$ is at most k for some $g' \in W$. This is true by Remark 2.14, because ω_Q is a canonical module for R_Q for all primes Q in a sufficiently small Zariski neighborhood of P.

(f) This is immediate from (c) and (e). The proof is complete. □

Remark 2.20. Just as the usual form of local duality may be applied to homomorphic images of Gorenstein rings or to homomorphic images $\overline{R}$ of Cohen-Macaulay rings R such that R has a canonical module, it should be clear that the results of Theorem 2.19 can be applied to modules over a homomorphic image $\overline{R}$ of a ring R satisfying the hypothesis of Theorem 2.19. One can consider the modules over $\overline{R}$ as R-modules. Theorem 5.6 provides an illustration of this technique.

Remark 2.21. There are many finitely generated modules M for which the conclusion in the last sentence of part (e) of Theorem 2.19, where i is allowed to take on all values in $\mathbb{Z}$, holds on a sufficiently small Zariski neighborhood $D(g)$ of P without any additional hypothesis on R. This holds, for example, if M_g has finite projective dimension over R_g for some $g \in R \setminus P$, which is equivalent to the statement that M_P has finite projective dimension over R_P.

3. Comparison with the results of Karen Smith

The results of Karen Smith [Sm18] were a great inspiration in developing the theory of §2. Many of the results of [Sm18] are connected with cohomology of sheaves on schemes and base change, but we focus in this section on the underlying results in commutative algebra that are used in proving the base change results. In the process, we discuss their connections with Theorem 2.19. We first note [Sm18, Theorem 2.1]:

Theorem 3.1 (K. E. Smith)**.** *Let A be a Noetherian domain, let $R = A[\![x_1, \dots, x_h]\!]$ be a power series ring over A, and let M be a finitely generated R-module. Denote by I the ideal $(x_1, \dots, x_h) \subsetneq R$. Then the local cohomology modules $H^i_I(M)$ are generically free over A and commute with base change for all i.*

We second note [Sm18, Corollary 1.3]:

Corollary 3.2 (K. E. Smith)**.** *Let A be a Noetherian reduced ring such that $A \hookrightarrow R$, where R is Noetherian, and let $I = (x_1, \dots, x_h)$ be an ideal of R such that the composite map $A \hookrightarrow R \to R/I$ is module-finite. Let M be an R-module such that M/IM is finitely generated over A. Then there exists $g \in A \setminus \bigcup_{\mathfrak{p} \in \mathrm{Min}(A)} \mathfrak{p}$ such that $H^j_I(M_g)$ is A_g-free for all j. Moreover, $H^j_I(M \otimes_A L) \cong H^j_I(M) \otimes_A L$ for every A_g-module L. In particular, this holds when L is an A-algebra such that $A \to L$ factors through A_g.*

We note the following. One may first localize so that A becomes a product of domains, and the result reduces to the domain case. Second, one may complete R with respect to I (which does not affect the local cohomology) and then the A-algebra map from the formal power series ring $A[\![X_1, \dots, X_h]\!]$ to R, defined by $X_i \mapsto x_i$, is module-finite. Then M becomes a finitely generated module over $A[\![X_1, \dots, X_h]\!]$, and the problem reduces to the case where $R = A[\![X_1, \dots, X_h]\!]$ with A a Noetherian domain and $I = (X_1, \dots, X_h)R$. Another key result in [Sm18] is that when $R = A[\![X_1, \dots, X_h]\!]$ and M is finitely generated over R as above, one has, after localizing at one element of $A \setminus \{0\}$, that $H^j_I(M) \cong \operatorname{Hom}_R\big(\operatorname{Ext}^{h-j}_R(M, R), H^h_I(R)\big)$ for all $j \geqslant 0$.

Discussion 3.3. Let $R = A[\![X_1, \dots, X_h]\!]$, where A is a Noetherian domain, and let M be a finitely generated R-module, as above. Let $P = (X_1, \dots, X_h)R$, $\omega = R$, and $W = A \setminus \{0\}$. Note that R_P is regular and, hence, ω_P is an canonical module for R_P. By Theorem 2.19, we see that, W-generically, $H^j_P(M)$ is free filterable relative to P and $H^j_P(M) \cong \operatorname{Hom}_R\big(\operatorname{Ext}^{h-j}_R(M, \omega), H^h_P(\omega)\big)$ for all $j \geqslant 0$. Now, as the composition $A \subseteq R \twoheadrightarrow R/P$ is an isomorphism, Proposition 2.4 tells us that, W-generically, $H^j_P(M)$ is A-free for all $j \geqslant 0$. To see the base change results by a different argument from the one used in [Sm18], observe that $H^j_P(M) \cong \operatorname{Tor}^R_{h-j}\big(M, H^h_P(\omega)\big)$ is the homology at the $h - j$ spot of the complex $G_\bullet \otimes_R H^h_P(R)$, where $G_\bullet$ is a free resolution of M by finitely generated free R-modules. As noted right above, W-generically, all $H^j_P(M)$ are A-free, which clearly applies to $H^h_P(R)$ as well. Thus, after localization at $g \in A \setminus \{0\}$, the complex $\big(G_\bullet \otimes_R H^h_P(R)\big)_g$ consists of free A_g-modules and has all its cohomology free over A_g. Now the homology of $\big(G_\bullet \otimes_R H^h_P(R)\big)_g$ commutes with $\_ \otimes_{A_g} L$ for any A_g-module L, which yields the base change result of [Sm18, Corollary 1.3]. Of course, in the more general situation of this paper, as in Theorem 2.19, one gets local cohomology with filtrations that have A-free factors W-generically. One cannot hope for more, since the local cohomology modules, typically, are not A-modules.

Discussion 3.4. Note that [Sm18] states many results in terms of $\operatorname{Hom}^{\mathrm{cts}}_A(M, A)$, which consists of the A-linear maps from the R-module M to A that vanish on $I^t M$ for some t. Here A, R, I and M are as in Corollary 3.2, but the situation reduces to the case where R is a power series over A. With $R = A[\![x_1, \dots, x_h]\!]$ and $I = (x_1, \dots, x_h)R$, we have that $H^h_I(R) \cong \operatorname{Hom}^{\mathrm{cts}}_A(R, A)$, which is the same as the module E in our Theorem 2.19 if one takes $\omega := R$ and $P := I$. Moreover, as noted in [Sm18, Proposition 3.1], the functor $M \mapsto \operatorname{Hom}_R(M, E)$ is naturally isomorphic to the functor $M \mapsto \operatorname{Hom}^{\mathrm{cts}}_A(M, A)$ on R-modules M: in essence, this is just the usual adjointness of tensor and Hom, restricted to maps that kill $I^t M$ for some t. It is then easy to see that the results of Theorem 2.19, when applied to the case where $R = A[\![x_1, \dots, x_h]\!]$, $P = (x_1, \dots, x_h)R$, $W = A \setminus \{0\}$ and $\omega = R$, are the same as the results of [Sm18], in light of the fact that, by Proposition 2.4, free filterable relative to P implies A-free.

Finally, we note that the result on generic local duality in [Sm18, Theorem 5.1] is stated there as follows:

Theorem 3.5 (K. E. Smith)**.** *Let R be a power series ring $A[\![x_1, \ldots, x_h]\!]$ over a Noetherian domain A, let $I = (x_1, \ldots, x_h)R$, and let M be a finitely generated R-module. Then, after replacing A by its localization at one element of $A \setminus \{0\}$, for all $i \geqslant 0$, there is a functorial isomorphism*

$$H^i_I(M) \cong \mathrm{Hom}^{\mathrm{cts}}_A\big(\,\mathrm{Ext}^{h-i}_R(M, R),\, A\big).$$

As noted in Discussion 3.4 just above, the R-module $\mathrm{Hom}^{\mathrm{cts}}_A\big(\,\mathrm{Ext}^{h-i}_R(M, R),\, A\big)$ is naturally isomorphic with $\mathrm{Hom}_R\big(\,\mathrm{Ext}^{h-i}_R(M, R),\, H^h_I(R)\big)$. Thus, our result on generic local duality in part (e) of Theorem 2.19, applied to $R = A[\![x_1, \ldots, x_h]\!]$, agrees with the result on generic local duality in [Sm18, Theorem 5.1].

4. $(R \setminus\!\!\setminus \mathfrak{P})$-PSEUDOFLAT MODULES

In this section, we discuss $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat R-modules. See Definition 1.1.

Proposition 4.1. *Let R be a ring, let $\mathfrak{P}$ be a proper ideal in R, and let $A := R/\mathfrak{P}$.*

- (a) *A direct limit of $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat R-modules is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat.*
- (b) *A flat A-module is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat.*
- (c) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules.*
  - (i) *If M' and M'' are $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat then so is M.*
  - (ii) *If M and M'' are $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat then so is M'.*
- (d) *An R-module with a countable ascending filtration whose factors are $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat.*
- (e) *An arbitrary direct sum of $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat R-modules is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat.*
- (f) *An R-module with a countable ascending filtration whose factors are flat A-modules is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat.*
- (g) *If an R-module is free filterable relative to $\mathfrak{P}$ then it is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat.*
- (h) *A module with a finite right resolution by $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat R-modules is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat.*
- (i) *If A is regular, an A-module M is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat if and only if it is A-flat.*
- (j) *If $R \to R'$ is flat over R and M is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat then $M \otimes_R R'$ is $(R' \setminus\!\!\setminus \mathfrak{P}R')$-pseudoflat.*

Proof. Parts (a), (b), (c) and (j) follow from the definition of $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat and basic facts about flat base change and the behavior of (possibly improper) regular sequences. Parts (d) and (e) follow from (c) and (a), while part (f) follows from (b) and (d). Part (f) implies part (g), while part (h) follows from part (c)(ii) by induction on the length of the resolution. To prove (i), note that if M is A-flat then it is $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat by part (b). Now suppose M is an A-module and $(R \setminus\!\!\setminus \mathfrak{P})$-pseudoflat. This is preserved when make a base change to any local ring of A, and so we may assume that A is regular local. The result now follows from the assertion that if A is regular local and every regular sequence on A is a possibly improper regular sequence on M, then M is A-flat. The argument is given in [HH92, 6.7, p. 77], where it is not ever used that the regular sequences $x_1, \ldots, x_k$ on A are proper regular sequences on M: one still has that $\operatorname{Tor}_i^A\big(A/(x_1, \ldots, x_k), M\big) = 0$ for $i \geqslant 1$ when $x_1, \ldots, x_k$ are possibly improper regular sequences on M. $\square$

We also note:

Proposition 4.2. *Let $\mathfrak{P}$ be a proper ideal in R and let $A := R/\mathfrak{P}$. Also, let R' be a flat extension of R, let $\underline{y} := y_1, \ldots, y_d \in R'$ be a sequence of elements whose images form a possibly improper regular sequence on $R'/\mathfrak{P}R'$, and let $\underline{y}^t = y_1^t, \ldots, y_d^t$. If*

$$(*) \qquad 0 \to M' \to M \to M'' \to 0$$

is an exact sequence of R-modules such that M'' is $(R \mathbin{\backslash\!\backslash} \mathfrak{P})$-pseudoflat, then for all $t \in \mathbb{N}$, the sequences

$$(*_t) \qquad 0 \to \frac{M' \otimes_R R'}{(\underline{y}^t)(M' \otimes_R R')} \to \frac{M \otimes_R R'}{(\underline{y}^t)(M \otimes_R R')} \to \frac{M'' \otimes_R R'}{(\underline{y}^t)(M'' \otimes_R R')} \to 0$$

and the sequence

$$(**) \qquad 0 \to H^d_{(\underline{y})}(M' \otimes_R R') \to H^d_{(\underline{y})}(M \otimes_R R') \to H^d_{(\underline{y})}(M'' \otimes_R R') \to 0$$

are exact. Hence, for any exact sequence $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$, with n finite, and given any $i_0 \in \{1, 2\}$, if M_i is $(R \mathbin{\backslash\!\backslash} \mathfrak{P})$-pseudoflat for all $i \neq i_0$, then the module M_{i_0} is $(R \mathbin{\backslash\!\backslash} \mathfrak{P})$-pseudoflat and the whole sequence remains exact when we apply either $\_ \otimes_R R'/(\underline{y}^t)$ or $H^d_{(\underline{y})}(\_ \otimes_R R')$.

Proof. Since the pseudoflatness conditions are preserved by the flat base change $R \to R'$ (cf. Proposition 4.1(j)), we may assume that $R' = R$. The sequences $(*_t)$ are standard and reduce by induction to the case $d = 1$. The sequence $(**)$ is the direct limit of the sequences $(*_t)$.

The final statement concerning the exact sequence $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$ follows at once by induction on n, the length of the exact sequence, with $n = 3$ proved just now in $(*_t)$ and $(**)$ above. The conclusion that M_{i_0} is $(R \mathbin{\backslash\!\backslash} \mathfrak{P})$-pseudoflat can be obtained by repeated applications of Proposition 4.1(c). $\square$

Corollary 4.3. *Let $\mathfrak{P}$ be a proper ideal in R and let $A := R/\mathfrak{P}$. Let R' be a flat extension of R, and let $\underline{y} = y_1, \ldots, y_d \in R'$ be a sequence of elements whose images in $R'/\mathfrak{P}R'$ form a possibly improper regular sequence. Suppose that*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots \subseteq M$$

is a finite or countably infinite filtration of M over R, so that $M = \bigcup_{n\geqslant 1} M_n$, and denote the factors as $N_i := M_i/M_{i-1}$, $i \geqslant 1$. If the modules N_i are $(R\setminus\!\!\setminus\mathfrak{P})$-pseudoflat for all $i \geqslant 1$, then M is $(R\setminus\!\!\setminus\mathfrak{P})$-pseudoflat and, for all t, the modules

$$\frac{M_i \otimes_R R'}{(\underline{y}^t)(M_i \otimes_R R')} \quad \text{(respectively, } H^d_{(\underline{y})}(M_i \otimes_R R')\text{)}, \quad i \geqslant 0,$$

give a corresponding filtration of $\dfrac{M \otimes_R R'}{(\underline{y}^t)(M \otimes_R R')}$ *(respectively, $H^d_{(\underline{y})}(M \otimes_R R')$) with factors* $\dfrac{N_i \otimes_R R'}{(\underline{y}^t)(N_i \otimes_R R')}$ *(respectively, $H^d_{(\underline{y})}(N_i \otimes_R R')$).*

Before stating the next corollary, we set up the following notation, which is convenient in describing elements of top local cohomology:

Notation 4.4. Let $\underline{y} = y_1, \ldots, y_d \in R$ and let M be an R-module, so that we may view $H^d_{(\underline{y})}(M)$ as $\varinjlim_t M/(\underline{y}^t)M$ as usual, where $(\underline{y}^t) := (y_1^t, \ldots, y_d^t)R$. For $u \in M$, we use the notation $[u;\, \underline{y}^t]$ for the natural image of $u + (\underline{y}^t)M \in M/(\underline{y}^t)M$ in $H^d_{(\underline{y})}(M)$. Note that when $\underline{y}$ is a regular sequence on M, we may think of $M/(\underline{y}^t)M$ as a natural submodule of $H^d_{(\underline{y})}(M)$. As long as there is no confusion, we also use the same notation $[u;\, \underline{y}^t]$ to denote the corresponding element in $H^d_{(\underline{y})}(M)_Q$ after localization, where Q is a prime ideal of R.

The following result plays a critical role in the proof of Key Lemma 7.13, and consequently in the proofs of Theorems 7.10 and 7.11.

Corollary 4.5. *Let R and $\mathfrak{P}$ be as in Corollary 4.3. Let M be an R-module, and let $u \in M$ be an element such that both M and M/Ru are $(R\setminus\!\!\setminus\mathfrak{P})$-pseudoflat. Also, let R' be a flat extension of R and let $\underline{y} := y_1, \ldots, y_d \in R'$ whose images form a possibly improper regular sequence on $R'/\mathfrak{P}R'$. Consider $u \otimes 1_{R'} \in M \otimes_R R'$ and, using Notation 4.4 above, consider $[u \otimes 1_{R'}; \underline{y}] \in H^d_{(\underline{y})}(M \otimes_R R') \cong M \otimes_R H^d_{(\underline{y})}(R')$. Then $\mathrm{Ann}_{R'}([u \otimes 1_{R'}; \underline{y}]) = \mathrm{Ann}_R(u)R' + (\underline{y})R'$.*

Proof. Since $\mathrm{Ann}_R u$ commutes with the flat base change $R \to R'$ and since the pseudoflatness conditions are preserved by that base change (cf. Proposition 4.1(j)), we may assume that $R' = R$. Note that the module Ru is $(R\setminus\!\!\setminus\mathfrak{P})$-pseudoflat as well by Proposition 4.1(c). Thus, by Corollary 4.3, we have an injection $Ru \otimes_R R/(\underline{y}) \hookrightarrow H^d_{(\underline{y})}(Ru) \hookrightarrow H^d_{(\underline{y})}(M)$ that takes $u \otimes \overline{1}$ to $[u; \underline{y}] \in H^d_{(\underline{y})}(M)$, where we agree that $u \otimes \overline{1} \in Ru \otimes_R R/(\underline{y})$. Thus $\mathrm{Ann}_R([u; \underline{y}]) = \mathrm{Ann}_R(u \otimes \overline{1})$, which is $\mathrm{Ann}_R\big(Ru \otimes_R R/(\underline{y})\big)$ since $u \otimes \overline{1}$ generates $Ru \otimes_R R/(\underline{y})$. But $\mathrm{Ann}_R\big(Ru \otimes_R R/(\underline{y})\big)$ is immediate from the fact that $Ru \otimes_R R/(\underline{y}) \cong R/\mathrm{Ann}_R(u) \otimes_R R/(\underline{y}) \cong R/\big(\mathrm{Ann}_R(u) + (\underline{y})\big)$. □

Remark 4.6. Of course, in Corollary 4.5, we may also apply the result with $\underline{y}$ replaced by $\underline{y}^t = y_1^t, \ldots, y_d^t$, which is also a regular sequence, and obtain $\mathrm{Ann}_{R'}([u \otimes 1_{R'}; \underline{y}^t]) = \mathrm{Ann}_R(u)R' + (\underline{y}^t)R'$.

The next corollary combines results in this section with the main result of §2 (Theorem 2.19):

Corollary 4.7. *Let R, P, E, W be as in Theorem 2.19. Let Δ be a finite set and consider a family of complexes $\{M_{i1} \to M_{i2} \to M_{i3}\}_{i \in \Delta}$, where for all $i \in \Delta$ and $1 \leqslant j \leqslant 3$ the M_{ij} are finitely generated R-modules. After we localize at one element of W and change notation to call the localized ring R, for all flat R-algebras R' and $\underline{y} = y_1, \ldots, y_d \in R'$ whose images form a possibly improper regular sequence on R'/PR', the functor $H^d_{(\underline{y})}\big(\mathrm{Hom}(\_\otimes_R R', E \otimes_R R')\big)$ commutes with taking homology at the M_{i2} spot for all $i \in \Delta$. Hence, $H^d_{(\underline{y})}\big(\mathrm{Hom}(\_\otimes_R R', E \otimes_R R')\big)$ commutes with taking homology of any finite collection of finite complexes of finitely generated R-modules after localizing at one element of W that is dependent on the set of complexes.*

Proof. The information about the complexes and their homology is determined by a finite family of short exact sequences. Thus, it suffices to prove that exactness is preserved when we apply the functor to one short exact sequence. By parts (a) and (d) of Theorem 2.19, after we localize at one element of W, when we apply $\mathrm{Hom}_{R'}(\_ \otimes_R R', E \otimes_R R')$ we get a short exact sequence of R'-modules that are all free filterable relative to PR' and hence are all $(R' \setminus\!\!\setminus PR')$-pseudoflat. The result now follows from part (g) of Proposition 4.1 and Proposition 4.2, applied over R'. ☐

Discussion 4.8 (A non-Noetherian acyclicty criterion). We first observe that the acyclicity criterion in [BuEi73] is valid without Noetherian assumptions. This is shown in [Nor76, §6], and a much more concise treatment is available in [Ho02]. One needs to use a notion of depth based on vanishing of Koszul homology: this agrees with the usual notion in the Noetherian case. Given a finitely generated ideal I of an arbitrary ring R and an R-module M, define the *Koszul depth* of M on $I = (r_1, \ldots, r_n)$ to be d, where d is the least value of $i \in \mathbb{N}$ such that $H_{n-i}(r_1, \ldots, r_n; M) \neq 0$. This is independent of the choice of generators of I.

Theorem 4.9 (Acyclicity Criterion)**.** *Let R be an arbitrary ring, not necessarily Noetherian. Let $G_\bullet$ denote a finite complex of finite rank nonzero free modules over R, say*

$$G_\bullet : \qquad 0 \to R^{b_n} \to \cdots \to R^{b_1} \to R^{b_0} \to 0.$$

Let $N \neq 0$ be an arbitrary R-module. Let α_i denote the matrix of the map $G_i \to G_{i-1}$, $1 \leqslant i \leqslant n$. Let $r_0 = 0$ and for $i \geqslant 1$, let r_i denote the determinantal rank of α_i modulo $\mathrm{Ann}_R N$. Then $G_\bullet \otimes_R N$ is acyclic if and only if the following two conditions hold:

(1) *$b_i = r_i + r_{i-1}$, for all $1 \leqslant i \leqslant n$.*

(2) *The Koszul depth of N on $I_{r_i}(\alpha_i) \geqslant i$, for all $1 \leqslant i \leqslant n$.*

Remark 4.10. Note that condition (1) is equivalent to the assumption that $r_n = b_n$, $r_{n-1} = b_n - b_{n-1}$, $r_{n-2} = b_n - b_{n-1} + b_{n-2}$, $\ldots$, $r_{n-i} = \sum_{t=0}^{i}(-1)^t b_{n-t}$, $\ldots$, and $r_1 = b_n - b_{n-1} + \cdots + (-1)^{n-1}b_1$.

Remark 4.11. Consider a complex $G_\bullet$ as in Theorem 4.9, and let the r_i be as in Remark 4.10. We claim that if R has Koszul depth at least one on each of the ideals $I_{r_i}(\alpha_i)$, where $1 \leqslant i \leqslant n$, then it is automatic that the ideals $I_{r_i+1}(\alpha_i)$ are all 0, $1 \leqslant i \leqslant n$. To see this, note that after a faithfully flat extension of R, each of the ideals $I_{r_i}(\alpha_i)$, $1 \leqslant i \leqslant n$, will contain a nonzerodivisor. Inverting the product of these does not affect the issue, so that we may assume that each $I_{r_i}(\alpha_i)$ is the unit ideal. We use induction on n. The case $n = 1$ is obvious. Assume $n \geqslant 2$. The issue is local on R, so that we may assume that R is qasilocal. After changes of basis in G_{n-1}, which do not affect the issue, we may assume that α_n has the block form $\begin{pmatrix} \mathrm{I}_{b_n} \\ 0 \end{pmatrix}$, where the 0 block has size $(b_{n-1} - b_n) \times b_n$. Now it is clear that $I_{r_n+1}(\alpha_n) = I_{b_n+1}(\alpha_n) = 0$. The fact that $G_\bullet$ is a complex implies that α_{n-1} has the block form $\begin{pmatrix} 0_{b_{n-2}\times b_n} & \alpha'_{n-1} \end{pmatrix}$, where α'_{n-1} is $b_{n-2} \times (b_{n-1} - b_n)$. That is, $G_\bullet$ is the direct sum of a complex of length $n-1$, namely

$$G'_\bullet : \qquad 0 \to R^{b_{n-1}-b_n} \xrightarrow{\alpha'_{n-1}} R^{b_{n-2}} \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} R^{b_1} \xrightarrow{\alpha_1} R^{b_0} \to 0$$

and a complex with nonzero terms only at the spots indexed by n and $n-1$, namely

$$0 \to R^{b_n} \xrightarrow{\mathrm{I}_{b_n}} R^{b_n} \to 0.$$

Then it is routine to see that $I_j(\alpha_{n-1}) = I_j(\alpha'_{n-1})$ for all j. In particular, we have $I_{r_{n-1}}(\alpha_{n-1}) = I_{b_{n-1}-b_n}(\alpha'_{n-1})$. Now the remaining claims, i.e., $I_{r_i+1}(\alpha_i) = 0$ for $i = 1, \ldots, n-1$, follow from the induction hypothesis on $G'_\bullet$.

We are now ready to state the result mentioned earlier.

Theorem 4.12. *Let R be a Noetherian ring, P a prime ideal of R, and $A := R/P$. Let $G_\bullet$ be complex of finitely generated free R-modules of finite length n such that $A \otimes_R G_\bullet$ is acyclic. Let $\mathcal{H}$ be an R-module and assume at least one of the following*

(1) *$\mathcal{H}$ is free filterable relative to P.*
(2) *$\mathcal{H}$ is a direct limit of modules each of which has a finite filtration with A-flat factors.*
(3) *$\mathcal{H}$ is $(R \setminus\!\setminus P)$-pseudoflat and P contains all primes in $\mathrm{Ass}_R(R)$ (the latter holds, for example, if R is a domain).*

Then $\mathcal{H} \otimes_R G_\bullet$ is acyclic.

In particular, under the hypotheses of Notation 2.17, if $\mathrm{frac}(A) \otimes_R G_\bullet$ *is acyclic, then given a finitely generated* R*-module* M*, after localizing at one element of* W*, we have that* $H^j_P(M) \otimes G_\bullet$ *is acyclic.*

Proof. Note that condition (1) implies condition (2). Assume that (2) holds, so that $\mathcal{H}$ is direct limit of modules $\mathcal{H}_\lambda$ each of which has a finite filtration by flat A-modules. Since $A \otimes_R G_\bullet$ is acyclic, so is the tensor product with any flat A-module, and then one has that the tensor product with each $\mathcal{H}_\lambda$ is acyclic by iterated use of the snake lemma. By taking a direct limit, one sees that the tensor product with $\mathcal{H}$ is acyclic.

We now prove that condition (3) is sufficient for acyclicity. Assume that $\mathcal{H}$ is $(R \setminus\!\!\setminus P)$-pseudoflat. We adopt the detailed description of $G_\bullet$ given in the statement of Theorem 4.9. Let the r_i be determined from the b_i as in Remark 4.10. The hypothesis that $A \otimes_R G_\bullet$ is acyclic shows that for each i, $1 \leqslant i \leqslant n$, the ideal of r_i minors of α_i, when passed to A, is either the unit ideal A or contains an A-regular sequence oflength at least i. The ideals $I_{r_i}(\alpha_i)$ over R therefore each contain an element $f_i \in R \setminus P$, which is a nonzerodivisor on R. By Remark 4.11 and Remark 4.10, these numbers r_i are indeed the determinantal rank of α_i on $\mathcal{H}$ for $1 \leqslant i \leqslant n$, which allows us to conclude that condition (1) of Theorem 4.9 holds for $\mathcal{H} \otimes_R G_\bullet$. Moreover, it follows that each $I_{r_i}(\alpha_i)$ contains i elements of R that map to a possibly improper regular sequence in A. Hence, these elements form a possibly improper regular sequence on $\mathcal{H}$, which verifies that condition (2) of Theorem 4.9 holds for $\mathcal{H} \otimes_R G_\bullet$. The acyclicity of $\mathcal{H} \otimes_R G_\bullet$ now follows from Theorem 4.9.

For the final statement on $H^j_P(M) \otimes G_\bullet$, first note that because $H^j_P(M)$ is supported only at P, localizing at an element $g \in W$ yields the same result for $H^j_P(M) \otimes G_\bullet$ if g is replaced by an element of $R \setminus P$ with the same image as g in R/P. Therefore we can carry out the proof with $W = R \setminus P$. The result follows because if $\mathrm{frac}(A) \otimes_R G_\bullet$ is acyclic, then after replacing R by its localization at one element of W, the complex $A \otimes_R G_\bullet$ becomes acyclic, and by Theorem 2.19, $H^j_P(M)$ is free filterable relative to P after localizing at one element of W. Now we may apply part (1) of this theorem. $\square$

5. Generic behavior of injective hulls

In this section we study generic behavior for homomorphic images of rings that satisfy Notation 2.17. Since our focus is on the homomorphic image, we change notations as indicated just below. We start with a ring that satisfies Notation 2.17, but we denote this ring by $\widetilde{R}$, so that Theorem 2.19 holds for $\widetilde{R}$ and $\widetilde{P} \in \mathrm{Spec}(\widetilde{R})$. The focus of this section is $R := \widetilde{R}/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\widetilde{R}$ such that $\mathfrak{a} \subseteq \widetilde{P}$. Denote $P := \widetilde{P}/\mathfrak{a} \in \mathrm{Spec}(R)$. Because we want R_P to have Krull dimension h, we will use $\widetilde{h}$ for the Krull dimension of $\widetilde{R}_{\widetilde{P}}$. In the first subsection we discuss what happens without taking a quotient. The second subsection deals with the case where we work with $R = \widetilde{R}/\mathfrak{a}$. The third subsection is focused on the case when R_P is S_2. Note that we do not need to change the notation for A, since $A = \widetilde{R}/\widetilde{P} \cong \dfrac{\widetilde{R}/\mathfrak{a}}{\widetilde{P}/\mathfrak{a}}$.

Notation 5.1. Let $\widetilde{R}$, $\widetilde{P}$, $A := \widetilde{R}/\widetilde{P}$, $\widetilde{h} := \operatorname{height}(\widetilde{P})$, and $\widetilde{\omega}$ be as in Notation 2.17. That is, let $\widetilde{R}$ be a Noetherian ring, $\widetilde{P}$ a prime ideal of $\widetilde{R}$ such that $\widetilde{R}_{\widetilde{P}}$ is Cohen-Macaulay of Krull dimension $\widetilde{h}$. Assume also that $\widetilde{\omega}$ is a finitely generated $\widetilde{R}$-module such that $\widetilde{\omega}_{\widetilde{P}}$ is a canonical module for $\widetilde{R}_{\widetilde{P}}$. Let $\widetilde{E} := H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})$ and $\widetilde{W} := \widetilde{R} \setminus \widetilde{P}$. For every prime $\mathfrak{p}$ of $\widetilde{R}$ let $\kappa_{\mathfrak{p}}$ denote $\widetilde{R}_{\mathfrak{p}}/\mathfrak{p}\widetilde{R}_{\mathfrak{p}}$, which is naturally isomorphic with $\operatorname{frac}(\widetilde{R}/\mathfrak{p})$. Also, denote $P := \widetilde{P}/\mathfrak{a}$, which is a prime of R. However, we do *not* assume that R_P is S_2 in this section until subsection 5.3.

Note that, for simplicity, we are taking $\widetilde{W}$ to be $\widetilde{R} \setminus \widetilde{P}$ itself rather than a multiplicative system in $\widetilde{R}$ or a subset of $\widetilde{R}$ that naturally maps onto $A \setminus \{0\}$.

Consider the following conditions:

- (E1) The regular locus in $\operatorname{Spec}(A)$ has non-empty interior, i.e., there exists $g_1 \in \widetilde{W}$ such that A_{g_1} is regular.
- (E2) There exists $g_2 \in \widetilde{W}$ such that $\widetilde{R}_{g_2}$ is Cohen-Macaulay and $\widetilde{\omega}_{g_2}$ is a global canonical module for $\widetilde{R}_{g_2}$.
- (E3) If R_P is S_2 then P is in the interior of the S_2 locus of R; that is, if $(\widetilde{R}/\mathfrak{a})_{\widetilde{P}}$ is S_2 then there exists $g_3 \in \widetilde{W}$ such that $(\widetilde{R}/\mathfrak{a})_{g_3}$ is S_2.

Remark 5.2. By [EGAIV67, Cor. 5.10.9], every local ring that is catenary and S_2 is equidimensional. By [EGAIV67, Prop. 6.11.8], the S_2 locus is open in an excellent ring.

Proposition 5.3. *If $\widetilde{R}_g$ is excellent for some $g \in \widetilde{W}$, then (E1), (E2) and (E3) all hold.*

Proof. It suffices to assume that $\widetilde{R}$ is excellent. Then (E1) is clear, since $A = \widetilde{R}/\widetilde{P}$ is an excellent domain. For (E2), we can localize at an element of $\widetilde{W}$ so that $\widetilde{R}$ is Cohen-Macaulay, and so that $\widetilde{\omega}$ is a global canonical module. See Remark 2.14. Finally, (E3) follows from Remark 5.2. □

5.1. **Generic behavior of injective hulls near $\widetilde{P}$ if $\widetilde{R}_{\widetilde{P}}$ is Cohen-Macaulay.** We shall show how to obtain injective hulls for quotients $\widetilde{R}/\widetilde{Q}$ over $\widetilde{R}$ for all prime ideals $\widetilde{Q}$ in an open neighborhood of $\widetilde{P}$ in $V(\widetilde{P})$.

Theorem 5.4. *Let $\widetilde{R}$, $\widetilde{P}$, $\widetilde{h}$, $A := \widetilde{R}/\widetilde{P}$, $\widetilde{W} := \widetilde{R} \setminus \widetilde{P}$, $\widetilde{\omega}$, and $\widetilde{E}$ be as in Notation 5.1. Assume that $\widetilde{R}$ is excellent or that conditions (E1) and (E2) above hold. Then we* *can localize at one element $g \in \widetilde{W}$ such that after replacing $\widetilde{R}$, $\widetilde{P}$, $\widetilde{\omega}$, and $\widetilde{E}$ by their localizations at g, all the following statements hold:*

(a) *The ring A is regular, the ring $\widetilde{R}$ is Cohen-Macaulay, and the module $\widetilde{\omega}$ is a canonical module for $\widetilde{R}$, i.e., for all $\widetilde{Q} \in \operatorname{Spec}(\widetilde{R})$, $\widetilde{\omega}_{\widetilde{Q}}$ is a canonical module for $\widetilde{R}_{\widetilde{Q}}$.*
(b) $\operatorname{Ann}_{\widetilde{E}}\widetilde{P} \cong A$, *so that we have an injection $0 \to A \hookrightarrow \widetilde{E}$.*

In all of the remaining parts, assume that we have localized at an element of $\widetilde{W}$ so that conditions (a) *and* (b) *hold. Moreover, in the remaining parts, we place the following condition* ($\sharp$) *on $\widetilde{Q}$ and $\underline{y} = y_1, \ldots, y_d$:*

($\sharp$) *The ideal $\widetilde{Q} \in \operatorname{Spec}(\widetilde{R})$ is in $V(\widetilde{P})$ and the sequence $\underline{y}$ in $\widetilde{Q}$ maps to a system of parameters for $A_{\widetilde{Q}} \cong A_Q$.*

Then we have the following:

(c) *After localization at one element of $\widetilde{W}$, for all $\widetilde{Q}$ and $\underline{y}$ satisfying* ($\sharp$), *the module $H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}$ is an injective hull for $\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}R_{\widetilde{Q}}$ over $\widetilde{R}_{\widetilde{Q}}$, i.e., $H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}} \cong \mathrm{E}_{\widetilde{R}}(\widetilde{R}/\widetilde{Q})$.*
(d) *For any given finitely generated $\widetilde{R}$-module M, after localizing at one element of $\widetilde{W}$, for all $\widetilde{Q}$ and $\underline{y}$ satisfying* ($\sharp$), *the natural map*

$$\Phi_M : H^d_{(\underline{y})}\big(\operatorname{Hom}_{\widetilde{R}}(M, \widetilde{E})\big)_{\widetilde{Q}} \to \operatorname{Hom}_{\widetilde{R}}\big(M, H^d_{(\underline{y})}(\widetilde{E})\big)_{\widetilde{Q}}$$

is an isomorphism.
(e) *After localizing at one element of $\widetilde{W}$, for all $\widetilde{Q}$ and $\underline{y}$ satisfying* ($\sharp$), *the natural map $H^d_{(\underline{y})}(\operatorname{Ann}_{\widetilde{E}}\widetilde{P})_{\widetilde{Q}} \hookrightarrow H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}$ is an injection, and a socle generator for $H^d_{(\underline{y})}(\operatorname{Ann}_{\widetilde{E}}\widetilde{P})_{\widetilde{Q}} \cong H^d_{(\underline{y})}(A)_{\widetilde{Q}}$ over $A_{\widetilde{Q}}$ maps to a socle generator for $H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}$ over $\widetilde{R}_{\widetilde{Q}}$.*

Proof. Part (a) is immediate from the hypotheses and Proposition 5.3. Part (b) follows from part (b) of Theorem 2.19 applied over $\widetilde{R}$.

(c) Let $\widetilde{Q}$ and $\underline{y} = y_1, \ldots, y_d$ satisfy ($\sharp$), so that $\dim(R_Q) = \widetilde{h} + d$. Note that $\widetilde{\omega}_Q$ is assumed to be a canonical module for $\widetilde{R}_{\widetilde{Q}}$ by part (b) above, and the maximal ideal of $\widetilde{R}_{\widetilde{Q}}$ is the radical of $\widetilde{P} + (\underline{y})$. Let $\underline{x} = x_1, \ldots, x_{\widetilde{h}}$ be a sequence of $\widetilde{h}$ elements of $\widetilde{R}$ whose images form a system of parameters for $\widetilde{R}_{\widetilde{P}}$. Localize at one element of $\widetilde{W}$ so that the radical of $(\underline{x})$ is $\widetilde{P}$ and, hence, the radical of $\big((\underline{x}) + (\underline{y})\big)R_Q$ is QR_Q. Then

$\mathrm{E}_{\widetilde{R}}(\widetilde{R}/\widetilde{Q}) \cong \mathrm{E}_{\widetilde{R}_{\widetilde{Q}}}(\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}\widetilde{R}_{\widetilde{Q}})$ can be realized as follows:

$$\begin{aligned}\mathrm{E}_{\widetilde{R}_{\widetilde{Q}}}(\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}\widetilde{R}_{\widetilde{Q}}) &\cong H_Q^{\dim(R_Q)}(\widetilde{\omega}_{\widetilde{Q}}) \cong H^{\widetilde{h}+d}_{(\underline{x},\,\underline{y})}(\widetilde{\omega}_{\widetilde{Q}}) \cong H^d_{(\underline{y})}\big(H^{\widetilde{h}}_{(\underline{x})}(\widetilde{\omega}_{\widetilde{Q}})\big)\\ &\cong H^d_{(\underline{y})}\big(H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})_{\widetilde{Q}}\big) = H^d_{(\underline{y})}(\widetilde{E}_{\widetilde{Q}}) \cong H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}.\end{aligned}$$

(d) Take a presentation $G_1 \to G_0 \to M \to 0$ where G_0 and G_1 are finitely generated free $\widetilde{R}$-modules. By Corollary 4.7 applied over $\widetilde{R}$, we may localize at one element of $\widetilde{W}$ such that exactness is preserved at G_0 and M when we apply $H^d_{(\underline{y})}\big(\operatorname{Hom}(\_,\widetilde{E})\big)_{\widetilde{Q}}$ for all $\widetilde{Q}$ and $\underline{y}$ satisfying ($\sharp$).

Note that $H^d_{(\underline{y})}(\_)_{\widetilde{Q}}$ is isomorphic with $H^d_{(\underline{y})}(\widetilde{R})_{\widetilde{Q}} \otimes_{\widetilde{R}} \_$. With this identification, the natural map Φ_M is induced by the bilinear map

$$H^d_{(\underline{y})}(\widetilde{R})_{\widetilde{Q}} \times \operatorname{Hom}_{\widetilde{R}}(M,\widetilde{E}) \to \operatorname{Hom}_{\widetilde{R}}\big(M, H^d_{(\underline{y})}(\widetilde{R})_{\widetilde{Q}} \otimes_{\widetilde{R}} \widetilde{E}\big)$$

whose value on (h, f) is the map $u \mapsto h \otimes f(u)$. It is straightforward to check that Φ_R is an isomorphism, and that $\Phi_{M\oplus M'}$ may be identified with $\Phi_M \oplus \Phi_{M'}$, so that Φ_G is an isomorphism for any finitely generated free $\widetilde{R}$-module G. For the general case, we make use of the presentation $G_1 \to G_0 \to M \to 0$ of M over $\widetilde{R}$. For all Q and $\underline{y}$ satisfying ($\sharp$), we have the commutative diagram below in which the rows are exact:

$$\begin{array}{ccccccc} 0 \longrightarrow & H^d_{(\underline{y})}\big(\operatorname{Hom}_R(M,\widetilde{E})\big)_{\widetilde{Q}} & \longrightarrow & H^d_{(\underline{y})}\big(\operatorname{Hom}_{\widetilde{R}}(G_0,\widetilde{E})\big)_{\widetilde{Q}} & \longrightarrow & H^d_{(\underline{y})}\big(\operatorname{Hom}_{\widetilde{R}}(G_1,\widetilde{E})\big)_{\widetilde{Q}} \\ & \downarrow{\scriptstyle \Phi_M} & & \downarrow{\scriptstyle \Phi_{G_0}} & & \downarrow{\scriptstyle \Phi_{G_1}} \\ 0 \longrightarrow & \operatorname{Hom}_{\widetilde{R}}\big(M, H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}\big) & \longrightarrow & \operatorname{Hom}_{\widetilde{R}}\big(G_0, H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}\big) & \longrightarrow & \operatorname{Hom}_{\widetilde{R}}\big(G_1, H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}\big) \end{array}$$

The exactness of the top row has been obtained, while the exactness of the bottom row follows from part (c) right above. Since we have already shown that both Φ_{G_0} and Φ_{G_1} are isomorphisms, it follows from the five lemma that the map Φ_M is an isomorphism.

(e) Applying Corollary 4.7 to the exact sequence $\widetilde{R} \to \widetilde{R}/\widetilde{P} \to 0$ over $\widetilde{R}$, we see that, after localization at one element of $\widetilde{W}$ as needed, there is an injection $H^d_{(\underline{y})}(\operatorname{Ann}_{\widetilde{E}}\widetilde{P})_{\widetilde{Q}} \hookrightarrow H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}$ for all Q and $\underline{y}$ satisfying ($\sharp$), in which all the maps are natural. Hence, a socle generator for $H^d_{(\underline{y})}(\operatorname{Ann}_{\widetilde{E}}\widetilde{P})_{\widetilde{Q}} \cong H^d_{(\underline{y})}(A)_{\widetilde{Q}}$ over $A_{\widetilde{Q}}$ maps to a socle generator for $H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}} \cong \mathrm{E}_{\widetilde{R}_{\widetilde{Q}}}(\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}\widetilde{R}_{\widetilde{Q}})$ over $\widetilde{R}_{\widetilde{Q}}$. □

5.2. **Generic behavior of injective hulls for images of Cohen-Macaulay rings.** In this subsection, the focus is the quotient ring $R := \widetilde{R}/\mathfrak{a}$, with $P := \widetilde{P}/\mathfrak{a}$, where

$\mathfrak{a} \subseteq \widetilde{P}$. We shall show how to obtain injective hulls for R/Q over R for all prime ideals $Q := \widetilde{Q}/\mathfrak{a}$ in open neighborhood of P, where $\mathfrak{a} \subseteq \widetilde{Q}$.

Remark 5.5. Let $\widetilde{R}$ be a Noetherian ring and N an $\widetilde{R}$-module. Throughout the next result, we freely use the natural identification $\operatorname{Hom}_{\widetilde{R}}(M, N_{\widetilde{Q}}) \cong \operatorname{Hom}_{\widetilde{R}}(M, N)_{\widetilde{Q}}$ when M is finitely generated (and so finitely presented). Note that N need not be finitely generated. We likewise use the identification $\operatorname{Hom}_{\widetilde{R}}(\widetilde{R}/\mathfrak{a}, N) \cong \operatorname{Ann}_N \mathfrak{a}$, so that we have $\operatorname{Ann}_{N_{\widetilde{Q}}} \mathfrak{a} \cong (\operatorname{Ann}_N \mathfrak{a})_{\widetilde{Q}}$.

Theorem 5.6. *Let $\widetilde{R}$, $\widetilde{P}$, $\widetilde{h}$, $A := \widetilde{R}/\widetilde{P}$, $\widetilde{W} := \widetilde{R} \setminus \widetilde{P}$, $\widetilde{\omega}$, and $\widetilde{E}$ be as above. Assume that $\widetilde{R}$ is excellent or that conditions (E1) and (E2) above hold. Let $\mathfrak{a} \subseteq \widetilde{P}$ be an ideal of $\widetilde{R}$. Denote $R = \widetilde{R}/\mathfrak{a}$, $P = \widetilde{P}/\mathfrak{a}$ and $E = \operatorname{Ann}_{\widetilde{E}} \mathfrak{a}$. For any prime $\widetilde{Q} \supseteq \widetilde{P}$, denote $Q = \widetilde{Q}/\mathfrak{a} \subseteq R$. Then we can localize at one element $\widetilde{g} \in \widetilde{W}$ such that, after replacing $\widetilde{R}$, $\widetilde{P}$, $\widetilde{\omega}$, R and $\widetilde{E}$ by their localizations, the following statements hold:*

(a) *The ring A is regular, the ring $\widetilde{R}$ is Cohen-Macaulay, and the module $\widetilde{\omega}$ is a global canonical module for $\widetilde{R}$.*
(b) *$A \cong \operatorname{Ann}_{\widetilde{E}} \widetilde{P} = \operatorname{Ann}_E \widetilde{P} = \operatorname{Ann}_E P$, so that we have an injection $A \hookrightarrow E \subseteq \widetilde{E}$. Write $\operatorname{Ann}_E P = Au$ for some $u \in E$.*

In all of the remaining parts, assume that we have localized at an element of $\widetilde{W}$ so that conditions (a) *and* (b) *hold. Note that localization of an R-module at an element $\widetilde{g} \in \widetilde{R} \setminus \widetilde{P}$ is the same as localization of that R-module at $g \in R \setminus P$, where g is the natural image of $\widetilde{g}$. In particular, the localization of an R-module at a prime $\widetilde{Q} \supseteq \widetilde{P}$ is the same as its localization at $Q := \widetilde{Q}/\mathfrak{a}$. Moreover, in the remaining parts, we place the following condition (#) on $\widetilde{Q}$ and $\underline{y} = y_1, \ldots, y_d$:*

(#) *The ideal $\widetilde{Q} \in \operatorname{Spec}(\widetilde{R})$ is in $V(\widetilde{P})$ and the sequence $\underline{y}$ in $\widetilde{Q}$ maps to a regular system of parameters for $A_{\widetilde{Q}} \cong A_Q$.*

Then we have the following:

(c) *After localizing at one element of $\widetilde{W}$, for all $\widetilde{Q}$ and $\underline{y}$ satisfying (#), the inclusion $E = \operatorname{Ann}_{\widetilde{E}} \mathfrak{a} \hookrightarrow \widetilde{E}$ induces the following injective homomorphism*
$$H^d_{(\underline{y})}(E)_Q \cong H^d_{(\underline{y})}(E)_{\widetilde{Q}} = H^d_{(\underline{y})}(\operatorname{Ann}_{\widetilde{E}} \mathfrak{a})_{\widetilde{Q}} \cong \operatorname{Ann}_{H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}} \mathfrak{a} \hookrightarrow H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}.$$
Therefore, $H^d_{(\underline{y})}(E)_Q$ is an injective hull for R_Q/QR_Q over R_Q, i.e., $H^d_{(\underline{y})}(E)_Q \cong \mathrm{E}_{R_Q}\big((R/Q)_Q\big)$.
(d) *After localizing at one element of $\widetilde{W}$, for all $\widetilde{Q}$ and $\underline{y}$ satisfying (#), the natural map*
$$H^d_{(\underline{y})}(\operatorname{Ann}_E P)_Q \hookrightarrow H^d_{(\underline{y})}(E)_Q$$

is an injection, under which a socle generator for $H^d_{(\underline{y})}(\mathrm{Ann}_E P)_Q \cong H^d_{(\underline{y})}(A)_Q$ *over the regular local ring* A_Q *maps to a socle generator for* $H^d_{(\underline{y})}(E)_Q \cong \mathrm{Ann}_{H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}}\mathfrak{a} \cong \mathrm{E}_{R_Q}\big((R/Q)_Q\big)$ *over* R_Q*. In particular, with* $u \in E = \mathrm{Ann}_{\widetilde{E}}\mathfrak{a}$ *as in part* (b) *above, the element* $[u;\underline{y}] \in H^d_{(\underline{y})}(E)_Q$ *is a socle generator.*

Proof. Parts (a) and (b) hold for the same reason as in Theorem 5.4.

(c) The induced isomorphism $H^d_{(\underline{y})}(E)_{\widetilde{Q}} = H^d_{(\underline{y})}(\mathrm{Ann}_{\widetilde{E}}\mathfrak{a})_{\widetilde{Q}} \cong \mathrm{Ann}_{H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}}\mathfrak{a}$ follows from part (d) of Theorem 5.4 with $M := \widetilde{R}/\mathfrak{a}$ and Remark 5.5. As we have noted earlier, for $\widetilde{R}$-modules killed by $\mathfrak{a}$ it does not matter whether we think of them as $\widetilde{R}$-modules and localize at $\widetilde{Q}$ or think of them as R-modules and localize at Q. Hence, $H^d_{(\underline{y})}(\mathrm{Ann}_{\widetilde{E}}\mathfrak{a}))_Q \cong \mathrm{Ann}_{H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}}}\mathfrak{a}$. Moreover, we have $H^d_{(\underline{y})}(\widetilde{E})_{\widetilde{Q}} \cong \mathrm{E}_{\widetilde{R}}(\widetilde{R}/\widetilde{Q})$ by part (c) of Theorem 5.4. Therefore,

$$H^d_{(\underline{y})}(E)_Q = H^d_{(\underline{y})}(\mathrm{Ann}_{\widetilde{E}}\mathfrak{a}))_Q \cong \mathrm{Ann}_{\mathrm{E}_{\widetilde{R}}(\widetilde{R}/\widetilde{Q})}\mathfrak{a} \cong \mathrm{E}_R(R/Q).$$

(d) Applying Corollary 4.7 to the exact sequence $\widetilde{R}/\mathfrak{a} \to \widetilde{R}/\widetilde{P} \to 0$ over $\widetilde{R}$, we see that, after localization at one element of $\widetilde{W}$ as needed, there is a natural injection $H^d_{(\underline{y})}(\mathrm{Ann}_{\widetilde{E}}\widetilde{P})_{\widetilde{Q}} \hookrightarrow H^d_{(\underline{y})}(E)_{\widetilde{Q}}$ for all $\widetilde{Q}$ and $\underline{y}$ satisfying (#). In terms of $A \cong Au = \mathrm{Ann}_E P \subseteq E$ and in terms of modules over R_Q, we have a natural injection $H^d_{(\underline{y})}(A)_Q \cong H^d_{(\underline{y})}(Au)_Q = H^d_{(\underline{y})}(\mathrm{Ann}_E P)_Q \hookrightarrow H^d_{(\underline{y})}(E)_Q$. Hence, a socle generator for $H^d_{(\underline{y})}(Au)_Q \cong H^d_{(\underline{y})}(A)_Q$ over A_Q maps to a socle generator for $H^d_{(\underline{y})}(E)_Q \cong \mathrm{E}_{R_Q}\big((R/Q)_Q\big)$ over R_Q. Corresponding to the socle generator $[1;\underline{y}] \in H^d_{(\underline{y})}(A)_Q$, the element $[u;\underline{y}] \in H^d_{(\underline{y})}(Au)_Q$ is a socle generator for $H^d_{(\underline{y})}(Au)_Q$. Therefore, the element $[u;\underline{y}] \in H^d_{(\underline{y})}(E)_Q$ is a socle generator for $H^d_{(\underline{y})}(E)_Q \cong \mathrm{E}_R(R/Q)$ over R_Q. ☐

5.3. **Generic behavior of injective hulls for S_2 quotients of Cohen-Macaulay rings.** In this subsection, we keep all of the conventions of Notation 5.1. In addition, we assume that R_P is S_2, where $R = \widetilde{R}/\mathfrak{a}$ and $P = \widetilde{P}/\mathfrak{a}$. We show that after localizing at one element of $\widetilde{W}$, we can obtain injective hulls of residue class fields of R at primes $Q := \widetilde{Q}/\mathfrak{a} \in V(\widetilde{P}/\mathfrak{a})$ from the top nonvanishing local cohomology module of a finitely generated R-module. We need some preliminaries.

Discussion 5.7. Let $\widetilde{R}$ be a (not necessarily excellent or local) Cohen-Macaulay ring with global canonical module $\widetilde{\omega}$, let $\mathfrak{a}$ be an ideal of $\widetilde{R}$ such that all minimal primes of $\mathfrak{a}$ have the same height, say k. Assume that $R := \widetilde{R}/\mathfrak{a}$ is S_2. Let $\omega := \mathrm{Ext}^k_{\widetilde{R}}(R, \widetilde{\omega})$. When R is Cohen-Macaulay, this is a canonical module for R. However, when R is S_2 and when all minimal primes of $\mathfrak{a}$ have the same height k, this module has some of the same behavior as in the Cohen-Macaulay case and, in some literature, is still referred to as a canonical module for R. In particular, without assuming that R is Cohen-Macaulay, we have the following (some of what we prove below is in the literature in one form or another, cf. [Ao80, Prop. 2], [HK71, Satz 5.12] or [Ao83, Thm. 1.2] for example, but we have given a brief, self-contained treatment):

(1) For every finitely generated R-module M, we have $\mathrm{Ext}^k_{\widetilde{R}}(M, \widetilde{\omega}) \cong \mathrm{Hom}_R(M, \omega)$. Moreover, $\mathrm{Ext}^k_{\widetilde{R}}(\omega, \widetilde{\omega}) \cong \mathrm{Hom}_R(\omega, \omega) \cong R$, in which $R \cong \mathrm{Hom}_R(\omega, \omega)$ can be induced by the homothety map.
(2) $H^{\dim(R_Q)}_Q(\omega_Q) \cong \mathrm{E}_R(R/Q) \cong \mathrm{E}_{R_Q}\big((R/Q)_Q\big)$ for all prime Q of R.
(3) For all finitely generated R-module M and for all prime Q of R, we have $H^{\dim(R_Q)}_Q(M_Q) \cong \mathrm{Hom}_{R_Q}\Big(\mathrm{Hom}_R(M, \omega)_Q,\, \mathrm{E}_{R_Q}\big((R/Q)_Q\big)\Big)$. In particular, we have $H^{\dim(R_Q)}_Q(R_Q) \cong \mathrm{Hom}_{R_Q}\Big(\omega_Q,\, \mathrm{E}_{R_Q}\big((R/Q)_Q\big)\Big)$ for all prime Q of R.

To prove (1), choose a maximal regular sequence $\underline{z} = z_1, \ldots, z_k$ in $\mathfrak{a}$. As $\underline{z} \in \mathfrak{a}$ and $\underline{z}$ is regular on $\widetilde{\omega}$, we have $\omega = \mathrm{Ext}^k_{\widetilde{R}}(R, \widetilde{\omega}) \cong \mathrm{Hom}_{\widetilde{R}}\big(R, \widetilde{\omega}/(z_1, \ldots, z_k)\widetilde{\omega}\big)$. But then, for any finitely generated R-module M, we get

$$\begin{aligned}\mathrm{Ext}^k_{\widetilde{R}}(M, \widetilde{\omega}) \cong \mathrm{Hom}_{\widetilde{R}}\big(M, \widetilde{\omega}/(\underline{z})\widetilde{\omega}\big) &\cong \mathrm{Hom}_{\widetilde{R}}\big(M, \mathrm{Hom}_{\widetilde{R}}(R, \widetilde{\omega}/(\underline{z})\widetilde{\omega})\big) \\ &\cong \mathrm{Hom}_{\widetilde{R}}(M, \omega) = \mathrm{Hom}_R(M, \omega).\end{aligned}$$

In particular, $\mathrm{Ext}^k_{\widetilde{R}}(\omega, \widetilde{\omega}) \cong \mathrm{Hom}_R(\omega, \omega)$. To show that the map $\theta\colon R \to \mathrm{Hom}_R(\omega, \omega)$ induced by homothety is an isomorphism, we need to prove that the kernel and cokernel are 0. If we localize at any prime $\mathfrak{p} := \widetilde{\mathfrak{p}}/\mathfrak{a} \in \mathrm{Spec}(R)$ of height 1 or 0, we see that $R_{\mathfrak{p}}$ is Cohen-Macaulay and, hence, $\omega_{\mathfrak{p}} = \mathrm{Ext}^{\dim(\widetilde{R}_{\widetilde{\mathfrak{p}}}) - \dim(R_{\mathfrak{p}})}_{\widetilde{R}_{\widetilde{\mathfrak{p}}}}(R_{\mathfrak{p}}, \widetilde{\omega}_{\widetilde{\mathfrak{p}}})$ is a true canonical module of $R_{\mathfrak{p}}$, because R is equidimensional. This shows that θ becomes an isomorphism whenever we localize at such a prime $\mathfrak{p}$. Thus, the kernel and cokernel of the map θ are supported only at primes of height 2 or greater. Note that both R and $\mathrm{Hom}_R(\omega, \omega)$ are S_2. Now we can see that both $\mathrm{Ker}(\theta)$ and $\mathrm{Coker}(\theta)$ must be zero. If the kernel is not 0, then localize at a minimal prime of its support, which must be in $\mathrm{Ass}_R\big(\mathrm{Ker}(\theta)\big) \subseteq \mathrm{Ass}_R(R)$, to get a contradiction. Hence θ is an injective map. Suppose that the cokernel is not 0. After the localization at a minimal prime its support, the cokernal becomes depth 0 while both R and $\mathrm{Hom}_R(\omega, \omega)$ have depth at least 2, which is a contradiction.

Now that we have (1), we can prove (2) as follows. Let $Q = \widetilde{Q}/\mathfrak{a}$ be any prime ideal of $R = \widetilde{R}/\mathfrak{a}$, where $\mathfrak{a} \subseteq \widetilde{Q} \in \mathrm{Spec}(\widetilde{R})$. Then $k = \dim(\widetilde{R}_{\widetilde{Q}}) - \dim(R_Q)$. Therefore,

$$\begin{aligned}H^{\dim(R_Q)}_{QR_Q}(\omega_Q) &\cong \mathrm{Hom}_{\widetilde{R}_{\widetilde{Q}}}\big(\mathrm{Ext}^k_{\widetilde{R}}(\omega, \widetilde{\omega})_{\widetilde{Q}},\, \mathrm{E}_{\widetilde{R}_{\widetilde{Q}}}(\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}_{\widetilde{Q}})\big) && \text{(duality over } \widetilde{R}_{\widetilde{Q}}) \\ &\cong \mathrm{Hom}_{\widetilde{R}_{\widetilde{Q}}}\big((\widetilde{R}/\mathfrak{a})_{\widetilde{Q}},\, \mathrm{E}_{\widetilde{R}_{\widetilde{Q}}}(\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}_{\widetilde{Q}})\big) && \text{(part (1) here)}\end{aligned}$$

$$\cong \mathrm{E}_{R_Q}\left((R/Q)_Q\right) \cong \mathrm{E}_R(R/Q).$$

To see (3), let M be an finitely generated R-module, and let $Q = \widetilde{Q}/\mathfrak{a}$ be as above. Then, by local duality over $\widetilde{R}_{\widetilde{Q}}$, we get

$$\begin{aligned} H^{\dim(R_Q)}_{QR_Q}(M_Q) &\cong \mathrm{Hom}_{\widetilde{R}_{\widetilde{Q}}}\left(\mathrm{Ext}^k_{\widetilde{R}}(M,\widetilde{\omega})_{\widetilde{Q}},\, \mathrm{E}_{\widetilde{R}_{\widetilde{Q}}}(\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}_{\widetilde{Q}})\right) && \text{(local duality)} \\ &\cong \mathrm{Hom}_{\widetilde{R}_{\widetilde{Q}}}\left(\mathrm{Hom}_R(M,\omega)_{\widetilde{Q}},\, \mathrm{E}_{\widetilde{R}_{\widetilde{Q}}}(\widetilde{R}_{\widetilde{Q}}/\widetilde{Q}_{\widetilde{Q}})\right) && \text{(part (1))} \\ &\cong \mathrm{Hom}_{R_Q}\left(\mathrm{Hom}_R(M,\omega)_{\widetilde{Q}},\, \mathrm{E}_{R_Q}(R_Q/Q_Q)\right). \end{aligned}$$

In particular, we obtain $H^{\dim(R_Q)}_Q(R_Q) \cong \mathrm{Hom}_{R_Q}\Big(\omega_Q,\, \mathrm{E}_{R_Q}\left((R/Q)_Q\right)\Big)$.

Theorem 5.8. *Let $\widetilde{R}$ be a (not necessarily excellent or local) ring, let $\widetilde{P} \in \mathrm{Spec}(\widetilde{R})$ and let $\widetilde{\omega}$ be a finitely generated $\widetilde{R}$-module. Assume that $\widetilde{R}_{\widetilde{P}}$ is Cohen-Macaulay and that $\widetilde{\omega}_{\widetilde{P}}$ is a canonical module for $\widetilde{R}_{\widetilde{P}}$. Fix an ideal $\mathfrak{a} \subseteq \widetilde{P}$ and denote $R := \widetilde{R}/\mathfrak{a}$, $\widetilde{P}/\mathfrak{a} =: P \in \mathrm{Spec}(R)$. Assume that R_P is S_2. Let $\widetilde{h} := \dim(\widetilde{R}_{\widetilde{P}})$, $h := \dim(R_P)$, and $k := \widetilde{h} - h$. Set $\omega := \mathrm{Ext}^k_{\widetilde{R}}(R, \widetilde{\omega})$, $E := H^h_P(\omega)$ and $\widetilde{E} := H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})$. Also, let $\widetilde{W} \subseteq \widetilde{R}$ be a subset that maps onto $A \setminus \{0_A\}$ under the natural map $\widetilde{R} \to \widetilde{R}/\widetilde{P}$. Similarly, let $W \subseteq R$ be a subset that maps onto $A \setminus \{0_A\}$ under the natural map $R \to R/P$. Then the following holds:*

- (a) *After replacing $\widetilde{R}$ and R with $\widetilde{R}_{\widetilde{g}}$ and $R_{\widetilde{g}}$ for some $\widetilde{g} \in \widetilde{R} \setminus \widetilde{P}$, we have $\mathrm{Ext}^k_{\widetilde{R}}(\omega, \widetilde{\omega}) \cong \mathrm{Hom}_R(\omega,\omega) \cong R$, in which the isomorphism $R \cong \mathrm{Hom}_R(\omega,\omega)$ can be induced by the homothety map.*
- (b) *$\widetilde{W}$-generically, we have $H^h_P(\omega) \cong \mathrm{Hom}_{\widetilde{R}}\left(\widetilde{R}/\mathfrak{a},\, H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})\right) \cong \mathrm{Ann}_{H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})}\,\mathfrak{a}$, i.e., $E \cong \mathrm{Hom}_{\widetilde{R}}\left(\widetilde{R}/\mathfrak{a},\, \widetilde{E}\right)$. Hence, W-generically, we have $\mathrm{Ann}_{H^h_P(\omega)} P \cong A$.*
- (c) *If $0 \to M' \to M \to M'' \to 0$ is a sequence of finitely generated R-modules that becomes exact after localization at P, then the induced sequence*

  $$0 \to \mathrm{Hom}_R(M'',\, E) \to \mathrm{Hom}_R(M,\, E) \to \mathrm{Hom}_R(M',\, E) \to 0$$

  is W-generically exact. Hence, for every finitely generated R-module M and for every $i \geqslant 1$, the module $\mathrm{Ext}^i_R\left(M, H^h_P(\omega)\right)$ is W-generically 0. Therefore, the functor $\mathrm{Hom}_R(\_,\, E)$ is W-generically exact on any given finite set of short exact sequences of finitely generated R-modules, and, hence, on any given finite set of finite long exact sequences of finitely generated R-modules. The choice of g depends on which finite set of finite exact sequences one chooses.
- (d) *Let $N := \mathrm{Hom}_R(M,E)$ or $N := H^i_P(M)$, where M is a finitely generated R-module. Then after localizing at one element of W* that is independent of *$n \in \mathbb{N}$, the filtration of N by the modules $\mathrm{Ann}_N P^n$, which ascends as n increases with $\bigcup_{n\in\mathbb{N}} \mathrm{Ann}_N P^n = N$, has factors that are finitely generated free modules over A.*

(e) *For every finitely generated R-module M, W-generically, we have $H^h_P(M) \cong \operatorname{Hom}_R\big(\operatorname{Hom}_R(M,\omega),\, H^h_P(\omega)\big)$. In particular, W-generically, we get $H^h_P(R) \cong \operatorname{Hom}_R\big(\omega,\, H^h_P(\omega)\big)$.*

Proof. The proof is an application of Theorem 2.19 to $\widetilde{R}$, together with Discussion 5.7. As in the proof of Theorem 2.19, it suffices to assume $\widetilde{W} = \widetilde{R}\setminus\widetilde{P}$ and $W = R\setminus P$. We may use $\widetilde{g}$ to denote an element of $\widetilde{W}$ and use $g \in W$ to denote the natural image of $\widetilde{g}$ under the natural map $\widetilde{R} \to R$. In the course of the proof we may repeatedly, but finitely many times, localize at one element $\widetilde{g} \in \widetilde{W}$. Each time, we make a change of terminology, and continue to use $\widetilde{R}$ and R to denote the resulting ring $\widetilde{R}_{\widetilde{g}}$ and $R_{\widetilde{g}}$. Likewise, we use $\widetilde{P}$ and P for their extensions to $\widetilde{R}_{\widetilde{g}}$ and $R_{\widetilde{g}} = R_g$, and for every module under consideration we use the same letter for the module after base change from $\widetilde{R}$ or R to $\widetilde{R}_{\widetilde{g}}$ or R_g (finitely generated modules under consideration are replaced by their localizations: this is the same as base change from $\widetilde{R}$ or R to $\widetilde{R}_{\widetilde{g}}$ or R_g). For an R-module M, we naturally identify $M_{\widetilde{g}}$ with M_g.

(a) Naturally, we may write $R_P = \widetilde{R}_{\widetilde{P}}/\mathfrak{a}_{\widetilde{P}}$ and $\omega_P = \operatorname{Ext}^k_{\widetilde{R}_{\widetilde{P}}}(R_P,\, \widetilde{\omega}_{\widetilde{P}})$. Since $\widetilde{R}_{\widetilde{P}}$ is Cohen-Macaulay with $\widetilde{\omega}_{\widetilde{P}}$ being a canonical module, and since R_P is S_2, we apply Discussion 5.7(1) to obtain $\operatorname{Ext}^k_{\widetilde{R}_{\widetilde{P}}}(\omega_{\widetilde{P}}, \widetilde{\omega}_{\widetilde{P}}) \cong \operatorname{Hom}_{R_P}(\omega_P, \omega_P) \cong R_P$, in which $R_P \cong \operatorname{Hom}_{R_P}(\omega_P, \omega_P)$ can be induced by the homothety map. Thus, after replacing $\widetilde{R}$ with its localization at one element of $\widetilde{W}$ and, accordingly, replacing R with its localization at the corresponding element of W, we obtain $\operatorname{Ext}^k_{\widetilde{R}}(\omega, \widetilde{\omega}) \cong \operatorname{Hom}_R(\omega,\omega) \cong R$, in which $R \cong \operatorname{Hom}_R(\omega,\omega)$ can be induced by the homothety map. So, $\widetilde{W}$-generically, $\operatorname{Ext}^k_{\widetilde{R}}(\omega, \widetilde{\omega}) \cong \widetilde{R}/\mathfrak{a}$.

(b) By Theorem 2.19(e) applied to $\widetilde{R}$, we see that, $\widetilde{W}$-generically,

$$\begin{aligned} H^h_P(\omega) &\cong \operatorname{Hom}_{\widetilde{R}}\big(\operatorname{Ext}^{\widetilde{h}-h}_{\widetilde{R}}(\omega,\, \widetilde{\omega}),\, H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})\big) && \text{(by 2.19(e))}\\ &\cong \operatorname{Hom}_{\widetilde{R}}\big(\widetilde{R}/\mathfrak{a},\, H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})\big) \cong \operatorname{Ann}_{H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})} \mathfrak{a} && \text{(by part (a)).} \end{aligned}$$

Hence, $\widetilde{W}$-generically, we have $\operatorname{Ann}_{H^h_P(\omega)} \widetilde{P} = \operatorname{Ann}_{H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})} \widetilde{P} \cong A$ according to part (b) of Theorem 2.19. Over R, we see that, W-generically, $\operatorname{Ann}_{H^h_P(\omega)} P \cong A$.

(c) Let $0 \to M' \to M \to M'' \to 0$ be a sequence of finitely generated R-modules that becomes exact after localization at P. Considered over $\widetilde{R}$, this is a sequence of finitely generated $\widetilde{R}$-modules that becomes exact after localization at $\widetilde{P}$. By part (a) of Theorem 2.19, the induced sequence

$$0 \to \operatorname{Hom}_{\widetilde{R}}(M'',\, \widetilde{E}) \to \operatorname{Hom}_{\widetilde{R}}(M,\, \widetilde{E}) \to \operatorname{Hom}_{\widetilde{R}}(M',\, \widetilde{E}) \to 0$$

is $\widetilde{W}$-generically exact, in which $\widetilde{E} := H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})$. Moreover, part (b) says that $E = \operatorname{Hom}_{\widetilde{R}}\big(\widetilde{R}/\mathfrak{a},\, \widetilde{E}\big)$. Thus, the induced sequence

$$0 \to \operatorname{Hom}_R(M'',\, E) \to \operatorname{Hom}_R(M,\, E) \to \operatorname{Hom}_R(M',\, E) \to 0$$

is W-generically exact.

Next, for any finitely generated R-module M, there exists a short exact sequence $0 \to L \to R^n \to M \to 0$ of R-modules for some $n \in \mathbb{N}$, which forces L to be finitely generated over R as well. Then, as proved above, the induced sequence

$$0 \to \operatorname{Hom}_R(M,\, E) \to \operatorname{Hom}_R(R^n,\, E) \to \operatorname{Hom}_R(L,\, E) \to 0$$

is W-generically exact, which forces $\operatorname{Ext}^1_R(M,\, E)$ to vanish W-generically. As for $i > 1$, we observe that $\operatorname{Ext}^i_R\big(M, H^h_P(\omega)\big) \cong \operatorname{Ext}^1_R\big(\Omega^{i-1}_R(M),\, H^h_P(\omega)\big)$ is W-generically 0 because $\Omega^{i-1}_R(M)$, an $(i-1)^{\text{st}}$ syzygy of M over R, finitely generated over R. Therefore, the functor $\operatorname{Hom}_R(\_,\, E)$ is W-generically exact on any given finite set of short exact sequences of finitely generated R-modules, and, hence, on any given finite set of finite long exact sequences of finitely generated R-modules. The choice of $g \in W$, at which we localize, depends on which finite set of finite exact sequences one chooses.

(d) Let M be a finitely generated R-module. In light of part (b) above, we may identify $\operatorname{Hom}_R(M, E)$ (respectively, $H^i_P(M)$) with $\operatorname{Hom}_{\widetilde{R}}(M, H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega}))$ (respectively, $H^i_{\widetilde{P}}(M)$). Now the claim follows from parts (d) and (f) of Theorem 2.19.

(e) As $\widetilde{R}_{\widetilde{P}}$ is Cohen-Macaulay and R_P is S_2, we apply Discussion 5.7(1), to $\widetilde{R}_{\widetilde{P}}$ and R_P, and obtain $\operatorname{Ext}^k_{\widetilde{R}_{\widetilde{P}}}(M_{\widetilde{P}}, \widetilde{\omega}_{\widetilde{P}}) \cong \operatorname{Hom}_{R_P}(M_P, \omega_P)$ over $\widetilde{R}_{\widetilde{P}}$. As all the modules are finitely generated, we see that $\operatorname{Ext}^k_{\widetilde{R}}(M, \widetilde{\omega}) \cong \operatorname{Hom}_R(M, \omega)$ $\widetilde{W}$-generically. By Theorem 2.19(e) applied to $\widetilde{R}$, we see that, $\widetilde{W}$-generically,

$$\begin{aligned} H^h_P(M) &\cong \operatorname{Hom}_{\widetilde{R}}\big(\operatorname{Ext}^k_{\widetilde{R}}(M, \widetilde{\omega}),\, H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})\big) && \text{(by 2.19(e))} \\ &\cong \operatorname{Hom}_{\widetilde{R}}\big(\operatorname{Hom}_R(M, \omega),\, H^{\widetilde{h}}_{\widetilde{P}}(\widetilde{\omega})\big) && \text{(by above)} \\ &\cong \operatorname{Hom}_R\big(\operatorname{Hom}_R(M, \omega),\, H^h_P(\omega)\big) && \text{(by part (b)).} \end{aligned}$$

Over R, this says that $H^h_P(M) \cong \operatorname{Hom}_R\big(\operatorname{Hom}_R(M, \omega),\, H^h_P(\omega)\big)$, W-generically. Now the claim for $H^h_P(R)$ is clear, and the proof is complete. □

Theorem 5.9. *Suppose that $\widetilde{R}$ is excellent or that conditions (E1), (E2) and (E3) all hold. Let $\widetilde{P}$ be prime in $\widetilde{R}$ with $\mathfrak{a} \subseteq \widetilde{P}$. Assume also that $\widetilde{R}_{\widetilde{P}}$ is Cohen-Macaulay, and that $\widetilde{R}$ has a finitely generated module $\widetilde{\omega}$ such that $\widetilde{\omega}_{\widetilde{P}}$ is a canonical module for $\widetilde{R}_{\widetilde{P}}$. Let $R := \widetilde{R}/\mathfrak{a}$, let $P := \widetilde{P}/\mathfrak{a}$, and assume as well that R_P is S_2. Let $\widetilde{P}$ have height $\widetilde{h}$ in $\widetilde{R}$ and let P have height h in R, which imply that $\mathfrak{a}_{\widetilde{P}}$ has height $k := \widetilde{h} - h$*

in $\widetilde{R}_{\widetilde{P}}$. *Let* $\omega := \mathrm{Ext}^k_{\widetilde{R}}(R, \widetilde{\omega})$. *Then we can localize at one element of* $\widetilde{R} \setminus \widetilde{P}$ *in such a way that all of the following statements hold:*

(a) *The ring* A *is regular, the ring* $\widetilde{R}$ *is Cohen-Macaulay, and the module* $\widetilde{\omega}$ *is a global canonical module for* $\widetilde{R}$.
(b) R *is* S_2, *locally equidimensional, and all minimal primes of* $\mathfrak{a}$ *are contained in* $\widetilde{P}$ *and have height* k.
(c) $H^h_P(\omega) \cong \mathrm{Hom}_{\widetilde{R}}\big(\widetilde{R}/\mathfrak{a}, H^{\tilde{h}}_{\widetilde{P}}(\widetilde{\omega})\big)$ *and, hence* $\mathrm{Ann}_{H^h_P(\omega)} P \cong \mathrm{Hom}_{\widetilde{R}}\big(\widetilde{R}/\widetilde{P}, H^{\tilde{h}}_{\widetilde{P}}(\widetilde{\omega})\big)$. *Moreover,* $\mathrm{Ann}_{H^h_P(\omega)} P \cong A$. *Write* $\mathrm{Ann}_{H^h_P(\omega)} P = Au$, *for some* $u \in H^h_P(\omega)$, *so that the image of* u *in* $H^h_P(\omega)_P$ *is a socle generator for* $H^h_P(\omega)_P \cong \mathrm{E}_{R_P}(\kappa_P)$ *over* R_P.
(d) *For all primes* $\widetilde{Q} \in V(\widetilde{P})$, *and for all* $\underline{y} := y_1, \ldots, y_d \in \widetilde{Q}$ *that maps to a system of parameters for* $A_{\widetilde{Q}}$, *we have* $H^d_{(\underline{y})}\big(H^h_P(\omega)\big)_Q \cong \mathrm{E}_{R_Q}(\kappa_Q) \cong H^{d+h}_Q(\omega)_Q$ *over* R_Q.
(e) *Further assume that the sequence* $\underline{y} := y_1, \ldots, y_d \in \widetilde{Q}$ *maps to a regular system of parameters for* $A_{\widetilde{Q}}$. *Then, with* u *as in part* (c) *above, the element* $[u; \underline{y}] \in H^d_{(\underline{y})}\big(H^h_P(\omega)\big)_Q$ *(see Notation 4.4) is a socle generator for the module* $H^d_{(\underline{y})}\big(H^h_P(\omega)\big)_Q \cong \mathrm{E}_{R_Q}(\kappa_Q)$ *over* R_Q.

Proof. In light of Proposition 5.3 and Remark 5.2, we may localize at one element of $\widetilde{W}$ and assume that $A = \widetilde{R}/\widetilde{P}$ is regular, that $\widetilde{R}$ is Cohen-Macaulay, that $\widetilde{\omega}$ is a global canonical module for $\widetilde{R}$, and that $R := \widetilde{R}/\mathfrak{a}$ is S_2 and, hence, locally equidimensional. Moreover, we can localize at one element of $\widetilde{W}$ so that all minimal primes of $\mathfrak{a}$ in $\widetilde{R}$ are contained in $\widetilde{P}$, and so we may assume that all minimal primes of $\mathfrak{a}$ have the same height, which is necessarily k, which will be the height of $\mathfrak{a}$ in $\widetilde{R}$. This verifies parts (a) and (b).

The isomorphisms in part (c) concerning $H^h_P(\omega)$ and $\mathrm{Ann}_{H^h_P(\omega)} P$ are verified in part (b) of Theorem 5.8. Moreover, the isomorphism $H^h_P(\omega)_P \cong \mathrm{E}_{R_P}(\kappa_P)$ is shown in Discussion 5.7(2). The rest of part (c), concerning u, is clear.

In part (d), note that part (c) says that $H^h_P(\omega) \cong \mathrm{Ann}_{\widetilde{E}}\, \mathfrak{a}$, where $\widetilde{E} := H^{\tilde{h}}_{\widetilde{P}}(\widetilde{\omega})$. Hence, the isomorphism $H^d_{(\underline{y})}\big(H^h_P(\omega)\big)_Q \cong \mathrm{E}_{R_Q}(\kappa_Q)$ follows immediately from part (c) of Theorem 5.6. On the other hand, the isomorphism $H^{\dim(R_Q)}_Q(\omega_Q) \cong \mathrm{E}_{R_Q}(\kappa_Q)$ has been verified in Discussion 5.7(2), with $\dim(R_Q) = d + h$.

Finally, part (e) follows from part (d) of Theorem 5.6, in light of the isomorphism $H^h_P(\omega) \cong \mathrm{Ann}_{\widetilde{E}}\, \mathfrak{a}$ as noted above. □

6. Canonical modules via étale extensions

There are no restrictions on characteristic in this section. We need the following result, whose history in the literature we describe below.

Theorem 6.1. *Let $(R, \mathfrak{m}, K)$ be an excellent Cohen-Macaulay local ring. Then the Henselization R^{h} of R has a canonical module.*

Discussion 6.2. Hinich proved [Hin93] that an approximation ring (also called a ring with approximation property) has a dualizing complex. In the Cohen-Macaulay case, the existence of a dualizing complex is equivalent to the existence of a canonical module. Rotthaus gave a more elementary proof for the Cohen-Macaulay local case in [Rott96]. She phrases the result in terms of rings with the complete approximation property, but since the treatment in [Swan98], following the ideas of Popescu, [Po85, Po86], resolved any doubts about the general case of Néron-Popescu desingularization, we know that every excellent Henselian ring is an approximation ring (in fact, a local ring is an approximation ring if and only if it is excellent and Henselian). It is explained in [Rott95] how this implies that every excellent Henselian ring has the complete approximation property as well. We also note that the Henselization of an excellent local ring is excellent [EGAIV67, Cor. 18.7.6].

We use Theorem 6.1 to prove the following result. First recall that a local map of local rings is called a *pointed* étale extension if it is a localization at a prime of a (finitely presented) étale extension and the induced map of residue class fields is an isomorphsim. In the result below, z is an indeterminate over R and if $f \in R[z]$, f' denotes the derivative of f with respect to z.

Theorem 6.3. *Let R be an excellent Noetherian ring, with no restriction on the characteristic of R, and let P be a prime ideal of R such that R_P is Cohen-Macaulay. Let z be an indeterminate over R. Then after replacing R by its localization at one element in $R \setminus P$, there exists a flat extension $\breve{R}$ of the form $\big(R[z]/fR[z]\big)_g$, where f is a monic polynomial in $R[z]$, $g \in R[z]$, and there exists an element $r \in R$ such that $f(r) \in P$ with the properties listed below. In describing these properties, for every prime Q of R such that $Q \supseteq P$, we use symbol $\breve{Q}$ to denote the prime ideal $\Big(\big(QR[z] + (z-r)R[z]\big)/fR[z]\Big)_g$ in $\big(R[z]/fR[z]\big)_g$.*

1. *R is Cohen-Macaulay.*
2. *$f'(r)$ is a unit of R.*
3. *$g(r)$ is a unit of R. Hence, $g \notin \breve{Q}$ for any choice of $Q \in V(P)$.*
4. *$\breve{Q}$ is a prime ideal of $\breve{R}$ lying over Q in R.*
5. *$\breve{R}$ is Cohen-Macaulay and has a global canonical module ω.*
6. *For all $Q \in V(P)$, $R_Q \to \breve{R}_{\breve{Q}}$ is a pointed étale extension.*

(7) *$\breve{R}/\breve{P} \cong R/P$, and so under the natural map* $\mathrm{Spec}(\breve{R}) \to \mathrm{Spec}(R)$ *induced by contraction, $V(\breve{P})$ maps homeomorphically to $V(P)$.*

Moreover, if we replace g by any multiple g_1 in $R_P[z]$ such that $g_1(r) \notin PR_P$ (or equivalently, $g_1 \in R_P[z] \setminus \big(PR_P[z] + (z-r)R_P[z]\big)$), then after replacing R by its localization at one element of $R \setminus P$, g_1 has coefficients in R, $g_1(r) \in R$ is a unit, and all of the above properties (1) – (7) *hold for the extension $R \to \big(R[z]/fR[z]\big)_{g_1}$.*

Proof. By Theorem 6.1, we may choose a canonical module ω_1 for the Henselization $(R_P)^{\mathrm{h}}$ of R_P, and represent it as the cokernel of a matrix $\mathcal{M}_1$. The ring $(R_P)^{\mathrm{h}}$ is a direct limit of pointed étale extensions of R_P. Hence, $\mathcal{M}_1$ will descend to a matrix $\mathcal{M}_2$ over a suitable pointed étale extension of R_P, and the cokernel of $\mathcal{M}_2$, call it ω_2, will have the property that $(R_P)^{\mathrm{h}} \otimes \omega_2 \cong \omega_1$. It follows that ω_2 is a canonical module for this pointed étale extension of R_P.

By the structure theorem for pointed étale extensions of a local ring, the local étale extension has the the from $\big(R_P[z]/(f)\big)_{\mathfrak{P}}$, where $f = f(z)$ is a monic polynomial in $\mathfrak{P} \subseteq R_P[z]$, and $\mathfrak{P}$ is a prime ideal of $R_P[z]$ of the form $(P + (z-r))R_P[z]$, where $r \in R_P$ represents a simple root ρ of the image of$f$ mod P in κ_P. The condition that ρ be a simple root is equivalent to assuming t hat $f'(r) \notin PR_P$.

Hence, we may assume that ω_2 is a module over an extension ring R_P of the form $\big(R_P[z]/fR_P[z]\big)_{\mathfrak{P}}$. However, instead of localizing at all elements not in $\mathfrak{P}$, we may descend to an algebra of the form $\big(R[z]/fR[z]\big)_g \cong R[z]_g/fR[z]_g$: for a suitable choice of $g \in R[z] \setminus \big(PR[z] + (z-r)R[z]\big)$, ω_2 will descend to a module ω over $\big(R[z]/fR[z]\big)_g$ such that $\omega_{\mathfrak{P}}$ is a canonical module for $\Big(\big(R[z]/fR[z]\big)_g\Big)_{\breve{P}} = \big(R_P[z]/fR_P[z]\big)_{\mathfrak{P}}$. The condition that $g \notin PR[z] + (z-r)R[z]$ is equivalent to the assumption that $g(r) \notin P$.

By replacing R with its localization at an element of $R \setminus P$, we may additionally assume that $f(z)$ is monic over R , that $g(z) \in R[z]$, that $r \in R$, that $f'(r)$ is a unit of R (since it is not in P) and that $g(r)$ is a unit of R. We are now in the situation described in the statement of the theorem, and we may make a preliminary choice of $\breve{R}$ to be $\big(R[z]/fR[z]\big)_g$. Note that the natural image of $\mathfrak{P}$, mod f, is the localization of $\breve{P}$ at itself. However, ω will not yet necessarily be a global canonical module over $\breve{R}$. However, by Proposition 5.3, we can localize further at multiple of g not in $PR[z] + (z-r)R[z]$ such that this is true, and we may localize further at one element of R not in P so that for the new choice of g we have that $g(r)$ is a unit in R. It is now straightforward to verify that the conditions (1) – (7) all hold, and that, after replacing R by its localization at one element not in P, they continue to hold if we localize further at polynomial in $R_P[z]$ that is not in $PR_P[z] + (z-r)R_P[z]$: we may localize at one element of R not in P so that the new polynomial g_1 is in $R[z]$ and so that $g_1(r)$ is a unit. □

7. The purity exponent

7.1. **Purity.** In this subsection, there are no assumptions on the characteristic of the ring and there are no finiteness assumptions on modules, unless such assumptions are explicitly stated. A map of R-modules $N \to M$ is called *pure* if for every R-module V, the induced map $V \otimes N \to V \otimes M$ is injective. In particular, $N \to M$ is injective. This is a weakening of the condition that N be a direct summand of M over R. In fact, if $0 \to N \to M \to C \to 0$ is exact and C is finitely presented, then $N \to M$ is pure if and only if it the map is split as a map of R-modules; that is, M is the internal direct sum of the image of N and a submodule N' so that the composite map $N \to M \twoheadrightarrow M/N'$ is an isomorphism. A direct limit of pure maps is pure. Purity is preserved by arbitrary base change and, in particular by localization at any multiplicative system of R. Moreover, $N \to M$ is pure if and only if for all maximal (respectively, all prime) ideals P of R, $N_P \to M_P$ is pure over R_P. For further information about purity, we refer the reader to [HoR74, §6], [HoR76, §5.(a)], and [HH95, Lemma 2.1].

7.2. **Purity exponents: definition and basic facts.** Throughout the remainder of this section, R is a Noetherian ring of prime characteristic $p > 0$, and c is an element in R. In the sequel, F^e_R or simply F^e will denote the e th iterate of the Frobenius endomoprhism on R, and ${}^eR$ will denote R viewed as an R-algebra via the structural homomorphism $R \xrightarrow{F^e} R$. An alternative notation for ${}^eR$ is $F^e_*(R)$. We write ${}^ec$ for the element corresponding to c in ${}^eR$. If $W \subseteq R$, we shall also write ${}^eW$ for the set $\{{}^ew : w \in W\}$. In particular, if P is a prime ideal of R, ${}^eP$ is the corresponding prime ideal of ${}^eR$, and ${}^eP$ is the radical of the extended ideal $(P){}^eR = {}^e\big(P^{[p^e]}\big)$.

We use the notation $\theta^R_{e,c}$ (or $\theta_{e,c}$ if the ring R is understood from context) for the R-linear map $R \to {}^eR$ such that $1 \mapsto {}^ec$.[7] The purity exponent $\mathfrak{e}_c$ of c is defined to be the least positive integer e such that the map $\theta^R_{e,c} : R \to {}^eR$ is pure over R, if such an exponent exists. If no such exponent exists, then $\mathfrak{e}_c$ is defined to be ∞. Note that R is F-pure if and only if $\mathfrak{e}_1 = 1$, and also if and only if $\theta_{e,1}$ is pure for some (equivalently, all) $e \geqslant 1$. If c' is a divisor of c we have that $\mathfrak{e}_{c'} \leqslant \mathfrak{e}_c$, since if $c = c'c''$, $\theta_{e,c}$ is the composition $({}^ec'' \cdot \_) \circ \theta_{e,c'}$. Hence, the finiteness of $\mathfrak{e}_c$ for any $c \neq 0$ implies that c is not a zerodivisor and that the ring is F-pure, and therefore reduced. Since these conditions are forced by the finiteness of $\mathfrak{e}_c$, we do not assume them in general, which may be useful when we consider what happens as we localize at R a varying prime ideal.

Note also that once $\theta_{e,c}$ is pure, this also holds for $\theta_{e',c}$ for all $e' \geqslant e$. It suffices to see this for $e' = e + 1$. But the map $\theta_{e,c}$ induces a ${}^1R$-pure map (consequently, an

[7]When R is reduced, we may identify ${}^ec \in {}^eR$ with $c^{1/p^e} \in R^{1/p^e}$; hence, we may write $\theta_{e,c}$ as $\theta_{e,c} : R \to R^{1/p^e}$ such that $1 \mapsto c^{1/p^e}$.

R-pure map) ${}^1\theta_{e,c} : {}^1R \to {}^{e+1}R$ such that ${}^1 1_{{}^1R} \mapsto {}^{e+1}c$, and we may compose with the R-pure map $\theta_{1,1}$ to obtain that $\theta_{e+1,c} = {}^1\theta_{e,c} \circ \theta_{1,1}$ is pure. This gives us the following:

Remark 7.1. For every $n \geqslant 0$, let $\mathscr{A}_n(R)$, usually shortened to $\mathscr{A}_n$, denote the set $\{r \in R : \theta_{n,r} \text{ is not pure}\}$. Then $\mathscr{A}_{n+1} \subseteq \mathscr{A}_n$ for all n. The purity exponent for c is finite if and only if $c \notin \bigcap_{n=1}^{\infty} \mathscr{A}_n$, in which case the purity exponent is the least integer e such that $c \notin \mathscr{A}_e$. We shall show shortly in Theorem 7.7 that when $(R, \mathfrak{m}, \kappa)$ is local, $\mathscr{A}_n$ is an ideal.

Definition 7.2. Let $c \in R$. For $\mathfrak{p} \in \operatorname{Spec}(R)$, we use $\mathfrak{e}_c(\mathfrak{p})$ to denote $\mathfrak{e}_{c/1}$ in the ring $R_{\mathfrak{p}}$. This defines a function $\mathfrak{e}_c : \operatorname{Spec}(R) \to \mathbb{N}_+ \cup \{\infty\}$ where $\mathfrak{p} \mapsto \mathfrak{e}_c(\mathfrak{p})$.

Hence, we have the following:

Proposition 7.3. *Let R be a Noetherian ring of prime characteristic p, and $c \in R$.*

(a) $\mathfrak{e}_c(P) \leqslant \mathfrak{e}_c(Q) \leqslant \mathfrak{e}_c$ *for all prime ideals $P \subseteq Q$ in R.*
(b) $\mathfrak{e}_c = \sup_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{e}_c(\mathfrak{p}) = \sup_{\mathfrak{p} \in \operatorname{Max}(R)} \mathfrak{e}_c(\mathfrak{p})$.

When $\mathfrak{e}_c$ is finite, it gives a "tight closure style" test exponent for membership in ideals:

Proposition 7.4. *Let R be a Noetherian ring of prime characteristic $p > 0$ and let $c \in R$ be such that $\mathfrak{e}_c < \infty$. Let $f \in R$ and let $I \subseteq R$ be an ideal. Then $f \in I$ if and only if $cf^{p^{\mathfrak{e}_c}} \in I^{[p^{\mathfrak{e}_c}]}$.*

Proof. Since $\theta_{\mathfrak{e}_c,c} : R \to {}^{\mathfrak{e}_c}R$, determined by $1 \mapsto {}^{\mathfrak{e}_c}c$, is pure, the map remains injective if we apply $R/I \otimes_R \_$ to obtain a map $\theta' : R/I \otimes_R R \to R/I \otimes_R {}^{\mathfrak{e}_c}R$. Now $cf^{p^{\mathfrak{e}_c}} \in I^{[p^{\mathfrak{e}_c}]}$ if and only if the class of f in R/I is contained in $\ker(\theta') = 0$ if and only if $f \in I$. ☐

Remark 7.5. The statement in Proposition 7.4 is particularly useful when R is very strongly F-regular (see subsection 7.4, because in that case $\mathfrak{e}(c) < \infty$ for all $c \in R^\circ$, where $R^\circ = R \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p}$ is the complement of the union of all minimal primes of R. Moreover, one of our main results, Theorem 7.11, shows that every excellent strongly F-regular is very strongly F-regular.

Notation 7.6. Let M be an R-module. We write $\operatorname{E}_R(M)$ for the injective hull over R of the R-module M, which is unique up to non-unique isomorphism. When $(R, \mathfrak{m}, \kappa)$ is local, we may use the notation E_R or even E to denote $\operatorname{E}_R(\kappa)$. For $e \in \mathbb{N}$, we write $q := p^e$ and $F_R^e(M) := {}^eR \otimes_R M$ and view $F_R^e(M)$ as an R-module via $r \cdot \sum_i {}^e r_i \otimes m_i = \sum_i {}^e(rr_i) \otimes m_i$ for all $r \in R$ and $\sum_i {}^e r_i \otimes m_i \in {}^eR \otimes_R M = F_R^e(M)$. Also, for $x \in M$, denote $x_M^q := 1 \otimes x \in {}^eR \otimes_R M = F_R^e(M)$. When the module M is clear in the context, we may write x_M^q as x^q.

Theorem 7.7. *Let notation be as in 7.6, with $(R, \mathfrak{m}, \kappa)$ local, and let v be a socle generator of $E = \operatorname{E}_R(\kappa)$. Then for all e, $\mathscr{A}_e$ is the annihilator in R of $v^q \in F_R^e(E)$. Or, equivalently, $\mathfrak{e}_c \leqslant e$ if and only if $\theta_{e,c}$ is pure if and only if $0 \neq cv^q \in F_R^e(E)$.*

Proof. By [HH95, Lemma 2.1(e)], the map $\theta_{e,c} : R \to {}^eR$, with $1 \mapsto {}^ec$, is pure if and only the induced map $\theta_{e,c} \otimes_R \mathrm{I}_E$ is injective, which holds if and only if the element $1 \otimes v$ in $R \otimes_R E$ does not map to 0, i.e., $0 \neq {}^ec \otimes v \in {}^eR \otimes_R E$. In terms of Notation 7.6, we see that $\theta_{e,c}$ is pure if and only if $0 \neq cv^q \in F^e_R(E)$. The rest of the claims, concerning $\mathscr{A}_e$ or $\mathfrak{e}_c$, are clear now. □

7.3. **Flat regular extensions.** We have the following:

Theorem 7.8. *Let $h : R \to S$ be a flat ring homomorphism and let $c \in R$.*

(a) *If $h : R \to S$ is faithfully flat (e.g., $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is local), then $\mathfrak{e}_c \leqslant \mathfrak{e}_{h(c)}$.*
(b) *If $(S/h^{-1}(Q)S)_Q$ is regular for all $Q \in \mathrm{Max}(S)$ (e.g., $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is local with regular closed fiber), then $\mathfrak{e}_{h(c)} \leqslant \mathfrak{e}_c$.*

Proof. In (a), if $S \to {}^eS$ with $1 \mapsto {}^eh(c)$ is pure over S, and so over R. Since $R \to S$ is faithfully flat, we have that $h : R \to S$ is pure over R, and so $R \to {}^eS$ with $1 \mapsto {}^ec$ is pure. Since this map factors $R \to {}^eR \to {}^eS$, where the first map is $\theta_{e,c}$, it follows that $\theta_{e,c}$ is pure over R.

(b) For any $\mathfrak{n} \in \mathrm{Max}(S)$, let $\mathfrak{m} = h^{-1}(\mathfrak{n})$ be its contraction to R. If $\mathfrak{e}_{h(c)}(\mathfrak{n}) \leqslant \mathfrak{e}_c(\mathfrak{m})$ for all $\mathfrak{n} \in \mathrm{Max}(S)$, then $\mathfrak{e}_{h(c)} = \sup_{\mathfrak{n} \in \mathrm{Max}(S)} \mathfrak{e}_{h(c)}(\mathfrak{n}) \leqslant \mathfrak{e}_c$. Thus, this reduces to the local case $h : (R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ with regular closed fiber.

In this case, let $\underline{y} = y_1, \ldots, y_d$ denote elements of S whose images in $S/\mathfrak{m}S$ are a regular system of parameters. By [HH94a, Lemma 7.10] this implies the following:

(1) The elements $\underline{y}$ form a regular sequence on S.
(2) With $\underline{y}^t := y_1^t, \ldots, y_d^t$, all of the modules $S/(\underline{y}^t)$ are faithfully flat over R.
(3) The direct limit $H^d_{(\underline{y})}(S)$ of the $S/(\underline{y}^t)$ is flat over R.
(4) The injective hull E_S of L over S is $H^d_{\underline{y}}(S) \otimes_R E_R$, where E_R denotes the injective hull of K over R.
(5) With notation as in 4.4, if u is a socle generator in E_R and $v := [1; \underline{y}] \in H^d_{(\underline{y})}(S)$, then $v \otimes u$ is a socle generator in $H^d_{\underline{y}}(S) \otimes_R E_R \cong E_S$.

Now we may identify $F^n_S\big(H^d_{(\underline{y})}(S) \otimes_R E_R\big)$ with

$$H^d_{(\underline{y}^{p^n})}(S) \otimes_R F^n_R(E_R) \cong H^d_{(\underline{y})}(S) \otimes_R F^n_R(E_R)$$

and $(v \otimes u)^{p^n}$ with $v^{p^n} \otimes u^{p^n}$, where $v^{p^n} = [1; \underline{y}^{p^n}]$. As $\underline{y}$ is regular on $S \otimes_R E_R$ and since $S/(\underline{y}^{p^n})$ is flat over R, the annihilator of $v^{p^n} \otimes u^{p^n}$ in S is $(\underline{y}^{p^n})S + \big(\mathrm{Ann}_R(u^{p^n})\big)S = (\underline{y}^{p^n})S + \mathscr{A}_n(R)S$. Therefore, $\mathscr{A}_n(S) = (\underline{y}^{p^n})S + \mathscr{A}_n(R)S$. By the faithful flatness of $S/(\underline{y}^{p^n})$ over R, $R/\mathscr{A}_n(R)$ injects into $S/(\underline{y}^{p^n}) \otimes_R \big(R/\mathscr{A}_n(R)\big)$. This implies that the contraction of $\mathscr{A}_n(S)$ to R is $\mathscr{A}_n(R)$. Hence, if $c \notin \mathscr{A}_n(R)$, then $h(c) \notin \mathscr{A}_n(S)$, as required. □

7.4. **Strongly F-regular and very strongly F-regular rings.** Let R° denote the complement of the union of all minimal primes of R, i.e., $R^\circ = R \setminus \bigcup_{\mathfrak{p}\in\mathrm{Min}(R)} \mathfrak{p}$. If $\mathfrak{e}_c$ is finite for all $c \in R^\circ$ then R is called *very strongly F-regular* in the terminology of [Hash10] and *F-pure regular* in the terminology of [DaSm16]. We will use the terminology "very strongly F-regular." If this condition holds for all local rings of R at maximal ideals (equivalently, prime ideals) then R is called *strongly F-regular.* This terminology is proposed in [Ho07], and it is also used in [Hash10]. We also note that R is strongly F-regular in this sense if and only if one of the following two equivalent conditions holds:

(1) For every R-module M and every R-submodule $N \subseteq M$ (with no assumption about finite generation), N is tightly closed in M.
(2) For every maximal ideal $\mathfrak{m}$ in R, the submodule 0 is tightly closed in the injective hull $\mathrm{E}_R(R/\mathfrak{m})$.

For further background on the theory of strongly F-regular rings, we refer the reader to [Ab01, AL03, HH89, HH94b, LS99, Sm00, HL02, SchSm10, Tu12, PT18, Yao06].

Our main goal in the sequel is to prove that strongly F-regular excellent rings are very strongly F-regular. This is an obvious question, raised, for example, in [DaSm16]. It is clear that a strongly F-regular ring is very strongly F-regular if R is local. It is also known that if R is F-finite or essentially of finite type over an excellent semilocal ring then strongly F-regular rings are very strongly F-regular (and under somewhat weaker hypotheses: see [DET23]). We refer to [HoY23, §2] for a thorough discussion of previously known results, many of which may be found in [Ho07], [Hash10], [DaSm16], [DET23] as well as in [HoY23].

Discussion 7.9. **Calculation of $\mathscr{A}_e$ from a canonical module.** Let $(R, \mathfrak{p}, \kappa)$ be Cohen-Macaulay with canonical module ω_R, $\underline{z} = z_1, \ldots, z_s$ be a system of parameters for R, and v be a socle generator in the local cohomology $E := H^s_{(\underline{z})}(\omega)$ corresponding to a socle element that is the image of u in $\omega_R/(\underline{z})\omega_R$, where $u \in \omega_R$. Then

$$\mathscr{A}_e = \varinjlim_t \Big((z_1^{qt}, \ldots, z_s^{qt}) F^e_R(\omega_R) :_R z^{qt-q} u^q \Big),$$

where $z = \prod_{j=1}^s z_j$ is the product of the z_j, and where u^q is in $F^e_R(\omega_R)$. Note that, by Theorem 7.7, $\mathscr{A}_e$ is $0 :_R v^q = \mathrm{Ann}_R(v^q)$, where $v^q \in F^e_R\big(H^s_{(\underline{z})}(\omega)\big) \cong H^s_{(\underline{z})}\big(F^e_R(\omega)\big)$.

7.5. **Semicontinuity.** Our object is to prove the following result.

Theorem 7.10. *Let R be a homomorphic image of an excellent Cohen-Macaulay Noetherian ring of prime characteristic $p > 0$ such that R is S_2 and let $c \in R$. Then, for any given $e \in \mathbb{N}_+ \cup \{\infty\}$, the set $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{e}_c(\mathfrak{p}) \leqslant e\}$ is Zariski open. In other words, the function $\mathfrak{e}_c : \mathrm{Spec}(R) \to \mathbb{N}_+ \cup \{\infty\}$ is upper semicontinuous.*

Before proving Theorem 7.10, we record an important consequence:

Theorem 7.11. *If an excellent Noetherian ring R of prime characteristic $p > 0$ is strongly F-regular, then it is very strongly F-regular.*

Proof. Let $c \in R^\circ$. Since the ring is strongly F-regular, for every $P \in \operatorname{Spec}(R)$, the purity exponent $\mathfrak{e}_c(P)$ is finite for $c/1 \in R_P$. By Theorem 7.10, this defines a Zariski open neighborhood $U_P := \{Q \in \operatorname{Spec}(R) : \mathfrak{e}_c(Q) \leqslant \mathfrak{e}_c(P)\}$ of P. The open sets U_P, $P \in \operatorname{Spec}(R)$, cover $\operatorname{Spec}(R)$ and have a finite subcover, U_{P_i}, $1 \leqslant i \leqslant m$. Let e be the maximum of the integers $\mathfrak{e}_c(P_i)$, $1 \leqslant i \leqslant m$. Then $(\theta_{e,c})_P$ is pure over R_P for all $P \in \operatorname{Spec}(R)$, and so $\theta_{e,c}$ is pure over R. See subsections 7.1 and 7.2. $\square$

To prove Theorem 7.10, we will need several preliminary results, including the following result on openness, based on an idea of Nagata. See [Mat87, Theorem 24.2] and [StProj, Lemma 5.16.5] for a generalization to Noetherian topological spaces. Note that there is no assumption about the characteristic of R.

Theorem 7.12. *Let R be any ring, and $U \subseteq \operatorname{Spec}(R)$. Consider the conditions:*

(i) *For P, $Q \in \operatorname{Spec}(R)$, if $P \subseteq Q$ and $Q \in U$, then $P \in U$.*
(ii) *For every $P \in U$, $U \cap V(P)$ contains a non-empty open subset of $V(P)$.*

Then

(a) *If U is open then U satisfies* (i) *and* (ii).
(b) *Assume that R is Noetherian. If U satisfies* (i) *and* (ii) *then U is open.*

To prove Theorem 7.10, it suffices to assume that $e \in \mathbb{N}_+$ and to show that the set $U_{c,e} := \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{e}_c(\mathfrak{p}) \leqslant e\}$ is open in $\operatorname{Spec}(R)$. Proposition 7.3(a) says that, for P, $Q \in \operatorname{Spec}(R)$, if $P \subseteq Q$ and $Q \in U_{c,e}$, then $P \in U_{c,e}$. To apply Theorem 7.12, it remains to show that, for every $P \in U_{c,e}$, $U_{c,e} \cap V(P)$ contains a non-empty open subset of $V(P)$. Theorem 7.10 now follows at once from the Key Lemma (i.e., Lemma 7.13) that immediately follows. In proving the Key Lemma, we will need to replace $\widetilde{R}$ by an étale extension, as in §6. We will also need to view R as $\widetilde{R}/\mathfrak{a}$, as in Theorem 5.9. It may be helpful to the reader to review the notation from §§5,6.

Lemma 7.13 (Key Lemma)**.** *Let R be a homomorphic image of an excellent ring $\widetilde{R}$ of prime characteristic $p > 0$, say $R = \widetilde{R}/\mathfrak{a}$, let $c \in R$, let e be a positive integer. Let $\widetilde{P} \in \operatorname{Spec}(\widetilde{R})$ be such that $\mathfrak{a} \subseteq \widetilde{P}$ and $\widetilde{R}_{\widetilde{P}}$ is Cohen-Macaulay. Let $P := \widetilde{P}/\mathfrak{a}$, and assume that R_P is S_2. Suppose that $\mathfrak{e}_c(P) \leqslant e$. Then there exists $g \in R \setminus P$ such that $\mathfrak{e}_c(Q) \leqslant e$ for all $Q \in D(g) \cap V(P)$.*

Proof. We may localize $\widetilde{R}$ repeatedly (but only finitely many times) at elements $g \in \widetilde{R} \setminus \widetilde{P}$, and we shall do this finitely many times in the course of the proof. Each time, we change notation and continue to use $\widetilde{R}$, $\widetilde{P}$, R, etc. In conquence, we my assume, for example, that $\widetilde{R}$ is Cohen-Macaulay, that R is S_2, and that $A := R/P$ is regular. We want to reduce to the case where $\widetilde{R}_{\widetilde{P}}$ has a canonical module (and that we may assume that it has the form $\widetilde{\omega}_{\widetilde{P}}$, where ω is a global canonical module for $\widetilde{R}$). To this end, we use Theorem 6.3 to replace $\widetilde{R}$ by a suitable étale extension $\breve{\widetilde{R}}$. In doing this, we may need to localize at another element of $\widetilde{R} \setminus \widetilde{P}$. Then $\breve{\widetilde{R}}$ has a global canonical module $\breve{\widetilde{\omega}}$ (we use this notation, but to be clear, at this point we do not have a module $\widetilde{\omega}$ over $\widetilde{R}$ that somehow gives rise to $\breve{\widetilde{\omega}}$). Let $\breve{R} := \breve{\widetilde{R}} \otimes_{\widetilde{R}} R$, which is still S_2, since this is an étale extension of R. The purity exponent e of c in R_P is the same as the purity exponent of the image of c in $\breve{R}_{\breve{P}}$, with notation as in Theorem 6.3, since pointed étale extensions are geometrically regular and we may apply Theorem 7.8. Suppose that we know the theorem for $\breve{R}$, so that $\mathfrak{e}_c(\breve{Q}) \leqslant e$ for all $\breve{Q}$ in an open neighborhood of $\breve{P}$ in $V(\breve{P})$. Since $R_Q \to \breve{R}_{\breve{Q}}$ is pointed étale and hence faithfully flat, we have $\mathfrak{e}_c(Q) \leqslant \mathfrak{e}_c(\breve{Q})$ by Theorem 7.8(a). It follows from Theorem 6.3(7) that $\mathfrak{e}_c(Q) \leqslant e$ as well for all Q in an open neighborhood of P in $V(P)$. Hence, it suffices to verify the Key Lemma after the pointed étale extension from $\widetilde{R}$ to $\breve{\widetilde{R}}$.

Therefore, in the remainder of the proof we may assume that $\widetilde{R}$ has a canonical module $\widetilde{\omega}$. For any prime $\widetilde{Q}$ of $\widetilde{R}$ in $V(\mathfrak{a})$, we write $Q = \widetilde{Q}/\mathfrak{a}$ and $\kappa_{\widetilde{Q}} := \widetilde{R}_{\widetilde{Q}}/\widetilde{Q}R_{\widetilde{Q}} \cong R_Q/QR_Q =: \kappa_Q$. After localization at one element of $\widetilde{W} := \widetilde{R} \setminus \widetilde{P}$, we may assume that all of the conclusions of Theorem 5.9 hold, and we shall use the notation of that theorem. Recall that for rings and modules annihilated by $\mathfrak{a}$, localization at an element of $\widetilde{W}$ yields the same result as localization at the image of that element in $W := R \setminus P$. After such a localization, we still denote the rings as $\widetilde{R}$ and R.

Let $\omega = \operatorname{Ext}^k_{\widetilde{R}}(R, \widetilde{\omega})$, where $k = \dim(\widetilde{R}_{\widetilde{P}}) - \dim(R_P)$, be defined as in Theorem 5.9, and let $E := H^h_P(\omega)$, where $h = \dim(R_P)$, so that E_P is an injective hull for κ_P over R_P. As in Theorem 5.9, after localization at an element in W, we have $\operatorname{Ann}_E P \cong A$. Write $\operatorname{Ann}_E P = Au$, for some $u \in E$. The image of u in E_P generates the socle. Note that in this theorem and its proof, $q = p^e$ is fixed, and the hypothesis that $\mathfrak{e}_c(P) \leqslant e$ tells us that $cu^q \neq 0$ in $F_{R_P}(E_P) \cong F^e_R(E)_P$. Up to radical, the ideal PR_P can be generated by h many elements in R_P. Thus, after localization at an element in W, we assume that P is the radical of an ideal generated by h many elements in R. Consequently, we may identify the following naturally isomorphic R-modules:

$$F^e_R(E) \cong F^e_R\big(H^h_P(\omega)\big) \cong H^h_{P^{[q]}}\big(F^e_R(\omega)\big) \cong H^h_P\big(F^e_R(\omega)\big).$$

Thus, we know that $0 \neq cu^q \in H^h_P\big(F^e_R(\omega)\big)_P$. Let $M := H^h_P\big(F^e_R(\omega)\big)$. Here, we think of $F^e_R(\omega)$ as simply a fixed, finitely generated R-module. By Theorem 2.19(f) and Proposition 2.5(c), after localization at one element of W, we have that M and $M/R(cu^q)$ are free filterable relative to P, so that we may apply Corollary 4.5 to $R'\big(1_{R'} \otimes_R (cu^q)\big) \subseteq R' \otimes_R M$, with any choice of R' that is flat over R.

Now fix an arbitrary $\widetilde{Q} \in V(\widetilde{P})$ and let $\underline{y} := y_1, \ldots, y_d \in \widetilde{R}$ map to a regular system of parameters in the regular local ring $\bar{A}_{\widetilde{Q}}$. By Theorem 5.9, we have $\mathrm{E}_{R_Q}(\kappa_Q) \cong H^d_{(\underline{y})}\big(H^h_P(\omega)\big)_Q$ over R_Q and, with notation as in 4.4, we may take $v := [u; \underline{y}]$ as a socle generator of $H^d_{(\underline{y})}\big(H^h_P(\omega)\big)_Q$.

We may identify $F^e_{R_Q}\big(\mathrm{E}_{R_Q}(\kappa_Q)\big) \cong F^e_{R_Q}\Big(H^d_{(\underline{y})}\big(H^h_P(\omega)\big)_Q\Big) \cong H^d_{(\underline{y}^q)}\big(F^e_R(E)\big)_Q$ and write $v^q = [u; \underline{y}]^q = [u^q; \underline{y}^q] \in H^d_{(\underline{y}^q)}\big(F^e_R(E)\big)_Q \cong F^e_{R_Q}\big(\mathrm{E}_{R_Q}(\kappa_Q)\big)$. Then $cv^q = [cu^q; \underline{y}^q] \in H^d_{(\underline{y}^q)}\big(F^e_R(E)\big)_Q$, where $cu^q \in F^e_R(E) = M$. We apply Corollary 4.5 with $R' = R_Q$, and with $\underline{y}^q$ replacing $\underline{y}$ as in Remark 4.6, to show that the annihilator of $[cu^q; \underline{y}^q]$ in R_Q is $\big(\mathrm{Ann}_R(cu^q)\big)R_Q + (\underline{y}^q)R_Q$. Since $cu^q \in F^e_R(E)$ does not become 0 after localization at P, we see that $\mathrm{Ann}_R(cu^q) \subseteq P$. Hence, we conclude that

$$\begin{aligned}\mathrm{Ann}_{R_Q}(cv^q) = \mathrm{Ann}_{R_Q}([cu^q; \underline{y}^q]) &= \big(\mathrm{Ann}_R(cu^q)\big)R_Q + (\underline{y}^q)R_Q \\ &\subseteq PR_Q + (\underline{y}^q)R_Q = \big(P + (\underline{y}^q)\big)R_Q \subseteq QR_Q.\end{aligned}$$

This means that $0 \neq cv^q \in H^d_{(\underline{y}^q)}\big(F^e_R(E)\big)_Q \cong F^e_{R_Q}\big(\mathrm{E}_{R_Q}(\kappa_Q)\big)$. By Theorem 7.7 applied to R_Q, we see that the purity exponent for the image of c in R_Q is at most e. In summary, there exists $g \in R \setminus P$ such that, for all $Q \in D(g) \cap V(P)$, we have $\mathfrak{e}_c(Q) \leqslant e$. □

Remark 7.14. The reason for assuming that R is S_2 in the Key Lemma 7.13 is that in this case, for every $P \in \mathrm{Spec}(R)$ there exists a finitely generated R-module ω such that the injective hull of the residue field for R_Q can be realized as the top local cohomology module of ω_Q over R_Q, for all prime ideals Q in an open neighborhood of P in $V(P)$. Then, when we apply the Frobenius functor F^e, we still have the top local cohomology of a finitely generated module. The proof works in a similar fashion whenever for every $P \in \mathrm{Spec}(R)$ there exists a finitely generated R-module Ψ such that the injective hull of κ_Q can be realized as the localization at Q of a top local cohomology with support in Q of Ψ, for all prime ideals Q in an open neighborhood of P in $V(P)$. The module Ψ then plays the role of ω in the argument. The proof also depends on the fact that, after replacing R with its localization at one element of $R \setminus P$, the top local cohomology $H_P^{\dim(R_P)}\big(F^e_R(\omega)\big)$ is generically free filterable relative to P. We do not know how to prove the needed facts about being generically free filterable relative to P unless we can realize the injective hulls that arise as localizations of cohomology modules.

Finally, we note the following consequence of Theorem 7.10:

Corollary 7.15. *Let R be a Noetherian ring of prime characteristic $p > 0$ such that R is S_2 and is a homomorphic image of Cohen-Macaulay excellent ring. Then the F-pure locus is open in R.*

Proof. Take $c = 1_R \in R$ and apply Theorem 7.10. The F-pure locus is the set of primes P such that the purity exponent of 1_{R_P} in R_P is at most 1. □

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

Email address: hochster@umich.edu

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303

Email address: yyao@gsu.edu