ON THE FROBENIUS COMPLEXITY OF DETERMINANTAL RINGS

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Abstract. We compute the Frobenius complexity for the determinantal ring of prime characteristic p obtained by modding out the 2×2 minors of an $m \times n$ matrix of indeterminates, where $m > n \geqslant 2$. We also show that, as $p \to \infty$, the Frobenius complexity approaches $m - 1$.

1. Introduction

For rings of prime characteristic, Frobenius operators have been studied extensively in the past few decades and used to describe the nature of singularities of such local rings, with emphasis on operators acting on the injective hull of the residue field or on local cohomology modules with support in the maximal ideal. They have played an important role in conjunction to tight closure theory or with understanding F-singularities in birational geometry, as in [BST, EH, HH, K, KSSZ, Ly, LS, Sc, Sh] to list just a few papers dealing with the subject. Recently, we have introduced a new concept measuring the abundance of the Frobenius operators on the injective hull of the residue field, called Frobenius complexity. In [EY], we laid the foundations of this invariant and computed it for a number of examples, which are determinantal rings obtained by modding out ideals of 2×2 minors in $m \times (m-1)$ matrix of indeterminates. In this paper, we further develop our methods to show how to compute the Frobenius complexity for rings defined by ideals of 2×2 minors of an $m \times n$ matrix of indeterminates, where $m > n \geqslant 2$. Our computations are algorithmic in nature. In addition, we also show that, as $p \to \infty$, the Frobenius complexity approaches $m - 1$. The paper is organized as follows. Section 1 introduces the terminology, notations and summarizes what is known about Frobenius complexity. Sections 2 and 3 provide a detailed analysis of the twisted construction for the Veronese ring of a polynomial ring in finitely many indeterminates and the computation of the complexity sequence for this skew-algebra. The last Section applies the work developed up to that point to obtain the main results of this paper, namely the Frobenius complexity for the determinantal ring of prime characteristic p obtained by modding out the 2×2 minors of an $m \times n$ matrix of indeterminates, where $m > n \geqslant 2$.

1.1. **Preliminaries.** Throughout this paper R is a commutative Noetherian ring, often local, of prime characteristic p. Let $q = p^e$, where $e \in \mathbb{N} = \{0, 1, \dots\}$. Consider the eth Frobenius homomorphism $F^e : R \to R$ defined $F(r) = r^q$, for all $r \in R$. For an R-module M, an eth Frobenius action (or Frobenius operator) on M is an additive map $\phi : M \to M$ such that $\phi(rm) = r^{p^e}\phi(m)$, for all $r \in R, m \in M$. For any $e \geqslant 0$, we let $R^{(e)}$ be the

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R-algebra defined as follows: as a ring $R^{(e)}$ equals R while the R-algebra structure is defined by $r \cdot s = r^q s$, for all $r \in R$, $s \in R^{(e)}$. Also, $R^{(e)}$ as an $R^{(e)}$-algebra is simply R as an R-algebra. Similarly, for an R-module M, we can define a new R-module structure on M by letting $r * m = r^{p^e} m$, for all $r \in R, m \in M$. We denote this R-module by $M^{(e)}$.

Consider now an eth Frobenius action, $\phi : M \to M$, on M, which is no other than an R-module homomorphism $\phi : M \to M^{(e)}$. Such an action naturally defines an R-module homomorphism $f_\phi : R^{(e)} \otimes_R M \to M$, where $f_\phi(r \otimes m) = r\phi(m)$, for all $r \in R, m \in M$. Here, $R^{(e)}$ has the usual structure (i.e., without twisting) as an R-module given by $R^{(e)} = R$ on the left, while on the right we have the twisted module structure via the Frobenius action.

Let $\mathscr{F}^e(M)$ be the collection of all eth Frobenius operators on M. The R-module structure on $\mathscr{F}^e(M)$ is given by viewing $M^{(e)}$ as an R-module without twisting, that is, $(r\phi)(x) = r\phi(x)$ for every $r \in R$, $\phi \in \mathscr{F}^e(M)$ and $x \in M$.

Definition 1.1. We define *the algebra of Frobenius operators* on M by

$$\mathscr{F}(M) = \oplus_{e \geqslant 0} \mathscr{F}^e(M),$$

with the multiplication on $\mathscr{F}(M)$ determined by composition of functions; that is, if $\phi \in \mathscr{F}^e(M), \psi \in \mathscr{F}^{e'}(M)$ then $\phi\psi := \phi \circ \psi \in \mathscr{F}^{e+e'}(M)$. Hence, in general, $\phi\psi \neq \psi\phi$.

Note that $\mathscr{F}^0(M) = \mathrm{End}_R(M)$, which is a subring of $\mathscr{F}(M)$. Naturally, each $\mathscr{F}^e(M)$ is a module over $\mathscr{F}^0(M)$. Since R maps canonically to $\mathscr{F}^0(M)$, this makes $\mathscr{F}^e(M)$ an R-module by restriction of scalars. Note that $(\phi \circ r)(m) = \phi(rm) = (r^q\phi)(m)$, for all $r \in R, m \in M$. Therefore, $\phi r = r^q \phi$, for all $r \in R$, $\phi \in \mathscr{F}^e(M)$, $q = p^e$.

1.2. **The Frobenius Complexity.** The main concept studied in this paper is the Frobenius complexity of a local ring R, which was introduced in [EY]. In fact, the results in this subsection, if not referenced otherwise, are taken from [EY]. We first need to review the definition of the complexity of a graded ring.

Definition 1.2. Let $A = \oplus_{e \geqslant 0} A_e$ be a $\mathbb{N}$-graded ring, not necessarily commutative.

(1) Let $G_e(A) = G_e$ be the subring of A generated by the elements of degree less or equal to e. We agree that $G_{-1} = A_0$.
(2) We use $k_e = k_e(A)$ to denote the minimal number of homogeneous generators of G_e as a subring of A over A_0. (So $k_{-1} = k_0 = 0$.) We say that A is *degree-wise finitely generated* if $k_e < \infty$ for all $e \geqslant 0$.
(3) For a degree-wise finitely generated ring A, we say that a set X of homogeneous elements of A *minimally generates* A if for all e, $X_{\leqslant e} = \{a \in X : \deg(a) \leqslant e\}$ is a minimal set of generators for G_e with $k_e = |X_{\leqslant e}|$ for every $e \geqslant 0$. Also, let $X_e = \{a \in X : \deg(a) = e\}$.

Proposition 1.3. *With the notations introduced above, let X be a set of homogeneous elements of A. Then*

(1) *The set X generates A as a ring over A_0 if and only if $X_{\leqslant e}$ generates G_e as a ring over A_0 for all $e \geqslant 0$ if and only if the image of X_e generates $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule for all $e \geqslant 0$.*

(2) *Assume that A is degree-wise finitely generated $\mathbb{N}$-graded ring and X generates A as a ring over A_0. The set X minimally generates A as a ring over A_0 if and only if $|X_e|$ is the minimal number of generators (out of all homogeneous generating sets) of $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule for all $e \geqslant 0$.*

Corollary 1.4. *Let A be a degree-wise finitely generated $\mathbb{N}$-graded ring and X a set of homogeneous elements of A. Then*

(1) *The minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule is $k_e - k_{e-1}$ for all $e \geqslant 0$.*
(2) *If X generates A as a ring over A_0 then $|X_e| \geqslant k_e - k_{e-1}$ for all $e \geqslant 0$.*

Let $f(n)$ and $g(n)$ be real-valued functions defined on the set of natural numbers. We say that $f(n) = O(g(n))$ if there exists $M > 0$ and a nonnegative integer n_0 such that $|f(n)| \leqslant M \cdot |g(n)|$ for all $n \geqslant n_0$.

Definition 1.5. Let A be a degree-wise finitely generated ring. The sequence $\{k_e\}_e$ is called the *growth* sequence for A. The *complexity sequence* is given by $\{c_e(A) = k_e - k_{e-1}\}_{e\geqslant 0}$. The *complexity* of A is

$$\inf\{n \in \mathbb{R}_{>0} : c_e(A) = O(n^e)\}$$

and it is denoted by $\operatorname{cx}(A)$. Therefore, if there is no $n > 0$ such that $c_e(A) = O(n^e)$, then $\operatorname{cx}(A) = \infty$.

Definition 1.6. Let A and B be $\mathbb{N}$-graded rings and $h\colon A \to B$ be a graded ring homomorphism. We say that h is *nearly onto* if $B = B_0[h(A)]$ (that is, B as a ring is generated by $h(A)$ over B_0).

Theorem 1.7. *Let A and B be $\mathbb{N}$-graded rings that are degree-wise finitely generated. If there exists a graded ring homomorphism $h\colon A \to B$ that is nearly onto, then $c_e(A) \geqslant c_e(B)$ for all $e \geqslant 0$.*

Definition 1.8. Let A be a $\mathbb{N}$-graded ring such that there exists a ring homomorphism $R \to A_0$, where R is a commutative ring. We say that A is a (left) *R-skew algebra* if $aR \subseteq Ra$ for all homogeneous elements $a \in A$. A right R-skew algebra can be defined analogously. In this paper, our R-skew algebras will be left R-skew algebras and therefore we will drop the adjective 'left' when referring it to them.

Corollary 1.9. *Let A be a degree-wise finitely generated R-skew algebra such that $R = A_0$. Then $c_e(A)$ equals the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as a left R-module for all e.*

We are now in position to state the definition of the Frobenius complexity of a local ring of prime characteristic.

Definition 1.10. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p. Denote by E the injective hull of the residue field of R. Denote $k_e(R) := k_e(\mathscr{F}(E))$, for all e, and call these numbers the *Frobenius growth sequence* of R. Then $c_e = c_e(R) := k_e(R) - k_{e-1}(R)$ defines the *Frobenius complexity sequence* of R.

We define the *Frobenius complexity* of the ring R by

$$\operatorname{cx}_F(R) = \log_p(\operatorname{cx}(\mathscr{F}(E))),$$

if $\mathrm{cx}(\mathscr{F}(E))$ is nonzero and finite. If the Frobenius growth sequence of the ring R is eventually constant (i.e., $\mathrm{cx}(\mathscr{F}(E)) = 0$), then the Frobenius complexity of R is set to be $-\infty$. If $\mathrm{cx}(\mathscr{F}(E)) = \infty$, the Frobenius complexity if R is set to be ∞.

Katzman, Schwede, Singh and Zhang have introduced an important $\mathbb{N}$-graded ring in their paper [KSSZ], which is an example of an R-skew algebra. We will study the complexity of this skew-algebra in this section, and apply these results to the complexity of the ring R in subsequent sections.

Definition 1.11 ([KSSZ]). Let $\mathscr{R}$ be an $\mathbb{N}$-graded commutative ring of prime characteristic p with $\mathscr{R}_0 = R$. Define $T(\mathscr{R}) := \oplus_{e \geqslant 0} \mathscr{R}_{p^e-1}$, which is an $\mathbb{N}$-graded ring by

$$a * b = ab^{p^e}$$

for all $a \in \mathscr{R}_{p^e-1}$, $b \in \mathscr{R}_{p^{e'}-1}$. The degree e piece of $T(\mathscr{R})$ is $T_e(\mathscr{R}) = \mathscr{R}_{p^e-1}$.

A number of results have been proved about the Frobenius complexity of a local ring and they are summarized below.

Theorem 1.12 ([EY, Corollary 2.12, Theorems 4.7 and 4.9]). *Let $(R, \mathfrak{m}, k)$ be a local ring.*

1. *If R is 0-dimensional then $\mathrm{cx}_F(R) = -\infty$.*
2. *If R is normal, complete and has dimension at most two, then $\mathrm{cx}_F(R) \leqslant 0$.*
3. *If R is normal, complete and such that the anticanonical cover is finitely generated over R, then $\mathrm{cx}_F(R) < \infty$.*

In addition the following holds.

Theorem 1.13 ([KSSZ, Proposition 4.1] and [EY, Theorem 4.5]). *If $(R, \mathfrak{m}, k)$ is normal and $\mathbb{Q}$-Gorenstein, then the order of its canonical module in the divisor class group is relatively prime to p if and only if $\mathrm{cx}_F(R) = -\infty$.*

Notation 1.14. As in [EY], we will also use the following notations and terminologies in the sequel:

1. For an integer $a \in \mathbb{N}$, if $a = c_n p^n + \cdots + c_1 p + c_0$ with $0 \leqslant c_i \leqslant p-1$ for all $0 \leqslant i \leqslant n$, then we use $a = \overline{c_n \cdots c_0}$ to denote the base p expression of a. Also, we write $a|_e$ to denote the remainder of a when dividing to p^e. Thus, if $a = \overline{c_n \cdots c_0}$ and $n \geqslant e-1$ then $a|_e = \overline{c_{e-1} \cdots c_0}$, which we refer to as the eth truncation of a. Put differently, $a|_e = a - \left\lfloor \frac{a}{p^e} \right\rfloor p^e$, in which $\left\lfloor \frac{a}{p^e} \right\rfloor$ is the floor function of $\frac{a}{p^e}$. When adding up integers $a_i \in \mathbb{N}$ with $1 \leqslant i \leqslant m$, all written in base p expressions, we can talk about the carry over to digit corresponding to p^e, which is simply $\left\lfloor \frac{a_1|_e + \cdots + a_m|_e}{p^e} \right\rfloor$. These notations depend on the choice of p, which should be clear from the context.
2. For any positive integers p and m (with p prime), denote by $M_{p,m}(i)$ (or simply $M(i)$ if p and m are understood) the rank of $(R[x_1, \ldots, x_m]/(x_1^p, \ldots, x_m^p))_i$ over R, for all $i \in \mathbb{Z}$. This is clearly independent of R. Observe that $M_{p,m} = 0$ exactly when $i > m(p-1)$ or $i < 0$. In fact, all $M_{p,m}(i)$ can be read off from the following Poincaré

series (actually a polynomial):

$$\sum_{i=-\infty}^{\infty} M_{p,m}(i)t^i = \sum_{i=0}^{m(p-1)} M_{p,m}(i)t^i = \left(\frac{1-t^p}{1-t}\right)^m = \left(1+\cdots+t^{p-1}\right)^m.$$

1.3. **Determinantal rings.** In this paper we consider the determinantal ring $K[X]/I$ where X is an $m \times n$ matrix of indeterminates and I is the ideal of all the 2×2 minors of X and K a field. This ring is isomorphic to the Segre product of $K[x_1, \dots, x_m]$ and $K[y_1, \dots, y_n]$.

Recall that, for $\mathbb{N}$-graded commutative rings $A = \oplus_{i\in\mathbb{N}} A_i$ and $B = \oplus_{i\in\mathbb{N}} B_i$ such that $A_0 = R = B_0$, their Segre product is

$$A \sharp B = \oplus_{i\in\mathbb{N}}(A_i \otimes_R B_i),$$

which is a ring under the natural operations.

Definition 1.15. Let $S_{m,n}$ denote the completion of $K[x_1, \dots, x_m] \sharp K[y_1, \dots, y_n]$ with respect to the ideal generated by all homogeneous elements of positive degree, in which K is a field and $m > n \geqslant 2$. It is easy to see that

$$\begin{aligned} S_{m,n} &\cong \prod_{\alpha\in\mathbb{N}^m,\, \beta\in\mathbb{N}^n,\, |\alpha|=|\beta|} Kx^\alpha y^\beta \\ &= \left\{ \sum_{|\alpha|=|\beta|} a_{\alpha,\beta}x^\alpha y^\beta \;\middle|\; a_{\alpha,\beta} \in K,\, \alpha\in\mathbb{N}^m,\, \beta\in\mathbb{N}^n \right\} \subset K[[x_1, \dots, x_m, y_1, \dots, y_n]]. \end{aligned}$$

Let $\mathscr{R}_{m,n}$ be the anticanonical cover of $S_{m,n}$.

The anticanonical cover of such a ring was described by Kei-ichi Watanabe.

Theorem 1.16 ([Wa, page 430])**.** *Let K be a field and $m > n \geqslant 2$. The anticanonical cover of the Segre product of $K[x_1, \dots, x_m]$ and $K[y_1, \dots, y_n]$ is isomorphic to*

$$\bigoplus_{i\in\mathbb{N}} \left(\bigoplus_{\alpha\in\mathbb{N}^m,\, \beta\in\mathbb{N}^n,\, |\alpha|-|\beta|=i(m-n)} Kx^\alpha y^\beta \right),$$

in which the grading is governed by i. Here, for $\alpha = (a_1, \dots, a_m)$ and $\beta = (b_1, \dots, b_n)$ we denote $x^\alpha = x_1^{a_1}\cdots x_m^{a_m}$ and $y^\beta = y_1^{b_1}\cdots y_n^{b_n}$.

It follows from Theorem 1.16 that

$$\mathscr{R}_{m,n} \cong \bigoplus_{i\in\mathbb{N}} \left(\prod_{\alpha\in\mathbb{N}^m,\, \beta\in\mathbb{N}^n,\, |\alpha|-|\beta|=i(m-n)} Kx^\alpha y^\beta \right),$$

in which the grading is governed by i.

Lemma 1.17 ([EY])**.** *Let A and B be degree-wise finitely generated $\mathbb{N}$-graded commutative rings and $h\colon A \to B$ be a graded ring homomorphism.*

1. *The homomorphism h is nearly onto if and only if B_i is generated by $h(A_i)$ as a B_0-module for all $i \in \mathbb{N}$ (that is, B is generated by $h(A)$ as a B_0-module).*

(2) *If A and B have prime characteristic p and h is nearly onto, then the induced graded homomorphism $T(h)\colon T(A) \to T(B)$ is nearly onto.*

Corollary 1.18. *Let A and B be $\mathbb{N}$-graded commutative rings of prime characteristic p. If there exists a graded ring homomorphism $h\colon A \to B$ that is nearly onto, then $c_e(T(A)) \geqslant c_e(T(B))$ for all $e \geqslant 0$.*

Proposition 1.19 (Compare with [EY, Proposition 5.5])**.** *Let K, $S_{m,n}$ and $\mathscr{R}_{m,n}$ be as in Definition 1.15 with $m > n \geqslant 2$. Then there are nearly onto graded ring homomorphisms from $\mathscr{R}_{m,n}$ to $V_{m-n}(K[x_1, \dots, x_m])$ and vice versa, in which $V_{m-n}(K[x_1, \dots, x_m])$ denotes the $(m-n)$-Veronese subring of $K[x_1, \dots, x_m]$.*

Proof. In light of Definition 1.15 and Theorem 1.16, we simply assume

$$\mathscr{R}_{m,n} = \bigoplus_{i \in \mathbb{N}} \left(\prod_{\alpha \in \mathbb{N}^m,\, \beta \in \mathbb{N}^n,\, |\alpha| - |\beta| = i(m-n)} K x^\alpha y^\beta \right).$$

Define $\phi\colon \mathscr{R}_{m,n} \to V_{m-n}(K[x_1, \dots, x_m])$ and $\psi\colon V_{m-n}(K[x_1, \dots, x_m]) \to \mathscr{R}_{m,n}$ by

$$\phi(f(x_1, \dots, x_m,\, y_1, \dots, y_n)) = f(x_1, \dots, x_m,\, 0, \dots, 0) \in K[x_1, \dots, x_m]$$
$$\text{and} \qquad \psi(g(x_1, \dots, x_m)) = g(x_1, \dots, x_m) \in \mathscr{R}_{m,n},$$

for all $f(x_1, \dots, x_m,\, y_1, \dots, y_n) \in \mathscr{R}_{m,n}$ and all $g(x_1, \dots, x_m) \in V_{m-n}(K[x_1, \dots, x_m])$.

It is routine to verify that both ϕ and ψ are graded ring homomorphisms. As $\phi \circ \psi$ is the identity map, we see that ϕ is onto and hence nearly onto. Finally, note that for every $i \in \mathbb{N}$, $(\mathscr{R}_{m,n})_i$ is generated by $\psi(V_{m-n}(K[x_1, \dots, x_m])_i) = \psi(K[x_1, \dots, x_m]_{i(m-n)})$ as a module over $(\mathscr{R}_{m,n})_0 = S_{m,n}$. So ψ is nearly onto, completing the proof. ☐

Theorem 1.20. *Let K, $S_{m,n}$ and $\mathscr{R}_{m,n}$ be as in Definition 1.15 with $m > n \geqslant 2$.*

(1) *Then $\mathscr{R}_{m,n}$ and $V_{m-n}(K[x_1, \dots, x_m])$ have the same complexity sequence.*
(2) *Assume that K has prime characteristic. Then $T(\mathscr{R}_{m,n})$ and $T(V_{m-n}(K[x_1, \dots, x_m]))$ have the same complexity sequence. Therefore*

$$\operatorname{cx}(\mathscr{F}(E_{m,n})) = \operatorname{cx}(T(\mathscr{R}_{m,n})) = \operatorname{cx}(T(V_{m-n}(K[x_1, \dots, x_m]))),$$

in which $E_{m,n}$ stands for the injective hull of the residue field of $S_{m,n}$. Consequently,

$$\operatorname{cx}_F(S_{m,n}) = \log_p \operatorname{cx}(T(V_{m-n}(K[x_1, \dots, x_m]))).$$

Proof. This follows from Corollary 1.18, Proposition 1.19 and [KSSZ, Theorem 3.3]. ☐

In summary, to compute the Frobenius complexity of $S_{m,n}$ with $m > n \geqslant 2$, it suffices to study $T(V_r(K[x_1, \dots, x_m]))$ with $r = m - n$ (hence $0 < r \leqslant m - 2$). The next section is devoted to the study of $T(V_r(K[x_1, \dots, x_m]))$, more generally with $1 \leqslant r,\, m \in \mathbb{N}$.

2. Investigating $T(V_r(R[x_1, \dots, x_m]))$

Let R be a commutative ring of prime characteristic p and r, m positive integers. In this section, we study $T(V_r(R[x_1, \dots, x_m]))$. In particular, we are interested in when it is finitely generated over R, as well as how to compute its complexity.

To simplify notation, denote the following (with R, p, m and r understood):

- $\mathscr{R} := R[x_1, \ldots, x_m]$.
- $\mathscr{V} := V_r(\mathscr{R}) = V_r(R[x_1, \ldots, x_m])$.
- $T := T(\mathscr{V}) = T(V_r(R[x_1, \ldots, x_m]))$.
- $G_e := G_e(T)$.
- $T_e := T_e(\mathscr{V}) = T_e(V_r(R[x_1, \ldots, x_m])) = \mathscr{R}_{r(p^e-1)} = (R[x_1, \ldots, x_m])_{r(p^e-1)}$. As there are several gradings going on, when we say the degree of a monomial, we agree that it refers to its (total) degree in $\mathscr{R} = R[x_1, \ldots, x_m]$. Thus a monomial in T_e is a monomial of (total) degree $r(p^e-1)$. Note that $T_e = \mathscr{R}_{r(p^e-1)}$ is an R-free (left) module with a basis consisting of monomials of (total) degree $r(p^e-1)$. In particular, $T_0 = R$.

Fix any $e \in \mathbb{N}$. We see that $G_{e-1} = G_{e-1}(T)$ is an R-free (left) module with a basis consisting of monomials that can be expressed as products (under $*$, the multiplication of T) of monomials of degree $r(p^i-1)$ where $i \leqslant e-1$. So all such monomials of total degree $r(p^e-1)$ form an R-basis of $(G_{e-1})_e$.

In conclusion, $\frac{T_e}{(G_{e-1})_e}$ is free as a left R-module with a basis given by monomials of degree $r(p^e-1)$ that cannot be written as products (under $*$) of monomials of degree $r(p^i-1)$, with $i \leqslant e-1$. We will refer to this basis as the *monomial basis* of $\frac{T_e}{(G_{e-1})_e}$. By Corollary 1.9, we see $c_e(T) = \operatorname{rank}_R(\frac{T_e}{(G_{e-1})_e})$.

As $c_0(T) = 0$ and $c_1(T) = \operatorname{rank}_R(T_1) = \operatorname{rank}_R(\mathscr{R}_{r(p-1)})$, we may assume $e \geqslant 2$ in the following discussion.

Let $\alpha = (a_1, \ldots, a_m) \in \mathbb{N}^m$ such that $|\alpha| := a_1 + \cdots + a_m = r(p^e-1)$, so that $x^\alpha := x_1^{a_1} \cdots x_m^{a_m}$ is a monomial in T_e (i.e., of degree $r(p^e-1)$). This monomial x^α belongs to $(G_{e-1})_e$ if and only if it can be decomposed as

$$x^\alpha = x^{\alpha'} * x^{\alpha''} = x^{\alpha' + p^{e'}\alpha''}$$

for some $x^{\alpha'} \in T_{e'}$, $x^{\alpha''} \in T_{e''}$ with $1 \leqslant e', e'' \leqslant e-1$ and $e' + e'' = e$. In other words, $x^\alpha \in (G_{e-1})_e$ if and only if there is an equation

$$\alpha = \alpha' + p^{e'}\alpha''$$

for some $\alpha', \alpha'' \in \mathbb{N}^m$, $1 \leqslant e' \leqslant e-1$, $e' + e'' = e$ with $|\alpha'| = r(p^{e'}-1)$ and $|\alpha''| = r(p^{e''}-1)$, which is equivalent to the existence of equations

$$a_i = a_i' + p^{e'}a_i'' \quad \text{for all} \quad i \in \{1, \ldots, m\}$$

for some $(a_1', \ldots, a_m')$, $(a_1'', \ldots, a_m'') \in \mathbb{N}^m$, $1 \leqslant e' \leqslant e-1$, $e' + e'' = e$ with $\sum_{i=1}^m a_i' = r(p^{e'}-1)$ and $\sum_{i=1}^m a_i'' = r(p^{e''}-1)$. Now it is routine to see that the above holds if and only if there exist $(a_1', \ldots, a_m') \in \mathbb{N}^m$ and $1 \leqslant e' \leqslant e-1$ with $\sum_{i=1}^m a_i' = r(p^{e'}-1)$ such that

$$a_i|_{e'} = a_i'|_{e'} \;\text{ and }\; a_i' \leqslant a_i \quad \text{for all} \quad i \in \{1, \ldots, m\},$$

see Notation 1.14(1) for the meaning of $a_i|_{e'}$. This in turn is equivalent to the existence of an integer $1 \leqslant e' \leqslant e-1$ such that

$$a_1|_{e'} + \cdots + a_m|_{e'} \leqslant r(p^{e'}-1),$$

which is equivalent to the existence of an integer $1 \leqslant e' \leqslant e-1$ such that

$$\left\lfloor \frac{a_1|_{e'} + \cdots + a_m|_{e'}}{p^{e'}} \right\rfloor \leqslant \left\lfloor \frac{r(p^{e'}-1)}{p^{e'}} \right\rfloor.$$

Note that the backward implications of the last two equivalences rely on the fact that $a_1|_{e'} + \cdots + a_m|_{e'}$ and $r(p^{e'}-1)$ are in the same congruence class modulo $p^{e'}$; the backward implications of the next to last equivalence also relies on the fact $a_i|_{e'} \equiv a_i \mod p^{e'}$ for all i, which allows us to reverse-engineer $(a'_1, \ldots, a'_m) \in \mathbb{N}^m$ as desired.

With the argument above, we establish the following result. For the sake of clarity, we state it in two parts. (The fact $a_1|_i + \cdots + a_m|_i \equiv r(p^i-1) \mod p^i$ is again needed in part (2) of the following proposition.)

Proposition 2.1. *Consider $T = T(V_r(R[x_1, \ldots, x_m]))$, in prime characteristic p.*

(1) *For any monomial $x_1^{a_1} \cdots x_m^{a_m} \in T_e$ with $e \geqslant 1$, the following are equivalent.*
   - $x_1^{a_1} \cdots x_m^{a_m} \in G_{e-1}(T)$.
   - *There exists an integer i, $1 \leqslant i \leqslant e-1$, such that the carry-over to the digit associated with p^i is less than or equal to $\left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor$ when $a_1 + \cdots + a_m$ is calculated in base p.*

(2) *For any monomial $x_1^{a_1} \cdots x_m^{a_m} \in T_e$ with $e \geqslant 1$, the following are equivalent.*
   - $x_1^{a_1} \cdots x_m^{a_m} \notin G_{e-1}(T)$.
   - $a_1|_i + \cdots + a_m|_i = r(p^i-1) + d_i p^i$ *with* $1 \leqslant d_i \in \mathbb{N}$ *for all* $1 \leqslant i \leqslant e-1$.
   - *The carry-over to the digit associated with p^i is greater than $\left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor$ for all $1 \leqslant i \leqslant e-1$ when $a_1 + \cdots + a_m$ is calculated in base p.*

Proposition 2.2. *For $T = T(V_r(R[x_1, \ldots, x_m]))$ and $e \geqslant 1$, $c_e(T)$ equals the number of monomials $x_1^{a_1} \cdots x_m^{a_m} \in T_e$ such that the carry-over to the digit associated with p^i is greater than $\left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor$ for all $1 \leqslant i \leqslant e-1$ when $a_1 + \cdots + a_m$ is calculated in base p.*

Using the criteria given in Proposition 2.1, we are able to determine precisely when $T(V_r(R[x_1, \ldots, x_m]))$ is finitely generated over $T_0 = R$.

Theorem 2.3. *Let $T = T(V_r(R[x_1, \ldots, x_m]))$, with r, m, R as above.*

(1) *If $r \geqslant m-1$, then T is generated by T_1 over T_0 (that is, $c_e(T) = 0$ for all $e \geqslant 2$).*
(2) *If $r < m-1$, then $c_e(T) > 0$ (i.e., T_e is not generated by lower degree) for all $e \geqslant 1$.*
(3) *The ring $T(V_r(R[x_1, \ldots, x_m]))$ is finitely generated over R if and only if $r \geqslant m-1$.*

Proof. Evidently, we only need to prove (1) and (2).

(1) Suppose, on the contrary, that for some $e \geqslant 2$ there exists a monomial $x_1^{a_1} \cdots x_m^{a_m} \in T_e$ that does not belong to $G_{e-1}(T)$. Then by Proposition 2.1

$$a_1|_i + \cdots + a_m|_i \geqslant r(p^i-1) + p^i$$

for all $1 \leqslant i \leqslant e-1$. However, the assumption $r \geqslant m-1$ implies

$$a_1|_i + \cdots + a_m|_i \leqslant m(p^i-1) \leqslant (r+1)(p^i-1) < r(p^i-1) + p^i.$$

We get a contradiction.

(2) As $c_1(T) > 0$ is clear, we assume $e \geqslant 2$. Consider

$$x_1^{p^e-1} \cdots x_{r-1}^{p^e-1} x_r^{p^e-p^{e-1}-1} x_{r+1}^{p^{e-1}-1} x_{r+2}^1 \in \mathscr{R}_{r(p^e-1)} = T_e.$$

Now it is routine to see that the carry-over to the digit associated with p^i is $\left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor + 1$ for all $1 \leqslant i \leqslant e-1$ when $a_1 = p^e - 1, \ldots, a_{r-1} = p^e - 1$, $a_r = p^e - p^{e-1} - 1$, $a_{r+1} = p^{e-1} - 1$, $a_{r+2} = 1$ and $a_i = 0$ (for $r+2 < i \leqslant m$) are added up in base p. This verifies $x_1^{p^e-1} \cdots x_{r-1}^{p^e-1} x_r^{p^e-p^{e-1}-1} x_{r+1}^{p^{e-1}-1} x_{r+2} \notin G_{e-1}(T)$ and hence $c_e(T) > 0$. $\square$

3. Computing $c_e(T(V_r(R[x_1, \ldots, x_m])))$

Let R, m, r, $\mathscr{R}$, $\mathscr{V}$ and T be as in last section and keep the notations. In particular, $T = T(V_r(R[x_1, \ldots, x_m]))$ is an $\mathbb{N}$-graded ring. For simplicity, denote $c_e(T)$ by $c_{m,r,e}$ or simply by c_e since r and m are understood. (It should be clear that $c_e(T(V_r(R[x_1, \ldots, x_m])))$ is independent of R. Also note that $c_1 = \operatorname{rank}_R(\mathscr{R}_{r(p-1)}) = \binom{r(p-1)+m-1}{m-1}$.)

Fix an integer $e \geqslant 2$. The goal is to count the number of monomials that produce the monomial basis of $\frac{T_e}{(G_{e-1})_e}$.

First, we set up some notations. Let $\alpha = (a_1, \ldots, a_m) \in \mathbb{N}^m$ with $|\alpha| := a_1 + \cdots + a_m = r(p^e - 1)$. For each $n \in [1, m] := \{1, \ldots, m\}$, write $a_n = \overline{\cdots a_{n,i} \cdots a_{n,0}}$ in base p expression. Then, for each $i \in [0, e-2] := \{0, \ldots, e-2\}$, denote

$$\alpha_i := (a_{1,i}, \ldots, a_{m,i}) \in \mathbb{N}^m,$$

which can be referred to as the vector of the digits corresponding to p^i. Also denote

$$\alpha_{e-1} := \left(\left\lfloor \frac{a_1}{p^{e-1}} \right\rfloor, \ldots, \left\lfloor \frac{a_m}{p^{e-1}} \right\rfloor \right) = \frac{1}{p^{e-1}} \left(a_1 - a_1|_{e-1}, \ldots, a_m - a_m|_{e-1}\right) \in \mathbb{N}^m.$$

Moreover, for each $i \in \{0, \ldots, e-1\}$, let $f_i(\alpha)$ denote the carry-over to the digit corresponding to p^i when computing $\sum_{i=1}^m a_i$ in base p. In other words,

$$f_i(\alpha) := \left\lfloor \frac{a_1|_i + \cdots + a_m|_i}{p^i} \right\rfloor.$$

Note that $f_0(\alpha) = 0$. Then denote $f(\alpha) := (f_{e-1}(\alpha), \ldots, f_0(\alpha)) \in \mathbb{N}^e$. Finally, denote

$$d(\alpha) := (d_{e-1}(\alpha), \ldots, d_0(\alpha)) := f(\alpha) - \left(\left\lfloor \frac{r(p^{e-1}-1)}{p^{e-1}} \right\rfloor, \ldots, \left\lfloor \frac{r(p^0-1)}{p^0} \right\rfloor \right) \in \mathbb{Z}^e,$$

so that $d_i(\alpha) = f_i(\alpha) - \left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor$ for all $i \in [0,\, e-1] := \{0, \dots, e-1\}$. Note that $d_0(\alpha) = 0$. Moreover, for all $i \in [0,\, e-2]$, we have

$$\begin{aligned} d_{i+1}(\alpha) &= \left\lfloor \frac{a_1|_{i+1} + \cdots + a_m|_{i+1}}{p^{i+1}} \right\rfloor - \left\lfloor \frac{r(p^{i+1}-1)}{p^{i+1}} \right\rfloor \\ &\overset{\dagger}{=} \left\lfloor \frac{|\alpha_i| + f_i(\alpha)}{p} \right\rfloor - \left\lfloor \frac{r(p-1) + \left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor}{p} \right\rfloor \\ &\overset{\ddagger}{=} \frac{1}{p}\left[(|\alpha_i| + f_i(\alpha)) - \left(r(p-1) + \left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor \right) \right] \\ &= \frac{1}{p}\big[|\alpha_i| + d_i(\alpha) - r(p-1) \big]. \end{aligned}$$

Note that $\overset{\dagger}{=}$ follows from how the carry overs to the digit corresponding to p^{i+1} are determined from the information on the digit corresponding to p^i plus the carry overs to the digit corresponding to p^i, while $\overset{\ddagger}{=}$ follows from the fact that $|\alpha_i| + f_i(\alpha) \equiv r(p-1) + \left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor$ mod p since they are all congruent to the (same) number representing the digit associated with p^i in the base p expression of $r(p^e-1)$ and $r(p^i-1)$.

Let $\alpha = (a_1, \dots, a_m) \in \mathbb{N}^m$ with $|\alpha| = r(p^e-1)$ as above and let $\delta = (d_{e-1}, \dots, d_0) \in \mathbb{Z}^e$ with $d_0 = 0$. By what we have established above, we see

$$\begin{aligned} d(\alpha) = \delta &\iff d_i(\alpha) = d_i,\ \forall i \in [1,\, e-1] \\ &\iff d_{i+1}(\alpha) = d_{i+1},\ \forall i \in [0,\, e-2] \\ &\iff \frac{1}{p}\big[|\alpha_i| + d_i(\alpha) - r(p-1) \big] = d_{i+1},\ \forall i \in [0,\, e-2] \\ &\iff |\alpha_i| + d_i(\alpha) - r(p-1) = d_{i+1}p,\ \forall i \in [0,\, e-2] \\ &\iff |\alpha_i| + d_i(\alpha) = r(p-1) + d_{i+1}p,\ \forall i \in [0,\, e-2] \\ &\overset{*}{\iff} |\alpha_i| + d_i = r(p-1) + d_{i+1}p,\ \forall i \in [0,\, e-2] \\ &\iff |\alpha_i| = r(p-1) + d_{i+1}p - d_i,\ \forall i \in [0,\, e-2]. \end{aligned}$$

Note that $\overset{*}{\Longrightarrow}$ holds because the assumption (i.e., antecedent) of this implication already implies $d(\alpha) = \delta$, while $\overset{*}{\Longleftarrow}$ follows from an easy induction on i (in light of the established equation $d_{i+1}(\alpha) = \frac{1}{p}\big[|\alpha_i| + d_i(\alpha) - r(p-1) \big]$). Furthermore, the assumption $|\alpha| = r(p^e-1)$ (together with $d(\alpha) = \delta$) translates to the following

$$|\alpha_{e-1}| + f_{e-1}(\alpha) = \left\lfloor \frac{a_1 + \cdots + a_m}{p^{e-1}} \right\rfloor = \left\lfloor \frac{r(p^e-1)}{p^{e-1}} \right\rfloor = r(p-1) + \left\lfloor \frac{r(p^{e-1}-1)}{p^{e-1}} \right\rfloor,$$

which is obtained by examining summations $a_1 + \cdots + a_m$ and $\overbrace{(p^e-1) + \cdots + (p^e-1)}^{r \text{ terms}}$ in base p. Therefore

$$|\alpha_{e-1}| = r(p-1) + \left\lfloor \frac{r(p^{e-1}-1)}{p^{e-1}} \right\rfloor - f_{e-1}(\alpha) = r(p-1) - d_{e-1}(\alpha) = r(p-1) - d_{e-1}.$$

In summary, with $\alpha \in \mathbb{N}^m$ and $\delta = (d_{e-1}, \ldots, d_0) \in \mathbb{Z}^e$ with $d_0 = 0$ as above, we conclude that $|\alpha| = r(p^e-1)$ and $d(\alpha) = \delta$ if and only if

$$(\star) \quad |\alpha_i| = r(p-1) + d_{i+1}p - d_i \text{ for all } i \in \{0, \ldots, e-2\} \quad \text{and} \quad |\alpha_{e-1}| = r(p-1) - d_{e-1}.$$

Now we are ready to formulate $c_{m,r,e} = c_e = c_e(T)$ for $T = T(V_r(R[x_1, \ldots, x_m]))$. This result generalizes [EY, Proposition 3.7]. Since $c_e = 0$ for all $e \geqslant 2$ when $m \leqslant r+1$ (cf. Theorem 2.3), the formula in the following proposition is most meaningful when $m-r-1 > 0$.

Proposition 3.1. *For $T = T(V_r(R[x_1, \ldots, x_m]))$, we have the following formula:*

$$\begin{aligned} c_e &= \sum_{\substack{(d_{e-1}, \ldots, d_1, d_0 = 0) \in \mathbb{N}^e \\ d_i \geqslant 1 \text{ for } 1 \leqslant i \leqslant e-1}} \left(P_m\left(r(p-1) - d_{e-1}\right) \prod_{i=0}^{e-2} M_{p,m}(r(p-1) + d_{i+1}p - d_i) \right) \\ &= \sum_{\substack{(d_{e-1}, \ldots, d_1, d_0 = 0) \in \mathbb{N}^e \\ 1 \leqslant d_i \leqslant m-r-1 \text{ for } 1 \leqslant i \leqslant e-1}} \left(\binom{r(p-1) - d_{e-1} + m - 1}{m-1} \prod_{i=0}^{e-2} M_{p,m}(r(p-1) + d_{i+1}p - d_i) \right) \end{aligned}$$

for all $e \geqslant 2$, where $P_m(i)$ denotes $\operatorname{rank}_R(R[x_1, \ldots, x_m]_i)$, i.e., $P_m(i) = \binom{m+i-1}{i} = \binom{m+i-1}{m-1}$, while $M_{p,m}(i)$ can be found in Notation 1.14(2).

Proof. Fix any $e \geqslant 2$ and adopt the notations set up above. Consider $x^\alpha = x_1^{a_1} \cdots x_m^{a_m} \in T_e$. By Proposition 2.1, $x^\alpha \notin G_{e-1}(T)$ if and only if

$$d_i(\alpha) \geqslant 1 \quad \text{for all} \quad i \in \{1, \ldots, e-1\}.$$

To determine c_e, we need to find the number of monomials with the above property, as stated in Proposition 2.2. This is equivalent to counting the number of $\alpha \in \mathbb{N}^m$ such that $|\alpha| = r(p^e-1)$ and $d_i(\alpha) \geqslant 1$ for all $i \in [1, e-1]$.

Fix any $\delta = (d_{e-1}, \ldots, d_0) \in \mathbb{N}^e$ with $d_0 = 0$ and $d_i \geqslant 1$ for all $i \in [1, e-1]$. We intend to find the number of $\alpha \in \mathbb{N}^m$ such that $|\alpha| = r(p^e-1)$ and $d(\alpha) = \delta$, which can be written as

$$\left\| \{\alpha \in \mathbb{N}^m : |\alpha| = r(p^e-1) \text{ and } d(\alpha) = \delta\} \right\|,$$

in which $\|X\|$ stands for the cardinality of any set X.

For each $i \in [0, e-2]$, the number of ways to realize $|\alpha_i| = r(p-1) + d_{i+1}p - d_i$ is given as follows (cf. $(\star)$):

$$\|\{\alpha_i \in [0, p-1]^m : |\alpha_i| = r(p-1) + d_{i+1}p - d_i\}\| = M_{p,m}(r(p-1) + d_{i+1}p - d_i).$$

The number of ways to realize $|\alpha_{e-1}| = r(p-1) - d_{e-1}$ is given as follows (cf. $(\star)$):

$$\|\{\alpha_{e-1} \in \mathbb{N}^m : |\alpha_{e-1}| = r(p-1) - d_{e-1}\}\| = P_m(r(p-1) - d_{e-1}).$$

Therefore, in light of ($\star$), the number of $\alpha \in \mathbb{N}^m$ such that $|\alpha| = r(p^e - 1)$ and $d(\alpha) = \delta$ is governed by the following formula:

$$\begin{aligned}
&\|\{\alpha \in \mathbb{N}^m : |\alpha| = r(p^e - 1) \text{ and } d(\alpha) = \delta\}\| \\
&\qquad\qquad = P_m\left(r(p-1) - d_{e-1}\right) \prod_{i=0}^{e-2} M_{p,m}(r(p-1) + d_{i+1}p - d_i).
\end{aligned}$$

Observe that if $m - r - 1 \leqslant 0$, then

$$\|\{\alpha \in \mathbb{N}^m : |\alpha| = r(p^e - 1) \text{ and } d(\alpha) = \delta\}\| = 0,$$

which follows from $M_{p,m}(r(p-1) + d_1 p - d_0) = 0$ since $r(p-1) + d_1 p - d_0 \geqslant r(p-1) + p = (r+1)(p-1) + 1 \geqslant m(p-1) + 1$; also see Theorem 2.3(1). We further observe that, whenever there exists $d_i > m - r - 1 > 0$ for some $i \in [1,\, e-1]$, then

$$\|\{\alpha \in \mathbb{N}^m : |\alpha| = r(p^e - 1) \text{ and } d(\alpha) = \delta\}\| = 0.$$

Indeed, pick the least $i \in [1,\, e-1]$ such that $d_i > m-r-1 > 0$ and we get $r(p-1) + d_i p - d_{i-1} \geqslant m(p-1) + 1$ and hence $M_{p,m}(r(p-1) + d_i p - d_{i-1}) = 0$. Put differently, when adding m many non-negative integers to $r(p^e - 1)$, the carry overs to digits associated with p^i cannot exceed $\left\lfloor \frac{r(p^i-1)}{p^i} \right\rfloor + m - r - 1$, for $i \in [1,\, e-1]$.

Finally, exhausting all $\delta = (d_{e-1}, \ldots, d_0) \in \mathbb{N}^e$ with $d_0 = 0$ and $d_i \geqslant 1$ for $i \in [1,\, e-1]$, we can formulate $c_e = c_{m,r,e} = c_e(T(R[x_1, \ldots, x_m]))$ as follows:

$$\begin{aligned}
c_{d,e} &= \sum_{\substack{(d_{e-1},\, \ldots,\, d_1,\, d_0 = 0) \in \mathbb{N}^e \\ d_i \geqslant 1 \text{ for } 1 \leqslant i \leqslant e-1}} \|\{\alpha \in \mathbb{N}^m : |\alpha| = r(p^e - 1) \text{ and } d(\alpha) = (d_{e-1},\, \ldots,\, d_1,\, d_0)\}\| \\
&= \sum_{\substack{(d_{e-1},\, \ldots,\, d_1,\, d_0 = 0) \in \mathbb{N}^e \\ d_i \geqslant 1 \text{ for } 1 \leqslant i \leqslant e-1}} \left(P_m\left(r(p-1) - d_{e-1}\right) \prod_{i=0}^{e-2} M_{p,m}(r(p-1) + d_{i+1}p - d_i) \right) \\
&= \sum_{\substack{(d_{e-1},\, \ldots,\, d_1,\, d_0 = 0) \in \mathbb{N}^e \\ 1 \leqslant d_i \leqslant m-r-1 \text{ for } 1 \leqslant i \leqslant e-1}} \left(\binom{r(p-1) - d_{e-1} + m - 1}{m-1} \prod_{i=0}^{e-2} M_{p,m}(r(p-1) + d_{i+1}p - d_i) \right),
\end{aligned}$$

which verifies the equations. $\square$

Next, we outline a method that allows us compute $c_e = c_{m,r,e} = c_e(T(V_r(R[x_1,\, \ldots,\, x_m])))$ for any m, r with $m \geqslant r + 2$, in which R may have any prime characteristic p. (Note that, if $m \leqslant r + 1$, then $c_e = 0$ for all $e \geqslant 2$, see Theorem 2.3.) The following generalizes [EY, Discussion 3.8].

Discussion 3.2. Fix any positive integers r, m such that $r + 1 < m$, any prime number p, and any ring R with characteristic p. Let $\mathscr{R} = R[x_1,\, \ldots,\, x_m]$. We describe a way to determine $c_e = c_{m,r,e} = c_e(T(V_r(\mathscr{R})))$ explicitly as follows:

For every $e \geqslant 0$, denote

$$X_e := \begin{bmatrix} X_{e,1} \\ \vdots \\ X_{e,m-r-1} \end{bmatrix}_{(m-r-1)\times 1},$$

in which

$$X_{e,n} := \sum_{\substack{(d_{e+1}=n,\, d_e,\, \dots,\, d_1,\, d_0=0)\in\mathbb{N}^{e+2} \\ 1\leqslant d_i\leqslant m-r-1 \text{ for } 1\leqslant i\leqslant e}} \prod_{i=0}^{e} M_{p,m}(r(p-1)+d_{i+1}p-d_i)$$

for all $n \in \{1, \dots, m-r-1\}$.

With these notations, it is straightforward to see that, for all $i \in [1, m-r-1]$,

$$X_{e+1,i} = \sum_{j=1}^{m-r-1} M_{p,m}(r(p-1)+ip-j)X_{e,j}$$

In other words, X_{e+1} can be computed recursively:

$$X_{e+1} = U \cdot X_e,$$

where

$$U := \left[u_{ij}\right]_{(m-r-1)\times(m-r-1)} \quad \text{with} \quad u_{ij} := M_{p,m}(r(p-1)+ip-j).$$

Therefore,

$$X_e = U^e \cdot X_0 \quad \text{for all} \quad e \geqslant 0.$$

With m, r and p given, both X_0 and U can be determined explicitly. Accordingly, we can compute $X_e = U^e \cdot X_0$ explicitly for all $e \geqslant 0$.

Finally, for all $e \geqslant 2$, we can determine $c_e = c_e(T(V_r(\mathscr{R})))$ explicitly, as follows:

$$\begin{aligned}
c_e &= \sum_{\substack{(d_{e-1},\, \dots,\, d_1,\, d_0=0)\in\mathbb{N}^{e} \\ 1\leqslant d_i\leqslant m-r-1 \text{ for } 1\leqslant i\leqslant e-1}} \left(P_m(r(p-1)-d_{e-1}) \prod_{i=0}^{e-2} M_{p,m}(r(p-1)+d_{i+1}p-d_i) \right) \\
&= \sum_{n=1}^{m-r-1} \left(P_m(r(p-1)-n) \sum_{\substack{(d_{e-1}=n,\, d_{e-2},\, \dots,\, d_1,\, d_0=0)\in\mathbb{N}^{e} \\ 1\leqslant d_i\leqslant m-r-1 \text{ for } 1\leqslant i\leqslant e-2}} \prod_{i=0}^{e-2} M_{p,m}(r(p-1)+d_{i+1}p-d_i) \right) \\
&= \sum_{n=1}^{m-r-1} P_m(r(p-1)-n)X_{e-2,n} = \sum_{n=1}^{m-r-1} \binom{r(p-1)-n+m-1}{m-1} X_{e-2,n} \\
&= Y_0 \cdot U^{e-2} \cdot X_0,
\end{aligned}$$

where $Y_0 := \left[\binom{r(p-1)-1+m-1}{m-1} \quad \cdots \quad \binom{r(p-1)-(m-r-1)+m-1}{m-1}\right]_{1\times(m-r-1)}$. Thus, the complexity $\operatorname{cx}(T(V_r(\mathscr{R})))$ can be computed.

Since the matrix U, as above, carries important information on the complexity $\operatorname{cx}(T(V_r(\mathscr{R})))$ (see Subsections 3.1 and 3.2 as well as Section 4), we make the following definition.

Definition 3.3. In what follows, we call

$$U(p,r,m) = U := \left[u_{ij}\right]_{(m-r-1)\times(m-r-1)} \quad \text{with} \quad u_{ij} := M_{p,m}(r(p-1)+ip-j)$$

as the *determining matrix* for p, r, m.

Theorem 3.4. *Consider $T = T(V_r(R[x_1,\dots,x_m]))$ as above with $m = r+2$. Then $c_e(T) = \binom{rp}{m-1}\binom{p+m-2}{m-1}^{e-2}\binom{p+m-3}{m-1}$ for all $e \geqslant 2$ and $\operatorname{cx}(T) = \binom{p+m-2}{m-1}$.*

Proof. Adopting all the notations introduced in Discussion 3.2, we see

$$\begin{aligned}
X_0 &= M_{p,m}(r(p-1)+p) = M_{p,m}(p-2) = P_m(p-2) = \binom{p+m-3}{m-1} > 0,\\
U &= M_{p,m}((r+1)(p-1)) = M_{p,m}(p-1) = P_m(p-1) = \binom{p+m-2}{m-1} > 0,\\
Y_0 &= P_m(r(p-1)-1) = \binom{r(p-1)-1+m-1}{m-1} = \binom{rp}{m-1} > 0.
\end{aligned}$$

Here we use the fact $M_{p,m}(i) = M_{p,m}(m(p-1)-i)$ for all i (cf. Lemma 3.5). Therefore, for all $e \geqslant 2$, we obtain

$$c_e = \binom{r(p-1)+m-2}{m-1}\binom{p+m-2}{m-1}^{e-2}\binom{p+m-3}{m-1},$$

which establishes

$$\operatorname{cx}(T(V_r(R[x_1,\dots,x_m]))) = \binom{p+m-2}{m-1}$$

when $m = r+2$. □

3.1. **The Frobenius complexity as $p \to \infty$.** We will maintain the notations from this section, including the condition $m \geqslant r+2$ and $r > 0$. We begin with some easy lemmas, with brief proofs included for the sake of completeness.

Lemma 3.5. *Fix an integer $m > 0$ and a prime number p.*

(1) $M_{p,m}(i) = M_{p,m}(m(p-1)-i)$.
(2) $M_{p,m}(i) \leqslant M_{p,m}(j)$ *if* $0 \leqslant i \leqslant j \leqslant \lceil m(p-1)/2 \rceil$ *or* $\lceil m(p-1)/2 \rceil \leqslant j \leqslant i \leqslant m(p-1)$.

Proof. (1) This follows from $\sum_{i=-\infty}^{\infty} M_{p,m}(i)t^i = (1+\cdots+t^{p-1})^m$, see Notation 1.14(2).

(2) This can be proved by induction on m, with the case of $m = 1$ being clear. For $m \geqslant 2$, we have

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} (M_{p,m}(i) - M_{p,m}(i-1))t^i &= (1-t) \sum_{i=-\infty}^{\infty} M_{p,m}(i)t^i \\
&= (1-t)\left(1+\cdots+t^{p-1}\right)^m \\
&= (1-t^p)\left(1+\cdots+t^{p-1}\right)^{m-1} \\
&= (1-t^p) \sum_{i=-\infty}^{\infty} M_{p,m-1}(i)t^i \\
&= \sum_{i=-\infty}^{\infty} (M_{p,m-1}(i) - M_{p,m-1}(i-p))t^i,
\end{aligned}$$

which yields $M_{p,m}(i) - M_{p,m}(i-1) = M_{p,m-1}(i) - M_{p,m-1}(i-p)$ for all i. From the induction hypothesis on $M_{p,m-1}(i)$ as well as part (1) above applied to $M_{p,m-1}(i)$, it is straightforward to see $M_{p,m}(i) - M_{p,m}(i-1) \geqslant 0$ if $i \leqslant \lceil m(p-1)/2 \rceil$ while $M_{p,m}(i) - M_{p,m}(i-1) \leqslant 0$ if $\lceil m(p-1)/2 \rceil \leqslant i-1$, which establishes the claim. $\square$

Lemma 3.6. *For any integers i and j such that $1 \leqslant i, j \leqslant m-r-1$, we have*

$$p - (m-r-1) \leqslant r(p-1) + pi - j \leqslant m(p-1) - (p - (m-r-1))$$

for all $p \gg 0$.

Proof. The linear functions $f(x) := x-(m-r-1)$, $g(x) := r(x-1)+ix-j = (r+i)x-(r+j)$ and $h(x) := m(x-1) - (x-(m-r-1)) = (m-1)x - (r+1)$ have slopes 1, $r+i$ and $m-1$ respectively, with $1 < r+i \leqslant m-1$ and $g(0) = -(r+j) \leqslant -(r+1) = h(0)$. Thus $f(p) \leqslant g(p) \leqslant h(p)$ for all $p \gg 0$. $\square$

Definition 3.7. For any $t \times s$ matrix $A = [a_{ij}]$ with nonnegative entries, where t, s are positive integers, define $|A| = \min\{a_{ij}\}$ and $\|A\| = \max\{a_{ij}\}$.

Lemma 3.8. *Given m and r, we have the following inequalities:*

$$\binom{m-1+p-(m-r-1)}{m-1} \leqslant |U| \leqslant \|U\| \leqslant \binom{m-1+\lceil \frac{m(p-1)}{2} \rceil}{m-1}$$

for the determining matrix $U = U(p, r, m)$ for all $p \gg 0$.

Proof. This is a consequence of Lemma 3.5 and Lemma 3.6. $\square$

Lemma 3.9. *Let A and B be matrices with nonnegative entries of sizes $l \times t$, respectively $t \times s$, with l, t, s positive integers. Then*

$$t\,|A| \cdot |B| \leqslant |A \cdot B| \leqslant \|A \cdot B\| \leqslant t\|A\| \cdot \|B\|.$$

Proof. This follows from matrix multiplication. $\square$

Now, let us recall that (cf. Discussion 3.2)

$$c_e = Y_0 \cdot X_{e-2} = Y_0 \cdot U^{e-2} \cdot X_0,$$

where

$$X_0 = \begin{bmatrix} X_{0,1} \\ \vdots \\ X_{0,m-r-1} \end{bmatrix}_{(m-r-1)\times 1} \qquad \text{with} \quad X_{0,i} = M_{p,m}(r(p-1)+ip)$$

and

$$Y_0 = \begin{bmatrix} \binom{r(p-1)-1+m-1}{m-1} & \cdots & \binom{r(p-1)-(m-r-1)+m-1}{m-1} \end{bmatrix}_{1\times(m-r-1)}.$$

Lemma 3.10. *For all p, m, r as above, both X_0 and Y_0 are non-zero (and non-negative).*

Proof. Indeed, $m \geqslant r+2$ implies $0 < r(p-1)+p \leqslant r(p-1)+2(p-1) \leqslant m(p-1)$, which implies $X_{0,1} = M_{p,m}(r(p-1)+p) > 0$.

On the other hand, $r(p-1)-1 \geqslant 0$ implies $r(p-1)-1+m-1 \geqslant m-1$, which implies $\binom{r(p-1)-1+m-1}{m-1} > 0$. $\square$

Moreover, both X_0 and Y_0 have all positive entries for $p \gg 0$. In fact we can be more precise.

Lemma 3.11. *If $p \geqslant m-r$, then both X_0 and Y_0 have all positive entries.*

Proof. If $p \geqslant m-r$, then $0 \leqslant r(p-1)+ip \leqslant m(p-1)$ and hence $M_{p,m}(r(p-1)+ip) > 0$, for all $i = 1, \ldots, m-r-1$.

On the other hand, note that $r(m-r)-m+1 = -(r-1)(r-m+1) \geqslant 0$ for all $r = 1, \ldots, m-2$. Consequently, if $p \geqslant m-r$ then for all $i = 1, \ldots, m-r-1$,

$$\begin{aligned} r(p-1)-i+m-1 &\geqslant r(p-1)-(m-r-1)+m-1 \\ &= (rp-m+1)+m-1 \geqslant (r(m-r)-m+1)+m-1 \geqslant m-1, \end{aligned}$$

which leads to $|Y_0| > 0$. $\square$

Proposition 3.12. *We have (with $U = U(p,r,m)$ being the determining matrix)*

$$(m-r-1)^{e-1} \cdot |Y_0| \cdot |U|^{e-2} \cdot |X_0| \leqslant c_e \leqslant (m-r-1)^{e-1} \cdot \|Y_0\| \cdot \|U\|^{e-2} \cdot \|X_0\|.$$

(Also, $(m-r-1)^{e-3} \cdot \|Y_0\| \cdot |U|^{e-2} \cdot \|X_0\| \leqslant c_e$.) Therefore we have

$$(m-r-1)\,|U| \leqslant \operatorname{cx}(T(V_r(\mathscr{R})) \leqslant (m-r-1)\|U\|$$

for all $p \gg 0$, where $\mathscr{R} = R[x_1, \ldots, x_m]$.

Proof. This follows from Lemma 3.9. $\square$

Corollary 3.13. *Let $\mathscr{R} = R[x_1, \ldots, x_m]$. If $p \gg 0$, then*

$$(m-r-1)\binom{m-1+p-(m-r-1)}{m-1} \leqslant \operatorname{cx}(T(V_r(\mathscr{R})) \leqslant (m-r-1)\binom{m-1+\lceil\frac{m(p-1)}{2}\rceil}{m-1}$$

and therefore $\lim_{p\to\infty} \log_p \operatorname{cx}(T(V_r(\mathscr{R})) = m-1$.

Proof. This follows from Lemma 3.8, Lemma 3.11, and Proposition 3.12. $\square$

This corollary motivates our definition of Frobenius complexity in characteristic zero, which is given in Section 4, see Definition 4.4.

3.2. **Perron-Frobenius.** We would like to summarize a few things about square matrices with positive real entries. Any such matrix admits a real positive eigenvalue λ such that all other eigenvalues in $\mathbb{C}$ have absolute values less than λ. We will refer to this eigenvalue as the *Perron root* or *Perron-Frobenius eigenvalue* of the matrix. This eigenvalue is a simple root of the characteristic polynomial of the matrix. Moreover, any real eigenvector for λ either has all entries positive or has all entries negative. See [Pe] and [Fr].

Let $p \gg 0$. Since the determining matrix $U(p, r, m) = U$ has only positive entries by Lemma 3.8, let λ be the Perron-Frobenius eigenvalue for U. There exists an invertible matrix P over $\mathbb{R}$ such that

$$U = PDP^{-1}$$

where D is the rational canonical form of U over $\mathbb{R}$. Without loss of generality, the left upper corner of D is λ and all the other entries of the first row or first column are 0; that is,

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & D_1 \end{bmatrix}_{(m-r-1)\times(m-r-1)}$$

with D_1 being an $(m-r-2) \times (m-r-2)$ matrix whose eigenvalues are all less than λ in absolute value. Hence the first column (resp. row) of P (resp. P^{-1}) is an eigenvector of U (resp. U^T) for λ. Thus, without loss of generality, we may assume that the first column of P and (consequently) the first row of P^{-1} have all positive entries.

Write $Y_0 P = [a,\ A]$ and $P^{-1} X_0 = [b,\ B]^T$ in block form, in which $a, b \in \mathbb{R}$ and A, B are $1 \times (m-r-2)$ matrices. Since both Y_0 and X_0 are non-zero (and clearly non-negative) by Lemma 3.10, both a and b are positive. From Discussion 3.2, we have

$$c_e = Y_0 U^{e-2} X_0 = (Y_0 P) D^{e-2} (P^{-1} X_0) = ab\lambda^{e-2} + A D_1^{e-2} B^T.$$

As all the eigenvalues (in $\mathbb{C}$) of D_1 have absolute values strictly less than λ, we see

$$\lim_{e\to\infty} \frac{\left|A D_1^{e-2} B^T\right|}{\lambda^e} = 0.$$

Thus

$$\operatorname{cx}(T(V_r(\mathscr{R}))) = \lambda.$$

(The above argument applies as long as p, m, r are such that U is all positive, since X_0 and Y_0 are always non-zero by Lemma 3.10.)

4. Frobenius complexity of determinantal rings

In this section, we combine what we have obtained to derive results on the Frobenius complexity of determinantal rings. In particular, we translate the results on $T(V_r(R[x_1, \ldots, x_m]))$ to $S_{m,n}$ with $m > n \geqslant 2$.

Theorem 4.1. *Let K, $S_{m,n}$ and $\mathscr{R}_{m,n}$ be as in Subsection 1.3 (cf. Definition 1.15) with $m > n \geqslant 2$. Further assume that K is a field of prime characteristic p. Let $E_{m,n}$ denote the injective hull of the residue field of $S_{m,n}$.*

(1) *The ring of Frobenius operators of* $S_{m,n}$ *(i.e.,* $\mathscr{F}(E_{m,n})$*) is never finitely generated over* $\mathscr{F}_0(E_{m,n})$*.*
(2) *When* $n = 2$*, we have* $\operatorname{cx}_F(S_{m,2}) = \log_p \binom{p+m-2}{m-1}$*.*
(3) *We have* $\lim_{p\to\infty} \operatorname{cx}_F(S_{m,n}) = m - 1$*.*
(4) *For* $p \gg 0$ *or whenever the determining matrix* $U = U(p, m, m - n)$ *has all positive entries, we have* $\operatorname{cx}_F(S_{m,n}) = \log_p(\lambda)$*, in which* λ *is the Perron root for* U*.*

Proof. (1) Since $m-n \leqslant m-2$, we see that $T(V_{m-n}(K[x_1, \dots, x_m]))$ is not finitely generated over $T_0(V_{m-n}(K[x_1, \dots, x_m]))$ by Theorem 2.3(2). Thus $\mathscr{F}(E_{m,n})$ is not finitely generated over $\mathscr{F}_0(E_{m,n})$ by Theorem 1.20(1).

(2) By Theorem 1.20(2) and Theorem 3.4,

$$\operatorname{cx}_F(S_{m,2}) = \log_p \operatorname{cx}(T(V_{m-2}(K[x_1, \dots, x_m]))) = \log_p \binom{p+m-2}{m-1}.$$

(3) This follows from Corollary 3.13.

(4) This is a straightforward consequence of the discussion in Subsection 3.2. □

Remark 4.2. We like to point out the following (maintaining the notations above):

(1) For every $m > 2$,

$$\lim_{e\to\infty} c_e(\mathscr{F}(E_{m,2})) = \lim_{e\to\infty} c_e(T(V_{m-2}(K[x_1, \dots, x_m]))) = \infty.$$

(2) Moreover, there exists an onto (hence nearly onto) graded ring homomorphism from $T(V_{m-n}(K[x_1, \dots, x_m]))$ to $T(V_{m-n}(K[x_1, \dots, x_{m-n+2}]))$. Thus by Corollary 1.18,

$$c_e(T(V_{m-n}(K[x_1, \dots, x_m]))) \geqslant c_e(T(V_{m-n}(K[x_1, \dots, x_{m-n+2}])))$$

for all $e \geqslant 0$. Hence $c_e(\mathscr{F}(E_{m,n})) \geqslant c_e(\mathscr{F}(E_{m-n+2,2}))$ for all $e \geqslant 0$ and consequently

$$\lim_{e\to\infty} c_e(\mathscr{F}(E_{m,n})) = \infty$$

for all $m > n \geqslant 2$.

4.1. **Example.** We will illustrate our method with a concrete example. We are going to use freely the notations established so far (especially the ones in Section 3).

Let $r = 2$, $m = 5$ and K be a field of characteristic $p = 3$. We are going to compute $c_e = c_e(T(V_2(K[x_1, \dots, x_5])))$, which in turn equals $c_e(\mathscr{F}(E_{5,3}))$ by Theorem 1.20. As in Discussion 3.2, we have

$$X_e = U^e \cdot X_0 \quad \text{for all} \quad e \geqslant 0,$$

in which

$$X_e = \begin{bmatrix} X_{e,1} \\ X_{e,2} \end{bmatrix},$$

$$X_0 = \begin{bmatrix} X_{0,1} \\ X_{0,2} \end{bmatrix} = \begin{bmatrix} M_{3,5}(7) \\ M_{3,5}(10) \end{bmatrix} = \begin{bmatrix} 30 \\ 1 \end{bmatrix},$$

$$U = \begin{bmatrix} M_{3,5}(6) & M_{3,5}(5) \\ M_{3,5}(9) & M_{3,5}(8) \end{bmatrix} = \begin{bmatrix} 45 & 51 \\ 5 & 15 \end{bmatrix}.$$

Note that U has all positive entries and the eigenvalues of U are $2(15+2\sqrt{30})$ and $2(15-2\sqrt{30})$.

At this point, we can apply the Theorem 4.1(4) above directly and determine the Frobenius complexity of $S_{5,3}$ by observing that the Perron root of U is $2(15+2\sqrt{30})$.

However, for illustrative purposes let us compute U^e. This is accomplished by diagonalizing U.

Skipping the details, we get

$$U^e = \begin{bmatrix} (15+4\sqrt{30})y_e + (-15+4\sqrt{30})z_e & 51(y_e - z_e) \\ 5(y_e - z_e) & (-15+4\sqrt{30})y_e + (15+4\sqrt{30})z_e \end{bmatrix},$$

in which

$$y_e := \frac{1}{\sqrt{15}} \cdot 2^{-\frac{7}{2}+e} \cdot (15+2\sqrt{30})^e \quad \text{and} \quad z_e := \frac{1}{\sqrt{15}} \cdot 2^{-\frac{7}{2}+e} \cdot (15-2\sqrt{30})^e.$$

Thus, for $e \geqslant 0$, we obtain

$$\begin{aligned} X_{e,1} &= 30((15+4\sqrt{30})y_e + (-15+4\sqrt{30})z_e) + 51(y_e + z_e), \\ X_{e,2} &= 150(y_e - z_e) + (-15+4\sqrt{30})y_e + (15+4\sqrt{30})z_e. \end{aligned}$$

Lastly, for $e \geqslant 2$, we have (cf. Discussion 3.2)

$$c_e = c_e(T(V_2(K[x_1,\dots,x_5]))) = \binom{7}{4} X_{e-2,1} + \binom{6}{4} X_{e-2,2},$$

which allows us to compute $c_e(T(V_2(K[x_1,\dots,x_5])))$ which equals $c_e(\mathscr{F}(E_{5,3}))$.

Therefore we are led to the following Proposition.

Proposition 4.3. *When $p = 3$, $\mathrm{cx}_F(S_{5,3}) = \log_3(2(15+2\sqrt{30}))$.*

At conclusion of the paper, we would like to introduce the definition of the Frobenius complexity for rings of characteristic zero, which is motivated by Corollary 3.13 and Theorem 4.1(3). As the definition involves rings that may not be local, we first extend our Definition 1.10 by defining the Frobenius complexity of a (not necessarily local) ring R of prime characteristic p as $\mathrm{cx}_F(R) := \log_p(\mathrm{cx}(\mathscr{C}(R)))$. (When $(R, \mathfrak{m}, k)$ is F-finite complete local, $\mathscr{C}(R)$ and $\mathscr{F}(E(k))$ are opposite as graded rings; so $\mathrm{cx}(\mathscr{C}(R)) = \mathrm{cx}(\mathscr{F}(E(k)))$ and we do have an extension of the definition.)

Definition 4.4. Let R be a ring (of characteristic zero) such that $R/pR \neq 0$ for almost all prime number p. When the limit $\lim_{p\to\infty} \mathrm{cx}_F(R/pR)$ exists, we call it *the Frobenius complexity* of R.

It is natural to ask under what conditions, if any at all, the Frobenius complexity exists. The case of $R = \mathbb{Z}[X_1, \dots, X_n]/I$ and $R = \mathbb{Z}[[X_1, \dots, X_n]]/I$ are particularly interesting. If R is a finitely generated algebra over a field k of characteristic zero, we could descend R to a finitely generated A-algebra R_A (where A is a subring of k that is finitely generated over $\mathbb{Z}$ containing the defining data of R) and study the the Frobenius complexity of R_A via reduction to prime characteristic p.

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References

[BST] M. Blickle, K. Schwede, K. Tucker, *F-signature of pairs and the asymptotic behavior of Frobenius splittings*. Adv. Math. **231** (2012), no. 6, 3232–3258.

[EH] F. Enescu, M. Hochster, *The Frobenius structure of local cohomology*, Algebra & Number Theory **2** (7) (2008), 721–754.

[EY] F. Enescu, Y. Yao, *The Frobenius complexity of a local ring of prime characteristic*, J. Algebra **459** (2016), 133–156.

[Fr] G. Frobenius, *Ueber Matrizen aus nicht negativen Elementen*, Sitzungsber. Königl. Preuss. Akad. Wiss. (1912), 456–477.

[HH] M. Hochster, C. Huneke, *Tight closure, invariant theory, and the BrianonSkoda theorem*, J. Amer. Math. Soc. **3** (1990) 31–116.

[K] M. Katzman, *Frobenius maps on injective hulls and their applications to tight closure*. J. Lond. Math. Soc. 81 (2010), 589–607.

[KSSZ] M. Katzman, K. Schwede, A. K. Singh, W. Zhang, *Rings of Frobenius operators*, Math. Proc. Cambridge Phil. Soc. **157** (2014), no. 1, 151–167.

[Ly] G. Lyubeznik, *F-modules: applications to local cohomology and D-modules in characteristic p¿0*, J. reine angew. Math. **491** (1997) 65–130.

[LS] G. Lyubeznik and K. E. Smith, *On the commutation of the test ideal with localization and completion*, Trans. Amer. Math. Soc. **353** (2001) 3149–3180.

[Sc] K. Schwede, *Test ideals in non-$\mathbb{Q}$-Gorenstein rings*, Trans. Amer. Math. Soc. 363 (11) (2011) 5925–5941.

[Sh] R. Y. Sharp, *An excellent F-pure ring of prime characteristic has a big tight closure test element'*, Trans. Amer. Math. Soc.. **362** (2010), no. 10, 5455–5481.

[Pe] O. Perron, *Zur Theorie der Matrices*, Math. Ann. **64** (1907), no. 2, 248–263.

[Wa] K.-i. Watanabe, *Infinite cyclic covers of strongly F-regular rings*, Contemporary Mathematics **159** (1994), 423–432.

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