◇ ◇ ◇ ◇ **MATH 8220: ABSTRACT ALGEBRA I** ◇ ◇ ◇ ◇
**HOMEWORK SETS AND EXAMS**

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2026 FALL SEMESTER
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Contents



Each homework set contains four (4) regular problems. When solving the problems, make sure your arguments are rigorous and complete.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

**Problem 1.1.** Consider $f(x) = 2x^3 - x^2 + x + 1$, $g(x) = x^3 + 2x^2 + 3x + 1 \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$.

(1) Determine whether $f(x)$ has a root in $\mathbb{Q}$.
(2) Determine whether $f(x)$ is irreducible in $\mathbb{Q}[x]$.
(3) Is $f(x)$ is irreducible in $\mathbb{Z}[x]$? If not, find a non-trivial factorization of $f(x)$ in $\mathbb{Z}[x]$.
(4) Determine whether $g(x)$ has a root in $\mathbb{Q}$.
(5) Determine whether $g(x)$ is irreducible in $\mathbb{Q}[x]$. Is $g(x)$ irreducible in $\mathbb{Z}[x]$?

*Solution.* (1) By the Rational Root Theorem, the (possible) rational roots of $f(x)$ must be from $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}\right\}$. Direct computation (of $f(\pm 1)$ and $f(\pm\frac{1}{2})$) shows

$$f(1) \neq 0, \quad f(-1) \neq 0, \quad f(\tfrac{1}{2}) \neq 0, \quad f(-\tfrac{1}{2}) = 0.$$

Therefore $f(x)$ has a root $-\frac{1}{2} \in \mathbb{Q}$. (In fact, $-\frac{1}{2}$ is the only rational root of $f(x)$.)

(2) Since $\deg(f(x)) > 1$ and $f(x)$ has a root in $\mathbb{Q}$, we see $f(x)$ is not irreducible in $\mathbb{Q}[x]$.

(3) Since $f(x) \in \mathbb{Z}[x]$ is not irreducible in $\mathbb{Q}[x]$, it is not irreducible in $\mathbb{Z}[x]$. By long division, we get $f(x) = (x + \frac{1}{2})(2x^2 - 2x + 2)$ over $\mathbb{Q}$. Correspondingly, we obtain

$$f(x) = (2x + 1)(x^2 - x + 1),$$

which is a non-trivial factorization of $f(x)$ in $\mathbb{Z}[x]$.

(4) By the Rational Root Theorem, the (possible) rational roots of $g(x)$ must be from $\left\{\frac{1}{1}, -\frac{1}{1}\right\}$. However, $g(1) \neq 0$ and $g(-1) \neq 0$. So $g(x)$ has no root in $\mathbb{Q}$.

(5) In light of (4) and the fact $\deg(g(x)) = 3$, we see $g(x)$ is irreducible in $\mathbb{Q}[x]$. Moreover, because $g(x)$ is primitive, $g(x)$ is irreducible also in $\mathbb{Z}[x]$.

**Problem 1.2.** Consider $h(x) = x^3 + \overline{2}x^2 + \overline{3}x + \overline{1} \in \mathbb{Z}_5[x]$, where $\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$.

(1) Determine whether $h(x)$ has a root in $\mathbb{Z}_5$.
(2) Determine whether $h(x)$ is irreducible in $\mathbb{Z}_5[x]$.
(3) Write $h(x)$ as a product of *monic* irreducible polynomials in $\mathbb{Z}_5[x]$. Explain why each of the factors is irreducible.

*Solution.* (1) By computing $h(\overline{0})$, $h(\overline{1})$, $h(\overline{2})$, $h(\overline{3})$ and $h(\overline{4})$ directly, we see

$$h(\overline{3}) = \overline{0}.$$

(2) Since $\deg(h(x)) \geqslant 2$ and $h(x)$ has a root in $\mathbb{Z}_5$, we see $h(x)$ is not irreducible in $\mathbb{Z}_5[x]$.

(3) Dividing $h(x)$ by $x - \overline{3}$ via long division, we get $h(x) = (x - \overline{3})(x^2 + \overline{3})$. By exhaustion, we see $x^2 + \overline{3}$ has no root in $\mathbb{Z}_5$, which implies $x^2 + \overline{3}$ is irreducible in $\mathbb{Z}_5[x]$. Thus

$$h(x) = (x - \overline{3})(x^2 + \overline{3}),$$

which is a product of monic irreducible polynomials in $\mathbb{Z}_5[x]$, as required.

**Problem 1.3.** Show that each the following polynomials is irreducible in $\mathbb{Q}[x]$.

(1) $f_1(x) = 3x^4 - 7x^3 + 7x^2 + 7$.
(2) $f_2(x) = 2x^4 - 90x^3 + 63x^2 - 84x + 105$.
(3) $f_3(x) = 2x^4 - 24x^3 + 48x^2 - 12x + 28$.

*Proof.* (1) Applying Eisenstein's Criterion with $p = 7$, we see $f_1(x)$ is irreducible in $\mathbb{Q}[x]$.

(2) Applying Eisenstein's Criterion with $p = 3$ to $f_2(x)$, we see $f_2(x)$ is irreducible in $\mathbb{Q}[x]$.

(3) As $f_3(x) = 2(x^4 - 12x^3 + 24x^2 - 6x + 14)$, it suffices to show $x^4 - 12x^3 + 24x^2 - 6x + 14$ is irreducible in $\mathbb{Q}[x]$. Applying Eisenstein's Criterion with $p = 2$ to $x^4 - 12x^3 + 24x^2 - 6x + 14$, we see $x^4 - 12x^3 + 24x^2 - 6x + 14$ is irreducible in $\mathbb{Q}[x]$. Thus $f_3(x)$ is also irreducible in $\mathbb{Q}[x]$. (Note that $f_3(x)$ is reducible in $\mathbb{Z}[x]$ though.) $\square$

**Problem 1.4.** Consider the polynomial $p(x) = x^3 + 2x^2 - 4x + 6$, which is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion. Let $u \in \mathbb{C}$ be a (fixed) root of $p(x)$. (Such a root exists in $\mathbb{C}$. In fact, $p(x)$ has at least one root in $\mathbb{R}$ by the Intermediate Value Theorem in calculus.) Consider $\mathbb{Q}[u] = \{a_0 + a_1u + a_2u^2 \mid a_i \in \mathbb{Q}\}$, which is a ring. In fact, $\mathbb{Q}[u]$ is a field. This exercise illustrates how to find the inverse of a (typical) non-zero element in $\mathbb{Q}[u]$. (Here $u$ is not an indeterminate, and $\mathbb{Q}[u]$ is not a polynomial ring.)

As an example, we compute the inverse of $2 + 3u$ and illustrate that it is indeed in $\mathbb{Q}[u]$. Consider the polynomial $f(x) = 3x + 2 \in \mathbb{Q}[x]$. Complete the following:

(1) Find $\gcd(p(x), f(x))$ by the Euclidean Algorithm (repeated division) for polynomials.
(2) Use your work in (1) to express $\gcd(p(x), f(x))$ as a linear combination of $p(x)$ and $f(x)$. That is, find $a(x)$, $b(x) \in \mathbb{Q}[x]$ such that $\gcd(p(x), f(x)) = a(x)p(x)+b(x)f(x)$.
(3) Compute $b(u)f(u)$.
(4) Finally, show $(2 + 3u)^{-1} \in \mathbb{Q}[u]$ by writing $(2 + 3u)^{-1}$ in the form of $a_0 + a_1u + a_2u^2$ with $a_i \in \mathbb{Q}$.

*Solution.* (1) To find $\gcd(p(x), f(x))$, we apply the Euclidean Algorithm as follows

$$\begin{aligned} p(x) &= (\tfrac{1}{3}x^2 + \tfrac{4}{9}x - \tfrac{44}{27})f(x) + \tfrac{250}{27}, \\ f(x) &= (\tfrac{27}{250}f(x))\tfrac{250}{27} + 0. \end{aligned}$$

The monic polynomial associated with $\frac{250}{27}$ is 1. Thus $\gcd(p(x), f(x)) = 1$.

(2) From the work in (1) above, we have $\frac{250}{27} = p(x) - (\frac{1}{3}x^2 + \frac{4}{9}x - \frac{44}{27})f(x)$. Therefore

$$\begin{aligned} 1 = \tfrac{27}{250} \cdot \tfrac{250}{27} &= \tfrac{27}{250}\left[p(x) - (\tfrac{1}{3}x^2 + \tfrac{4}{9}x - \tfrac{44}{27})f(x)\right] \\ &= \tfrac{27}{250}p(x) - \tfrac{27}{250}(\tfrac{1}{3}x^2 + \tfrac{4}{9}x - \tfrac{44}{27})f(x) \\ &= \tfrac{27}{250}p(x) + (-\tfrac{9}{250}x^2 - \tfrac{6}{125}x + \tfrac{22}{125})f(x). \end{aligned}$$

(3) Evaluating $1 = \frac{27}{250}p(x) + (-\frac{9}{250}x^2 - \frac{6}{125}x + \frac{22}{125})f(x)$ at $x = u$ and observing $p(u) = 0$, we see

$$1 = \tfrac{27}{250}p(u) + (-\tfrac{9}{250}u^2 - \tfrac{6}{125}u + \tfrac{22}{125})f(u) = (-\tfrac{9}{250}u^2 - \tfrac{6}{125}u + \tfrac{22}{125})(2 + 3u).$$

(4) Thus $\boxed{(2 + 3u)^{-1} = \frac{22}{125} - \frac{6}{125}u - \frac{9}{250}u^2 \in \mathbb{Q}[u]}$, as required.

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