◇ ◇ ◇ ◇ **MATH 8220: ABSTRACT ALGEBRA I** ◇ ◇ ◇ ◇

HOMEWORK SETS AND EXAMS

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Contents

Each homework set contains four (4) regular problems. When solving the problems, make sure your arguments are rigorous and complete.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

Problem 1.1. Consider $f(x) = 2x^3 - x^2 + x + 1$, $g(x) = x^3 + 2x^2 + 3x + 1 \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$.

(1) Determine whether $f(x)$ has a root in $\mathbb{Q}$.
(2) Determine whether $f(x)$ is irreducible in $\mathbb{Q}[x]$.
(3) Is $f(x)$ is irreducible in $\mathbb{Z}[x]$? If not, find a non-trivial factorization of $f(x)$ in $\mathbb{Z}[x]$.
(4) Determine whether $g(x)$ has a root in $\mathbb{Q}$.
(5) Determine whether $g(x)$ is irreducible in $\mathbb{Q}[x]$. Is $g(x)$ irreducible in $\mathbb{Z}[x]$?

Problem 1.2. Consider $h(x) = x^3 + \overline{2}x^2 + \overline{3}x + \overline{1} \in \mathbb{Z}_5[x]$, where $\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$.

(1) Determine whether $h(x)$ has a root in $\mathbb{Z}_5$.
(2) Determine whether $h(x)$ is irreducible in $\mathbb{Z}_5[x]$.
(3) Write $h(x)$ as a product of *monic* irreducible polynomials in $\mathbb{Z}_5[x]$. Explain why each of the factors is irreducible.

Problem 1.3. Show that each the following polynomials is irreducible in $\mathbb{Q}[x]$.

(1) $f_1(x) = 3x^4 - 7x^3 + 7x^2 + 7$.
(2) $f_2(x) = 2x^4 - 90x^3 + 63x^2 - 84x + 105$.
(3) $f_3(x) = 2x^4 - 24x^3 + 48x^2 - 12x + 28$.

Problem 1.4. Consider the polynomial $p(x) = x^3 + 2x^2 - 4x + 6$, which is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion. Let $u \in \mathbb{C}$ be a (fixed) root of $p(x)$. (Such a root exists in $\mathbb{C}$. In fact, $p(x)$ has at least one root in $\mathbb{R}$ by the Intermediate Value Theorem in calculus.) Consider $\mathbb{Q}[u] = \{a_0 + a_1u + a_2u^2 \mid a_i \in \mathbb{Q}\}$, which is a ring. In fact, $\mathbb{Q}[u]$ is a field. This exercise illustrates how to find the inverse of a (typical) non-zero element in $\mathbb{Q}[u]$. (Here u is not an indeterminate, and $\mathbb{Q}[u]$ is not a polynomial ring.)

As an example, we compute the inverse of $2 + 3u$ and illustrate that it is indeed in $\mathbb{Q}[u]$. Consider the polynomial $f(x) = 3x + 2 \in \mathbb{Q}[x]$. Complete the following:

(1) Find $\gcd(p(x), f(x))$ by the Euclidean Algorithm (repeated division) for polynomials.
(2) Use your work in (1) to express $\gcd(p(x), f(x))$ as a linear combination of $p(x)$ and $f(x)$. That is, find $a(x)$, $b(x) \in \mathbb{Q}[x]$ such that $\gcd(p(x), f(x)) = a(x)p(x) + b(x)f(x)$.
(3) Compute $b(u)f(u)$.
(4) Finally, show $(2 + 3u)^{-1} \in \mathbb{Q}[u]$ by writing $(2 + 3u)^{-1}$ in the form of $a_0 + a_1u + a_2u^2$ with $a_i \in \mathbb{Q}$.

PROBLEMS HINTS SOLUTIONS

Problem 2.1. Let $F \subseteq K$ be a field extension such that $[K : F] < \infty$. Let $\alpha \in K$ and $p(x)$ be the minimal polynomial of α over F.

(1) Prove that if $\deg(p(x)) > \frac{1}{2}[K : F]$ then $F(\alpha) = K$.
(2) Prove that if $[K : F]$ is a prime number and $\alpha \in K \setminus F$ then $F(\alpha) = K$.

Problem 2.2. Let $F \subseteq K$ be a field extension, $\omega \in K$ and $p(x) \in F[x]$. Prove that, if $p(x)$ is monic and irreducible in $F[x]$ such that $p(\omega) = 0$, then $p(x)$ is the minimal polynomial of ω over F.

Problem 2.3. Consider $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{R}$. Show $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Problem 2.4. Consider the field extension $\mathbb{Q} \subseteq \mathbb{Q}(u) \subseteq \mathbb{C}$ in which $u \in \mathbb{C}$ is a (fixed) root of $p(x) = x^3 + 2x^2 - 4x + 6$ and $\mathbb{Q}(u) = \mathbb{Q}[u] = \{a_0 + a_1 u + a_2 u^2 \mid a_i \in \mathbb{Q}\}$; see Problem 1.4. Express $(3 - 2u + u^2)^{-1}$ in the form of $a_0 + a_1 u + a_2 u^2$ with $a_i \in \mathbb{Q}$.

PROBLEMS HINTS SOLUTIONS