◇ ◇ ◇ ◇ **MATH 8220: ABSTRACT ALGEBRA I** ◇ ◇ ◇ ◇

HOMEWORK SETS AND EXAMS

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Contents

Each homework set contains four (4) regular problems. When solving the problems, make sure your arguments are rigorous and complete.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

Problem 1.1. Consider $f(x) = 2x^3 - x^2 + x + 1$, $g(x) = x^3 + 2x^2 + 3x + 1 \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$.

(1) Determine whether $f(x)$ has a root in $\mathbb{Q}$.
(2) Determine whether $f(x)$ is irreducible in $\mathbb{Q}[x]$.
(3) Is $f(x)$ is irreducible in $\mathbb{Z}[x]$? If not, find a non-trivial factorization of $f(x)$ in $\mathbb{Z}[x]$.
(4) Determine whether $g(x)$ has a root in $\mathbb{Q}$.
(5) Determine whether $g(x)$ is irreducible in $\mathbb{Q}[x]$. Is $g(x)$ irreducible in $\mathbb{Z}[x]$?

Hint. This should be straightforward. Use the results covered in class.

Problem 1.2. Consider $h(x) = x^3 + \overline{2}x^2 + \overline{3}x + \overline{1} \in \mathbb{Z}_5[x]$, where $\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$.

(1) Determine whether $h(x)$ has a root in $\mathbb{Z}_5$.
(2) Determine whether $h(x)$ is irreducible in $\mathbb{Z}_5[x]$.
(3) Write $h(x)$ as a product of *monic* irreducible polynomials in $\mathbb{Z}_5[x]$. Explain why each of the factors is irreducible.

Hint. Well, $\mathbb{Z}_5$ is a *finite* field. A monic polynomial has leading coefficient 1 by definition.

Problem 1.3. Show that each the following polynomials is irreducible in $\mathbb{Q}[x]$.

(1) $f_1(x) = 3x^4 - 7x^3 + 7x^2 + 7$.
(2) $f_2(x) = 2x^4 - 90x^3 + 63x^2 - 84x + 105$.
(3) $f_3(x) = 2x^4 - 24x^3 + 48x^2 - 12x + 28$.

Hint. Use Eisenstein's Criterion.

Problem 1.4. Consider the polynomial $p(x) = x^3 + 2x^2 - 4x + 6$, which is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion. Let $u \in \mathbb{C}$ be a (fixed) root of $p(x)$. (Such a root exists in $\mathbb{C}$. In fact, $p(x)$ has at least one root in $\mathbb{R}$ by the Intermediate Value Theorem in calculus.) Consider $\mathbb{Q}[u] = \{a_0 + a_1u + a_2u^2 \mid a_i \in \mathbb{Q}\}$, which is a ring. In fact, $\mathbb{Q}[u]$ is a field. This exercise illustrates how to find the inverse of a (typical) non-zero element in $\mathbb{Q}[u]$. (Here u is not an indeterminate, and $\mathbb{Q}[u]$ is not a polynomial ring.)

As an example, we compute the inverse of $2 + 3u$ and illustrate that it is indeed in $\mathbb{Q}[u]$. Consider the polynomial $f(x) = 3x + 2 \in \mathbb{Q}[x]$. Complete the following:

(1) Find $\gcd(p(x), f(x))$ by the Euclidean Algorithm (repeated division) for polynomials.
(2) Use your work in (1) to express $\gcd(p(x), f(x))$ as a linear combination of $p(x)$ and $f(x)$. That is, find $a(x)$, $b(x) \in \mathbb{Q}[x]$ such that $\gcd(p(x), f(x)) = a(x)p(x)+b(x)f(x)$.
(3) Compute $b(u)f(u)$.
(4) Finally, show $(2 + 3u)^{-1} \in \mathbb{Q}[u]$ by writing $(2 + 3u)^{-1}$ in the form of $a_0 + a_1u + a_2u^2$ with $a_i \in \mathbb{Q}$.

Hint. Just complete (1), (2), (3) and (4) as instructed. In (3), just evaluate the equation derived in (2) at $x = u$. You do **not** need to prove any of the claims in the first paragraph (such as that $\mathbb{Q}[u]$ is a field, etc.). **The solution is very short actually.**

PROBLEMS HINTS SOLUTIONS

Problem 2.1. Let $F \subseteq K$ be a field extension such that $[K:F] < \infty$. Let $\alpha \in K$ and $p(x)$ be the minimal polynomial of α over F.

(1) Prove that if $\deg(p(x)) > \frac{1}{2}[K:F]$ then $F(\alpha) = K$.
(2) Prove that if $[K:F]$ is a prime number and $\alpha \in K \setminus F$ then $F(\alpha) = K$.

Hint. In both (1) and (2), use the relation among $[K:F]$, $[K:F(\alpha)]$ and $[F(\alpha):F]$. Feel free to use the **fact** (an easy fact from linear algebra, in disguise) that if $F \subseteq K_1 \subseteq K_2$ is a field extension such that $[K_1:F] = [K_2:F] < \infty$ then $K_1 = K_2$.

Problem 2.2. Let $F \subseteq K$ be a field extension, $\omega \in K$ and $p(x) \in F[x]$. Prove that, if $p(x)$ is monic and irreducible in $F[x]$ such that $p(\omega) = 0$, then $p(x)$ is the minimal polynomial of ω over F.

Hint. Let $m(x)$ be the minimal polynomial of ω over F. Can you show $p(x) = m(x)$?

Problem 2.3. Consider $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{R}$. Show $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Hint. Here is a direct and short approach: To show $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \supseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$, it suffices to show that both $\sqrt{2}$ and $\sqrt{3}$ are in $\mathbb{Q}(\sqrt{2}+\sqrt{3})$. Note that $(\sqrt{2}+\sqrt{3})^n$ are all in $\mathbb{Q}(\sqrt{2}+\sqrt{3})$. Can you "get" both $\sqrt{2}$ and $\sqrt{3}$ from the powers of $\sqrt{2}+\sqrt{3}$? Do the algebra! (Other ways are available, and are welcome.)

Problem 2.4. Consider the field extension $\mathbb{Q} \subseteq \mathbb{Q}(u) \subseteq \mathbb{C}$ in which $u \in \mathbb{C}$ is a (fixed) root of $p(x) = x^3 + 2x^2 - 4x + 6$ and $\mathbb{Q}(u) = \mathbb{Q}[u] = \{a_0 + a_1 u + a_2 u^2 \mid a_i \in \mathbb{Q}\}$; see Problem 1.4. Express $(3 - 2u + u^2)^{-1}$ in the form of $a_0 + a_1 u + a_2 u^2$ with $a_i \in \mathbb{Q}$.

Hint. See Problem 1.4. Here it takes more steps to complete the Euclidean Algorithm.

PROBLEMS HINTS SOLUTIONS