◇ ◇ ◇ ◇ **MATH 4441/6441: MODERN ALGEBRA I** ◇ ◇ ◇ ◇

HOMEWORK SETS AND EXAMS

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Contents

There are four (4) problems in each homework set. Math 6441 students need to do all 4 problems while Math 4441 students need to do any three (3) problems out the four. If a Math 4441 student submits all 4 problems, then one of the lowest score(s) is dropped. There is a bonus point for Math 4441 students solving all 4 problems correctly/perfectly.

When solving homework problems, make sure that your arguments and computations are rigorous, accurate, and complete. Present your step-by-step work in your solutions/proofs.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

$(G, *) \ldots H \leqslant G \ldots |G| = [G : H]\cdot|H| \ldots a^{|G|} = e \ldots \varphi\colon G \to G',\ \varphi(ab) = \varphi(a)\varphi(b) \ldots N \trianglelefteq G \ldots G/N \ldots G/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi)$

Problem 1.1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6, 7\}$ and $C = \{3, 6, 7, 8\}$ be sets.

(1) Compute $(A \setminus B) \cap C$ and $A \setminus (B \cap C)$. Are they equal?
(2) Compute $(A \cap B) \cup C$ and $A \cap (B \cup C)$. Are they equal?

Solution. (1) First, we see

$$A \setminus B = \{1, 4\},$$
$$B \cap C = \{3, 6, 7\}.$$

In light in the computation above, we get

$$(A \setminus B) \cap C = \{1, 4\} \cap \{3, 6, 7, 8\} = \varnothing$$
$$A \setminus (B \cap C) = \{1, 2, 3, 4\} \setminus \{3, 6, 7\} = \{1, 2, 4\}.$$

Thus $(A \setminus B) \cap C \neq A \setminus (B \cap C)$ (in this particular case).

(2) We have

$$A \cap B = \{2, 3\},$$
$$B \cup C = \{2, 3, 5, 6, 7, 8\}.$$

In light in the computation above, we see

$$(A \cap B) \cup C = \{2, 3\} \cup \{3, 6, 7, 8\} = \{2, 3, 6, 7, 8\}$$
$$A \cap (B \cup C) = \{1, 2, 3, 4\} \cap \{2, 3, 5, 6, 7, 8\} = \{2, 3\}.$$

Thus $(A \cap B) \cup C \neq A \cap (B \cup C)$ (in this particular case).

Problem 1.2. Let A, B and C be sets. Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof. It suffices to show $x \in A \setminus (B \cup C) \Leftrightarrow x \in (A \setminus B) \cap (A \setminus C)$. Indeed, we have

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\ &\iff x \in A \text{ and } [x \notin B \text{ and } x \notin C] \\ &\iff [x \in A \text{ and } x \notin B] \text{ and } [x \in A \text{ and } x \notin C] \\ &\iff x \in A \setminus B \text{ and } x \in A \setminus C \\ &\iff x \in (A \setminus B) \cap (A \setminus C). \end{aligned}$$

□

Problem 1.3. For each function f_i, determine whether it is injective but not surjective, surjective but not injective, bijective, or neither injective nor surjective. Explain why.

(1) $f_1 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $f_1(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$, where $\mathbb{R}_{\geqslant 0} = \{x \in \mathbb{R} \mid x \geqslant 0\} = [0, \infty)$.
(2) $f_2 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $f_2(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$.
(3) $f_3 \colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with $f_3(x) = x^4$ for all $x \in \mathbb{R}$.
(4) $f_4 \colon \mathbb{R} \to \mathbb{R}$ with $f_4(x) = 10^{x^2}$ for all $x \in \mathbb{R}$. (Here 10^{x^2} stands for $10^{(x^2)}$, not $(10^x)^2$.)

Solution. (1) The function f_1 is injective but not surjective. It is injective (i.e., 1-1) because for if $x_1, x_2 \in \mathbb{R}_{\geqslant 0}$ such that $x_1 \neq x_2$ (meaning x_1, x_2 are distinct non-negative real numbers) then $x_1^2 \neq x_2^2$ hence $f_1(x_1) \neq f_1(x_2)$. The function f_1 is not onto $\mathbb{R}$ since, for $-1 \in \mathbb{R}$, there is no $x \in \mathbb{R}_{\geqslant 0}$ such that $f_1(x) = -1$.

(2) The function f_2 is bijective. This is because for for every $y \in \mathbb{R}_{\geqslant 0}$ (hence $y \geqslant 0$), there exists $\sqrt{y} \in \mathbb{R}_{\geqslant 0}$ such that $f_1(\sqrt{y}) = (\sqrt{y})^2 = y$ and, moreover, $\sqrt{y}$ is the only preimage (in $\mathbb{R}_{\geqslant 0}$, the domain of f_2) of y under f_2.

(3) The function f_3 is surjective but not injective. It is surjective because for every $y \in \mathbb{R}_{\geqslant 0}$ (hence $y \geqslant 0$), there exists $\sqrt[4]{y} \in \mathbb{R}$ such that $f_3(\sqrt[4]{y}) = (\sqrt[4]{y})^4 = y$. It is not injective since $f_3(1) = 1 = f_3(-1)$ while $1 \neq -1$.

(4) The function f_4 is neither injective nor surjective. It is not injective since $f_4(-2) = 10^4 = f_4(2)$ while $-2 \neq 2$. To see why f_4 is not surjective, just notice that there is no $x \in \mathbb{R}$ such that $f_4(x) = -13$.

Problem 1.4. Let A, B and C be sets.

(1) Find a concrete example of A, B and C such that $(A \cup B) \cap C \subsetneq A \cup (B \cap C)$.
(2) Prove $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

Solution/Proof. (1) Let $A = B = \{1\}$ and $C = \varnothing$, for example. Then

$$B \cap C = \varnothing \quad \text{and} \quad A \cup B = \{1\}.$$

Consequently, we see

$$A \cup (B \cap C) = \{1\} \cup \varnothing = \{1\} \quad \text{and} \quad (A \cup B) \cap C = \{1\} \cap \varnothing = \varnothing,$$

which illustrates $(A \cup B) \cap C \subsetneq A \cup (B \cap C)$ (for these specific sets A, B and C). (`However, one should not conclude that the statement` $(A \cup B) \cap C \subsetneq A \cup (B \cap C)$ `holds for all sets` A, B `and` C `--- just consider the case when` $A = \varnothing$ `for example.`)

(2) It suffices to show $x \in (A \cup B) \cap C \Rightarrow x \in A \cup (B \cap C)$. Indeed, we have

$$\begin{aligned} x \in (A \cup B) \cap C &\iff x \in A \cup B \text{ and } x \in C \\ &\iff [x \in A \text{ or } x \in B] \text{ and } x \in C \\ &\iff [x \in A \text{ and } x \in C] \text{ or } [x \in B \text{ and } x \in C] \\ &\stackrel{*}{\Longrightarrow} x \in A \text{ or } [x \in B \text{ and } x \in C] \\ &\iff x \in A \text{ or } x \in B \cap C \\ &\iff x \in A \cup (B \cap C). \end{aligned}$$

Note the implication $\stackrel{*}{\Longrightarrow}$ in the above argument—the reverse implication fails in general (cf. part (1) above). This completes the proof of $(A \cup B) \cap C \subseteq A \cup (B \cap C)$, as required. $\square$

PROBLEMS HINTS SOLUTIONS