◇ ◇ ◇ ◇ **MATH 4441/6441: MODERN ALGEBRA I** ◇ ◇ ◇ ◇

HOMEWORK SETS AND EXAMS

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Contents

There are four (4) problems in each homework set. Math 6441 students need to do all 4 problems while Math 4441 students need to do any three (3) problems out the four. If a Math 4441 student submits all 4 problems, then one of the lowest score(s) is dropped. There is a bonus point for Math 4441 students solving all 4 problems correctly/perfectly.

When solving homework problems, make sure that your arguments and computations are rigorous, accurate, and complete. Present your step-by-step work in your solutions/proofs.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

$(G,*) \ldots H \leqslant G \ldots |G| = [G:H]\cdot|H| \ldots a^{|G|} = e \ldots \varphi\colon G \to G',\ \varphi(ab) = \varphi(a)\varphi(b) \ldots N \trianglelefteq G \ldots G/N \ldots G/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi)$

Problem 1.1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6, 7\}$ and $C = \{3, 6, 7, 8\}$ be sets.

(1) Compute $(A \setminus B) \cap C$ and $A \setminus (B \cap C)$. Are they equal?
(2) Compute $(A \cap B) \cup C$ and $A \cap (B \cup C)$. Are they equal?

Problem 1.2. Let A, B and C be sets. Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Problem 1.3. For each function f_i, determine whether it is injective but not surjective, surjective but not injective, bijective, or neither injective nor surjective. Explain why.

(1) $f_1 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $f_1(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$, where $\mathbb{R}_{\geqslant 0} = \{x \in \mathbb{R} \mid x \geqslant 0\} = [0, \infty)$.
(2) $f_2 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $f_2(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$.
(3) $f_3 \colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with $f_3(x) = x^4$ for all $x \in \mathbb{R}$.
(4) $f_4 \colon \mathbb{R} \to \mathbb{R}$ with $f_4(x) = 10^{x^2}$ for all $x \in \mathbb{R}$. (Here 10^{x^2} stands for $10^{(x^2)}$, not $(10^x)^2$.)

Problem 1.4. Let A, B and C be sets.

(1) Find a concrete example of A, B and C such that $(A \cup B) \cap C \subsetneq A \cup (B \cap C)$.
(2) Prove $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

PROBLEMS HINTS SOLUTIONS

Problem 2.1. Let $X = \{a, b\}$, $Y = \{1, 2\}$ and $Z = \{x, y, z\}$.

(1) Find all functions from X to Y.
(2) Find all injective functions from X to Y, if they exist.
(3) Write down all surjective functions from Y to Z, if they exist.
(4) Write down all **non**-injective functions from Y to Z, if they exist.

Problem 2.2. Let S_3 denote the set of all bijective functions from $X = \{1, 2, 3\}$ to itself. Let $\varphi \in S_3$ and $\psi \in S_3$ be defined as follows

$$\varphi\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3 \qquad \text{and} \qquad \psi\colon\ 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2.$$

(1) Determine $\varphi \circ \psi$ and $\psi \circ \varphi$ explicitly. Are they equal?
(2) Determine φ^{-1} and ψ^{-1} explicitly.
(3) Determine φ^2 and φ^3 explicitly. Is anyone of the two equal to I_X?
(4) Determine ψ^2 and ψ^3 explicitly. Is anyone of the two equal to I_X?

Problem 2.3. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions, in which X, Y and Z are non-empty sets.

(1) If both f and g are surjective (i.e., onto), prove that $g \circ f$ is surjective.
(2) **Disprove**: If $g \circ f$ is surjective (i.e., onto), then both f and g are surjective.

Problem 2.4. Let $f,\ f_1,\ f_2\colon X \to Y$ and $g,\ g_1,\ g_2\colon Y \to Z$ be functions, in which X, Y and Z are (non-empty) sets.

(1) Prove that if g is 1–1 (i.e., injective) and $g \circ f_1 = g \circ f_2$, then $f_1 = f_2$.
(2) **Disprove** the statement: If $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$.