⋄ ⋄ ⋄ ⋄ MATH 4441/6441: MODERN ALGEBRA I ⋄ ⋄ ⋄ ⋄
HOMEWORK SETS AND EXAMS

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Contents

There are four (4) problems in each homework set. Math 6441 students need to do all 4 problems while Math 4441 students need to do any three (3) problems out the four. If a Math 4441 student submits all 4 problems, then one of the lowest score(s) is dropped. There is a bonus point for Math 4441 students solving all 4 problems correctly/perfectly.

When solving homework problems, make sure that your arguments and computations are rigorous, accurate, and complete. Present your step-by-step work in your solutions/proofs.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

$(G,*) \dots H \leqslant G \dots |G| = [G:H]\cdot|H| \dots a^{|G|} = e \dots \varphi\colon G \to G',\ \varphi(ab) = \varphi(a)\varphi(b) \dots N \trianglelefteq G \dots G/N \dots G/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi)$

Problem 1.1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6, 7\}$ and $C = \{3, 6, 7, 8\}$ be sets.

(1) Compute $(A \setminus B) \cap C$ and $A \setminus (B \cap C)$. Are they equal?
(2) Compute $(A \cap B) \cup C$ and $A \cap (B \cup C)$. Are they equal?

Hint. All should be straightforward. Determine each set by listing its elements explicitly. (Note that $A \setminus B$ may be also denoted by $A - B$.)

Problem 1.2. Let A, B and C be sets. Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Hint. How to show two sets are equal?

Problem 1.3. For each function f_i, determine whether it is injective but not surjective, surjective but not injective, bijective, or neither injective nor surjective. Explain why.

(1) $f_1 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $f_1(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$, where $\mathbb{R}_{\geqslant 0} = \{x \in \mathbb{R} \mid x \geqslant 0\} = [0, \infty)$.
(2) $f_2 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $f_2(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$.
(3) $f_3 \colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with $f_3(x) = x^4$ for all $x \in \mathbb{R}$.
(4) $f_4 \colon \mathbb{R} \to \mathbb{R}$ with $f_4(x) = 10^{x^2}$ for all $x \in \mathbb{R}$. (Here 10^{x^2} stands for $10^{(x^2)}$, not $(10^x)^2$.)

Hint. Your solution may follow this example: Let $f_5 \colon \mathbb{R} \to \mathbb{R}$ with $f_5(x) = |x|$ for all $x \in \mathbb{R}$. Then f_5 is neither injective nor surjective. It is not injective because $f_5(1) = f_5(-1)$ while $1 \neq -1$. It is not surjective because there is no $x \in \mathbb{R}$ such that $f_5(x) = -2$

Problem 1.4. Let A, B and C be sets.

(1) Find a concrete example of A, B and C such that $(A \cup B) \cap C \subsetneq A \cup (B \cap C)$.
(2) Prove $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

Hint. (1) For example, you may try letting $A = \{1, 2\}$, $B = \{\dots\}$ and $C = \{\dots\}$. You may even start with $A = \{1\}$. (Note that $X \subsetneq Y$ means $X \subseteq Y$ and $X \neq Y$.)

(2) How to show that a set is a subset of another set? Let $x \in (A \cup B) \cap C$ be an (arbitrary) element. Try to show $x \in A \cup (B \cap C)$.

PROBLEMS HINTS SOLUTIONS

Problem 2.1. Let $X = \{a, b\}$, $Y = \{1, 2\}$ and $Z = \{x, y, z\}$.

(1) Find all functions from X to Y.
(2) Find all injective functions from X to Y, if they exist.
(3) Write down all surjective functions from Y to Z, if they exist.
(4) Write down all **non**-injective functions from Y to Z, if they exist.

Hint. (1) Describe the functions from X to Y in the following format:

$$f_1\colon\ a \mapsto \boxed{?},\ b \mapsto \boxed{?}; \qquad f_2\colon\ a \mapsto \boxed{?},\ b \mapsto \boxed{?}; \qquad \dots \qquad \dots \qquad \dots \qquad \dots$$

You need to exhaust all functions and write them down explicitly.

(4) This should be straightforward. Don't miss anyone.

Problem 2.2. Let S_3 denote the set of all bijective functions from $X = \{1, 2, 3\}$ to itself. Let $\varphi \in S_3$ and $\psi \in S_3$ be defined as follows

$$\varphi\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3 \qquad \text{and} \qquad \psi\colon\ 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2.$$

(1) Determine $\varphi \circ \psi$ and $\psi \circ \varphi$ explicitly. Are they equal?
(2) Determine φ^{-1} and ψ^{-1} explicitly.
(3) Determine φ^2 and φ^3 explicitly. Is anyone of the two equal to I_X?
(4) Determine ψ^2 and ψ^3 explicitly. Is anyone of the two equal to I_X?

Hint. You may determine/describe a function in S_3 in the format of $1 \mapsto \boxed{?}$, $2 \mapsto \boxed{?}$, $3 \mapsto \boxed{?}$. Recall that φ^2 denotes $\varphi \circ \varphi$, while ψ^3 denotes $\psi \circ \psi \circ \psi$. Also, I_X stands for the identity function on X.

Problem 2.3. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions, in which X, Y and Z are non-empty sets.

(1) If both f and g are surjective (i.e., onto), prove that $g \circ f$ is surjective.
(2) **Disprove**: If $g \circ f$ is surjective (i.e., onto), then both f and g are surjective.

Hint. (1) Let $z \in Z$ (be an arbitrary element). Try to show the existence of (some element) $x \in X$ such that $g \circ f(x) = z$.

(2) Find explicit/concrete X, Y, Z, f and g such that the statement fails. It would suffice to consider finite sets.

Problem 2.4. Let f, f_1, $f_2\colon X \to Y$ and g, g_1, $g_2\colon Y \to Z$ be functions, in which X, Y and Z are (non-empty) sets.

(1) Prove that if g is 1–1 (i.e., injective) and $g \circ f_1 = g \circ f_2$, then $f_1 = f_2$.
(2) **Disprove** the statement: If $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$.

Hint. (1) Let $x \in X$ (be an arbitrary element). You need to show $f_1(x) = f_2(x)$. Make use of the assumption that g is 1–1 and $g \circ f_1 = g \circ f_2$.

(2) Just disprove it by an explicit counterexample. It would suffice to consider finite sets.