MATH 8221: ABSTRACT ALGEBRA II
HOMEWORK SETS AND EXAMS

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Contents

Note. Each homework set contains four (4) regular problems. When solving the problems, please make sure that your arguments are rigorous and complete.

There is also a set of problems for extra credits; see the last page of this file.

In this course, a ring may not be commutative and may not have unity. By default, a module over a ring R means a left R-module.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

$M \oplus N$ ...... $h \in \operatorname{Hom}_R(M,N) \implies M/\operatorname{Ker}(h) \cong \operatorname{Im}(h)$ ...... $R^{\oplus X} \twoheadrightarrow M$ ...... $M \cong \frac{R}{(d_1)} \oplus \cdots \oplus \frac{R}{(d_r)} \oplus R^m$ ...... $M \otimes_R N$

Problem 1.1. Let R be a ring (not necessarily with 1), $r \in R$, M an R-module (i.e., a left R-module), and $x \in M$.

(1) Prove that $\mathrm{Ann}_R(x)$ is a left ideal of R.
(2) Prove that $\mathrm{Ann}_R(M)$ is an ideal (i.e., a two-sided ideal) of R.
(3) Prove the one that (always) holds: $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(M)$ or $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(x)$.
(4) **Disprove** $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(rx)$ with a concrete counterexample.

Proof. (1) Since $0_R \cdot x = 0_M$, we see $0_R \in \mathrm{Ann}_R(x)$; therefore $\mathrm{Ann}_R(x) \neq \varnothing$. Let $a, b \in \mathrm{Ann}_R(x)$, so that $ax = 0_M = bx$; and let $s \in R$. Then

$$(a-b)x = ax - bx = 0_M - 0_M = 0_M \quad \text{and} \quad (sa)x = s(ax) = s0_M = 0_M,$$

which implies $a - b \in \mathrm{Ann}_R(x)$ and $sa \in \mathrm{Ann}_R(x)$. Therefore $\mathrm{Ann}_R(x)$ is a left ideal of R.

(2) `(This is similar to (1) above.)` As $0_R \cdot y = 0_M$ for all $y \in M$, we see that $0_R \in \mathrm{Ann}_R(M)$ and, hence, $\mathrm{Ann}_R(M) \neq \varnothing$. Let $a, b \in \mathrm{Ann}_R(M)$ and $s \in R$, so that $ay = 0_M = by$ for all $y \in M$. Then

$$\left.\begin{aligned} (a-b)y = ay - by = 0_M - 0_M &= 0_M \\ (sa)y = s(ay) = s0_M &= 0_M \\ \text{and} \quad (as)y = a(sy) &= 0_M \end{aligned}\right\} \quad \text{for all } y \in M.$$

Thus $a - b$, sa and as are all in $\mathrm{Ann}_R(M)$. Therefore $\mathrm{Ann}_R(M)$ is a two-sided ideal of R.

(3) We prove $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(x)$ as follows: Let $a \in \mathrm{Ann}_R(M)$. By definition, $ay = 0_M$ for all $y \in M$. In particular, $ax = 0_M$ and hence $a \in \mathrm{Ann}_R(x)$.

(4) Let $R = M_2(\mathbb{Z})$, the ring of all 2×2 matrices over $\mathbb{Z}$; and let $M = R$, which is naturally a left R-module. Then for $x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in M$ and $r = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in R$ (so that $rx = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in M$), it is routine to verify $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Ann}_R(x)$ but $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \notin \mathrm{Ann}_R(rx)$. □

Problem 1.2. Let R be a commutative ring (not necessarily with 1), $r \in R$, M an R-module (i.e., a left R-module), and $x \in M$.

(1) Prove that $\mathrm{Ann}_R(x)$ is an ideal of R.
(2) Prove $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(rx)$. (Compare with Problem 1.1(4) above.)

Proof. (1) This follows from Problem 1.1(1) and the fact that a left ideal of a commutative ring is automatically a (two-sided) ideal. `(One may prove this from scratch.)`

(2) Let $a \in \mathrm{Ann}_R(x)$, so that $ax = 0_M$ by definition. Then

$$a(rx) = (ar)x = (ra)x = r(ax) = r0_M = 0_M,$$

which shows $a \in \mathrm{Ann}_R(rx)$. Consequently, $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(rx)$, as required. □

Problem 1.3. Let R be a ring with 1 (i.e., $1_R \in R$), M a (left) R-module, and $x \in M$.

(1) Prove the following: $x = 0_M \iff \mathrm{Ann}_R(x) = R$.
(2) Prove the following: $M = \{0_M\} \iff \mathrm{Ann}_R(M) = R$. (Compare with Problem 1.5.)

Proof. (1) First, we have the following (which does not rely on the assumption $1 \in R$)

$$\begin{aligned} x = 0_M &\implies rx = 0_M \text{ for all } r \in R \\ &\implies r \in \mathrm{Ann}_R(x) \text{ for all } r \in R \\ &\implies R \subseteq \mathrm{Ann}_R(x) \implies R = \mathrm{Ann}_R(x). \end{aligned}$$

Conversely, we have

$$\operatorname{Ann}_R(x) = R \overset{1\in R}{\Longrightarrow} 1_R \in \operatorname{Ann}_R(x) \implies 1_R \cdot x = 0_M \implies x = 0_M.$$

(2) This equivalence may be proved as follows:

$$\begin{aligned} M = \{0_M\} &\iff y = 0_M \text{ for all } y \in M \overset{1.3(1)}{\iff} \operatorname{Ann}_R(y) = R \text{ for all } y \in M \\ &\iff ry = 0_M \text{ for all } y \in M \text{ and for all } r \in R \\ &\iff r \in \operatorname{Ann}_R(M) \text{ for all } r \in R \\ &\iff R \subseteq \operatorname{Ann}_R(M) \iff R = \operatorname{Ann}_R(M). \end{aligned}$$

This completes the proof. □

Problem 1.4. Let R be a ring, M an R-module, and x, $y \in M$.

(1) Show $\operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y) \subseteq \operatorname{Ann}_R(x+y)$.
(2) **Prove or disprove**: $\operatorname{Ann}_R(x) = \operatorname{Ann}_R(-x)$.
(3) **Prove or disprove**: $\operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y) = \operatorname{Ann}_R(x+y)$.

Proof. (1) Let $r \in \operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y)$, so that $r \in \operatorname{Ann}_R(x)$ and $r \in \operatorname{Ann}_R(y)$. Then $rx = 0_M = ry$, and consequently,

$$r(x+y) = rx + ry = 0_M + 0_M = 0_M,$$

which shows $r \in \operatorname{Ann}_R(x+y)$. Thus $\operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y) \subseteq \operatorname{Ann}_R(x+y)$.

(2) We prove $\operatorname{Ann}_R(x) = \operatorname{Ann}_R(-x)$ as follows: For any $r \in R$,

$$\begin{aligned} r \in \operatorname{Ann}_R(x) \iff rx = 0_M &\iff -(rx) = 0_M \\ &\iff r(-x) = 0_M \iff r \in \operatorname{Ann}_R(-x). \end{aligned}$$

(3) We disprove "$\operatorname{Ann}_R(x)\cap\operatorname{Ann}_R(y) = \operatorname{Ann}_R(x+y)$" with the following (counter)example:

$$R = \mathbb{Z}, \qquad M = R, \qquad x = 2 \in M, \quad \text{and} \quad y = -2 \in M.$$

(Note that M is naturally a left module over R.) Then $\operatorname{Ann}_R(x) = \{0\} = \operatorname{Ann}_R(y)$ and $\operatorname{Ann}_R(x+y) = R$ since $x + y = 0$. So $\operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y) = \{0\} \subsetneq R = \operatorname{Ann}_R(x+y)$. □

Problem 1.5 (Extra Credit, 1 point)**.** Let M be a (left) module over a ring R (not necessarily with 1). **Prove or disprove**: $M = \{0_M\} \iff \operatorname{Ann}_R(M) = R$. (Compare with Problem 1.3.)

Solution. We disprove the claim "$M = \{0_M\} \iff \operatorname{Ann}_R(M) = R$" with the following counterexample:

$$R = 2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\} \quad \text{and} \quad M = \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}.$$

Note that M is naturally a (left) R-module. (`Indeed,` M `is naturally a` $\mathbb{Z}$`-module and` R `is a subring of` $\mathbb{Z}$`. For example, for` $4 \in R$ `and` $\overline{1} \in M$`, we have` $4 \cdot \overline{1} = \overline{1} + \overline{1} + \overline{1} + \overline{1} = \overline{4} = \overline{0} \in \mathbb{Z}_2$`, in which` $\overline{0}$ `is the zero element of` $M = \mathbb{Z}_2$`.)`

Then, for all $r = 2n \in R$, $n \in \mathbb{Z}$, we see $r \cdot \overline{0} = \overline{0} = \overline{r} = r \cdot \overline{1}$. That is, $r \in \operatorname{Ann}_R(M)$ for all $r \in R$. Thus $R \subseteq \operatorname{Ann}_R(M)$ and hence $R = \operatorname{Ann}_R(M)$. But $M \neq \{0_M\}$.

Problem 2.1. Let $R = M_2(\mathbb{Z})$, the ring of all 2×2 matrices over $\mathbb{Z}$; and let $M = R$, which is naturally a (left) R-module. Also consider

$$x = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right) \in M, \quad N_1 = \{\left(\begin{smallmatrix} m & 0 \\ n & 0 \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \subseteq M \quad \text{and} \quad N_2 = \{\left(\begin{smallmatrix} m & n \\ 0 & 0 \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \subseteq M.$$

(1) Determine/describe $\mathrm{Ann}_R(x)$ and Rx explicitly. No need to justify.
(2) Out of N_1 and N_2, which one, if any, is an R-submodule of M? No need to justify.
(3) In case N_i is an R-submodule of M, find $\mathrm{Ann}_R(N_i)$ explicitly. No need to justify.
(4) If N_i is not an R-submodule of M, explain why it is not an R-submodule of M.

Solution. (1) Routine examination (details omitted) should produce results as follows

$$\mathrm{Ann}_R(x) = \{\left(\begin{smallmatrix} 0 & b \\ 0 & d \end{smallmatrix}\right) \mid a, b \in \mathbb{Z}\} \quad \text{and} \quad Rx = \{\left(\begin{smallmatrix} a & 2a \\ c & 2c \end{smallmatrix}\right) \mid a, b \in \mathbb{Z}\}.$$

(2) It is straightforward to see that (only) N_1 is an R-submodule of M.
(3) For the R-submodule N_1, it is routine to deduce that

$$\mathrm{Ann}_R(N_1) = \{0_R\} = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$$

(4) For example, with $r = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in R$ and $y = \left(\begin{smallmatrix} 3 & 4 \\ 0 & 0 \end{smallmatrix}\right) \in N_2$, we have

$$ry = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 4 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 3 & 4 \end{smallmatrix}\right) \notin N_2.$$

Thus, N_2 is *not* an R-submodule of M. This concludes the solution.

Problem 2.2. Let R be a ring (that may not be commutative or have unity), $a \in R$, and M an R-module. We define/denote $(0 :_M a)$ to be $\{x \in M \mid ax = 0_M\}$.

(1) True or false: $(0 :_M a) = M \iff a \in \mathrm{Ann}_R(M)$. No justification is necessary.
(2) **Disprove** with a counterexample: $(0 :_M a)$ is (always) an R-submodule of M.
(3) Prove that, if R is commutative, then $(0 :_M a)$ is an R-submodule of M.

Solution/Proof. (1) True. Indeed, $(0 :_M a) = M \iff aM = \{0_M\} \iff a \in \mathrm{Ann}_R(M)$.

(2) Similar to Problem 2.1(4), let $R = M_2(\mathbb{Z}) = M$, $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in R$, $x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in M$, and $r = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in R$. It is routine to see that $ax = 0_M$ and $a(rx) \neq 0_M$. In other words, we have

$$x \in (0 :_M a), \quad \text{but} \quad rx \notin (0 :_M a).$$

This shows that $(0 :_M a)$ is not an R-submodule of M.

(3) Clearly, $(0 :_M a) \subseteq M$. Since $a0_M = 0_M$, we see $0_M \in (0 :_M a)$ and hence $(0 :_M a) \neq \varnothing$. Next, let $x, y \in (0 :_M a)$ and $r \in R$. Then $ax = 0_M = ay$. As R is commutative, we have

$$a(x - y) = ax - ay = 0_M - 0_M = 0_M,$$
$$a(rx) = (ar)x = (ra)x = r(ax) = r0_M = 0_M.$$

Therefore $x - y \in (0 :_M a)$ and $rx \in (0 :_M a)$. So $(0 :_M a)$ is an R-submodule of M. □

Problem 2.3. Let R be a ring, M be an R-module, and N_1, N_2 be R-submodules of M.

(1) True or false: both $\mathrm{Ann}_R(N_1)$ and $\mathrm{Ann}_R(N_2)$ are (2-sided) ideals of R.
(2) Prove $\mathrm{Ann}_R(N_1 + N_2) = \mathrm{Ann}_R(N_1) \cap \mathrm{Ann}_R(N_2)$.
(3) Prove $\mathrm{Ann}_R(N_1 \cap N_2) \supseteq \mathrm{Ann}_R(N_1) + \mathrm{Ann}_R(N_2)$.

Solution/Proof. (1) True. This follows immediately from Problem 1.1(2).

(2) This may be proved as follows

$$\begin{aligned} r \in \operatorname{Ann}_R(N_1+N_2) &\iff rx = 0_M,\ \forall x \in N_1+N_2 \iff r(x_1+x_2) = 0_M,\ \forall x_i \in N_i \\ &\iff rx_1 = 0_M = rx_2,\ \forall x_i \in N_i \quad (\text{since } x_i \in N_i \subseteq N_1+N_2) \\ &\iff r \in \operatorname{Ann}_R(N_1) \text{ and } r \in \operatorname{Ann}_R(N_2) \\ &\iff r \in \operatorname{Ann}_R(N_1) \cap \operatorname{Ann}_R(N_2) \end{aligned}$$

(3) Let $t \in \operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2)$, so that $t = t_1 + t_2$ with $t_i \in \operatorname{Ann}_R(N_i)$. Then, for all $x \in N_1 \cap N_2$ (so that $x \in N_1$ and $x \in N_2$), we see

$$tx = (t_1+t_2)x = t_1x + t_2x = 0_M + 0_M = 0_M.$$

Therefore, $t \in \operatorname{Ann}_R(N_1 \cap N_2)$. This shows $\operatorname{Ann}_R(N_1 \cap N_2) \supseteq \operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2)$. □

Problem 2.4. (1) Prove that $\mathbb{C}$ is a finitely generated $\mathbb{R}$-module.

(2) Prove that $\mathbb{Q}$ is *not* a finitely generated $\mathbb{Z}$-module.

Proof. (1) We see that $\mathbb{C}$ is generated by $S = \{1, i\}$ as an $\mathbb{R}$-module. `(Indeed, for every` $z \in \mathbb{C}$`, we have` $z = a + bi = a \cdot 1 + b \cdot i$ `for some` $a, b \in \mathbb{R}$ `and hence` $z \in (S)$`.)`

(2) Suppose, on the contrary, that $\mathbb{Q}$ is a finitely generated module over $\mathbb{Z}$. To be explicit, say $\mathbb{Q}$ is generated by $\{x_1 = \frac{a_1}{b_1}, \ldots, x_g = \frac{a_g}{b_g}\}$ with $a_i, b_i \in \mathbb{Z}$, $b_i \neq 0$ and $g \in \mathbb{N}$. That is, $\mathbb{Q} = \{\sum_{i=1}^{g} n_i x_i \mid n_i \in \mathbb{Z}\}$, as $\mathbb{Z}$ has unity. Let $m = \operatorname{lcm}(b_1, \ldots, b_g)$, which is a positive integer. Consider $\frac{1}{m+1} \in \mathbb{Q}$, which should satisfy

$$\frac{1}{m+1} = \sum_{i=1}^{g} k_i x_i = \sum_{i=1}^{g} \frac{k_i a_i}{b_i} \quad \text{for some} \quad k_i \in \mathbb{Z},\ i = 1, \ldots, g.$$

Now, as $\frac{m}{b_i} \in \mathbb{Z}$ for all $i = 1, \ldots, g$ (because of the choice of m), we see

$$\frac{m}{m+1} = m \cdot \frac{1}{m+1} = m \cdot \sum_{i=1}^{g} \frac{k_i a_i}{b_i} = \sum_{i=1}^{g} \frac{m k_i a_i}{b_i} = \sum_{i=1}^{g} \frac{m}{b_i} k_i a_i \in \mathbb{Z}.$$

But the above is a contradiction. Therefore, $\mathbb{Q}$ is *not* a finitely generated $\mathbb{Z}$-module. □

Problem 2.5 (Extra Credit, 1 point)**.** Let R, M, N_1 and N_2 be as in Problem 2.3. **Prove or disprove**: $\operatorname{Ann}_R(N_1 \cap N_2) = \operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2)$. (Compare with Problem 2.3(3).)

Solution. We disprove the claim with the following counterexample:

$$R = M_2(\mathbb{Z}) = M, \quad N_1 = \{\left(\begin{smallmatrix} m & 0 \\ n & 0 \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \leqslant M \quad \text{and} \quad N_2 = \{\left(\begin{smallmatrix} 0 & m \\ 0 & n \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \leqslant M.$$

It is routine to see $\operatorname{Ann}_R(N_1) = \{0_R\} = \operatorname{Ann}_R(N_2)$. Hence $\operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2) = \{0_R\}$. But $\operatorname{Ann}_R(N_1 \cap N_2) = R$, since $N_1 \cap N_2 = \{0_M\}$.

`(The claim can also be disproved with vector spaces: Let` $R = \mathbb{Q}$`,` $M = \{(a, b) \mid a, b \in \mathbb{Q}\}$`,` $N_1 = \{(a, 0) \mid a \in \mathbb{Q}\}$ `and` $N_2 = \{(0, b) \mid b \in \mathbb{Q}\}$`. Then` $\operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2) = \{0_R\} \subsetneq R = \operatorname{Ann}_R(N_1 \cap N_2)$`. Note that` $N_1 \cap N_2 = \{0_M\}$`.)`

PROBLEMS HINTS SOLUTIONS

Problem 3.1. Let R be a ring and M an R-module. Let A, B, C be R-submodules of M.

(1) Prove $(A+B)\cap C \supseteq (A\cap C)+(B\cap C)$.

(2) Assume $A\subseteq C$ hence $A\cap C = A$. Prove $(A+B)\cap C = (A\cap C)+(B\cap C)$.

Proof. (1) We have the following inclusions

$$(A\cap C)+(B\cap C)\subseteq A+B \qquad \text{and} \qquad (A\cap C)+(B\cap C)\subseteq C+C = C,$$

which imply that $(A\cap C)+(B\cap C)\subseteq (A+B)\cap C$.

(2) Let $x\in(A+B)\cap C$. Thus $x\in C$ and $x = a+b$ for some $a\in A$ and $b\in B$. Then $x+(-a)\in C$ since $x\in C$ and $-a\in A\subseteq C$. Consequently, we have

$$B\ni b = x-a = x+(-a)\in C, \quad \text{which implies} \quad b\in B\cap C.$$

Hence $x = a+b\in A+(B\cap C) = (A\cap C)+(B\cap C)$. This completes the proof. □

Problem 3.2. Let R be a ring (not necessarily with unity), M and N be R-modules, $X\subseteq M$ such that X generates M, and f, $g\in \mathrm{Hom}_R(M,N)$. **Prove or disprove**: If $f|_X = g|_X$ then $f = g$. (You may assume $X\neq\varnothing$, as this is clear when $X=\varnothing$.)

Proof. We prove the claim $f=g$ as follows: For all $x\in X$, we have

$$(f-g)(x) = f(x)-g(x) = 0_N, \quad \text{implying} \quad x\in \mathrm{Ker}(f-g).$$

Thus $X\subseteq \mathrm{Ker}(f-g)$. As X generates M, we see $M\subseteq \mathrm{Ker}(f-g)$ hence $\mathrm{Ker}(f-g) = M$. (Note that $\mathrm{Ker}(f-g)$ is an R-submodule of M.) Consequently, for all $m\in M$,

$$f(m)-g(m) = (f-g)(m) = 0_N, \quad \text{i.e.,} \quad f(m) = g(m).$$

This verifies $f = g$, establishing the claim. (This could also be proved in a 'more elementary' fashion: For every $m\in M = (X)$, write $m = \sum_{x\in X}(r_x x + n_x x)$, with $r_x\in R$ and $n_x\in\mathbb{Z}$ (almost all zero). Consequently, we see

$$\begin{aligned} f(m) = f\left(\sum_{x\in X}(r_x x+n_x x)\right) &= \sum_{x\in X}\big(r_x f(x)+n_x f(x)\big) \\ &= \sum_{x\in X}\big(r_x g(x)+n_x g(x)\big) = g\left(\sum_{x\in X}(r_x x+n_x x)\right) = g(m). \end{aligned}$$

As $f(m) = g(m)$ for all $m\in M$, this verifies $f=g$.) □

Problem 3.3. Let R be a **commutative** ring and let M, N be R-modules. Note that $\mathrm{Hom}_R(M,N)$ is an R-module (such that, for all $r\in R$ and $h\in\mathrm{Hom}_R(M,N)$, we define $r*h\in\mathrm{Hom}_R(M,N)$ by $(r*h)(m) = rh(m)$ for all $m\in M$).

(1) **Prove ot disprove**: $\mathrm{Ann}_R(M)\subseteq \mathrm{Ann}_R(\mathrm{Hom}_R(M,N))$.

(2) **Prove ot disprove**: $\mathrm{Ann}_R(N)\subseteq \mathrm{Ann}_R(\mathrm{Hom}_R(M,N))$.

Proof. Denote the zero element of $\mathrm{Hom}_R(M,N)$ by 0_H. As we know, this zero map is given by $0_H(x) = 0_N$ for all $x\in M$.

(1) We prove the claim. Let $r\in\mathrm{Ann}_R(M)$, $h\in\mathrm{Hom}_R(M,N)$ and $m\in M$. Then

$$(r*h)(m) = r(h(m)) = h(rx) = h(0_M) = 0_N = 0_H(m).$$

As $m\in M$ is arbitrary, we see that $r*h = 0_H$. Then, as $h\in\mathrm{Hom}_R(M,N)$ is arbitrary, we see $r\in\mathrm{Ann}_R(\mathrm{Hom}_R(M,N))$. Finally, as $r\in\mathrm{Ann}_R(M)$ is arbitrary, we prove the inclusion $\mathrm{Ann}_R(M)\subseteq\mathrm{Ann}_R(\mathrm{Hom}_R(M,N))$.

(2) We prove the claim. For all $r \in \mathrm{Ann}_R(N)$, $h \in \mathrm{Hom}_R(M, N)$ and $m \in M$, we have

$$(r * h)(m) = r(h(m)) = 0_N = 0_H(m),$$

which verifies $\mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(\mathrm{Hom}_R(M, N))$, as required. □

Problem 3.4. Let R be a (not necessarily commutative) ring and let N be an R-module. We already know that $(\mathrm{Hom}_R(R, N), +)$ is an abelian group. For every $r \in R$ and for every $h \in \mathrm{Hom}_R(R, N)$, we define $r * h : R \to N$ as $(r * h)(x) = h(xr)$ for all $x \in R$.

(1) Show that $r * h \in \mathrm{Hom}_R(R, N)$ for all $r \in R$ and for all $h \in \mathrm{Hom}_R(R, N)$.

(2) Prove that $\mathrm{Hom}_R(R, N)$ is an R-module (under the scalar multiplication $r * h$).

Proof. (1) Let $s \in R$ and $h \in \mathrm{Hom}_R(R, N)$. For all x, y, $r \in R$, we have

$$(s * h)(x + y) = h((x + y)s) = h(xs + ys) = h(xs) + h(ys) = (s * h)(x) + (s * h)(y),$$
$$(s * h)(rx) = h((rx)s) = h(r(xs)) = r[h(xs)] = r[(s * h)(x)],$$

which verifies that $s * h$ is an R-linear homomorphism, i.e., $s * h \in \mathrm{Hom}_R(R, N)$.

(2) Let s, $t \in R$ and f, $g \in \mathrm{Hom}_R(R, N)$. For all $x \in R$, we have

$$\begin{aligned}
[s * (f + g)](x) &= (f + g)(xs) = f(xs) + g(xs) \\
&= (s * f)(x) + (s * g)(x) = [(s * f) + (s * g)](x), \\
[(s + t) * g](x) &= g(x(s + t)) = g(xs + xt) = g(xs) + g(xt) \\
&= (s * g)(x) + (t * g)(x) = [(s * g) + (t * g)](x), \\
[(st) * g](x) &= g(x(st)) = g((xs)t) = (t * g)(xs) = [s * (t * g)](x), \\
(1_R * g)(x) &= g(x1_R) = g(x), \text{ in the situation where } 1_R \in R.
\end{aligned}$$

Therefore, $s * (f + g) = (s * f) + (s * g)$, $(s + t) * g = (s * g) + (t * g)$, $(st) * g = s * (t * g)$ and $1_R * g = g$ (if $1_R \in R$). This proves that $\mathrm{Hom}_R(R, N)$ is an R-module. □

Problem 3.5 (Extra Credit, 1 point)**.** Let R and M be as in Problem 3.1. **Prove or disprove**: $(A + B) \cap C = (A \cap C) + (B \cap C)$ for all R-submodules A, B, C of M.

Solution. This can be disproved by the following counterexample: Let R be any non-zero ring (e.g., $R = \mathbb{Z}$) and $M = R \oplus R = \{(a, b) \mid a, b \in R\}$. Let

$$A = \{(a, 0) \mid a \in R\}, \quad B = \{(0, b) \mid b \in R\} \quad \text{and} \quad C = \{(c, c) \mid c \in R\},$$

which are all R-submodules of M. It is easy to see

$$A + B = M, \quad A \cap C = \{0_M\} \quad \text{and} \quad B \cap C = \{0_M\}.$$

Thus $(A + B) \cap C = C \neq \{0_M\} = (A \cap C) + (B \cap C)$. `(Compare with Problem 2.5.)`

PROBLEMS HINTS SOLUTIONS

Problem 4.1. Let R be a ring with unity and let M be an R-module. For every $x \in M$, define $h_x : R \to M$ by $h_x(s) = sx$, $\forall s \in R$. By Problem 3.4, $\mathrm{Hom}_R(R, M)$ is an R-module.

(1) For every $x \in M$, prove that $h_x \in \mathrm{Hom}_R(R, M)$.

Define $\varphi\colon M \to \mathrm{Hom}_R(R, M)$ by $\varphi(x) = h_x$, $\forall x \in M$. Complete the following as well:

(2) **Prove or disprove**: $\varphi \in \mathrm{Hom}_R\big(M, \mathrm{Hom}_R(R, M)\big)$, i.e., φ is R-linear.
(3) **Prove or disprove**: φ is an injective (i.e., 1-1) function.
(4) **Prove or disprove**: φ is a surjective (i.e., onto) function.
(5) **Prove or disprove**: φ is an R-linear isomorphism, so that $M \cong \mathrm{Hom}_R(R, M)$.

Proof. (1) The claim that $h_x \in \mathrm{Hom}_R(R, M)$ holds because, for all a, b, $r \in R$,

$$h_x(a+b) = (a+b)x = ax + bx = h_x(a) + h_x(b)$$
$$\text{and} \qquad h_x(ra) = (ra)x = r(ax) = rh_x(a).$$

(2) We prove the claim. Let x, $y \in M$ and $r \in R$ be arbitrary. For all $s \in R$, we have

$$\begin{aligned}\varphi(x+y)(s) &= h_{x+y}(s) = s(x+y) = sx + sy \\ &= h_x(s) + h_y(s) = (h_x + h_y)(s) = [\varphi(x) + \varphi(y)](s)\end{aligned}$$
$$\text{and} \quad \varphi(rx)(s) = h_{rx}(s) = s(rx) = (sr)x = h_x(sr) = (r * h_x)(s) = [r * \varphi(x)](s),$$

which verifies $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r * \varphi(x)$, with $*$ defined as in Problem 3.4. Hence $\varphi\colon M \to \mathrm{Hom}_R(R, M)$ is a homomorphism, i.e., $\varphi \in \mathrm{Hom}_R\big(M, \mathrm{Hom}_R(R, M)\big)$.

(3) We prove the claim. For x, $y \in M$, if $\varphi(x) = \varphi(y)$, i.e., $h_x = h_y$, then

$$x = 1_R x = h_x(1_R) = h_y(1_R) = 1_R y = y.$$

This verifies that φ is injective. `(This can also be done by examining` $\mathrm{Ker}(\varphi)$`.)`

(4) We prove the claim. Let $g \in \mathrm{Hom}_R(R, M)$ be arbitrary. Then we have

$$g(s) = g(s1_R) = sg(1_R) = h_{g(1_R)}(s) \quad \text{for all } s \in R,$$

which indicates $g = h_{g(1_R)} = \varphi\big(g(1_R)\big)$, with $g(1_R) \in M$. This shows that φ is onto.

(5) By the above, we see that φ is an R-linear isomorphism. Hence, $M \cong \mathrm{Hom}_R(R, M)$. ☐

Problem 4.2. Let R be a ring and $h \in \mathrm{Hom}_R(M, N)$, where M and N are R-modules. Let A, B be subsets of M. Prove that $h(A) \subseteq h(B) \iff A \subseteq B + \mathrm{Ker}(h)$. Here, for any X, $Y \subseteq M$, define $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$.

Proof. "$\Longrightarrow$" Assume $h(A) \subseteq h(B)$ and let $a \in A$. Then $h(a) \in h(A) \subseteq h(B)$, meaning that there exists $b \in B$ such that $h(b) = h(a)$, which forces $a - b \in \mathrm{Ker}(h)$. Consequently, $a = b + (a - b) \in B + \mathrm{Ker}(h)$. This verifies the claim that $A \subseteq B + \mathrm{Ker}(h)$.

"$\Longleftarrow$" Assume $A \subseteq B + \mathrm{Ker}(h)$ and assume $\alpha \in h(A)$. Then $\alpha = h(a)$ for some $a \in A$, which implies $a \in B + \mathrm{Ker}(h)$. Thus $a = b + k$ for some $b \in B$ and $k \in \mathrm{Ker}(h)$. Therefore, $\alpha = h(a) = h(b+k) = h(b) + h(k) = h(b) + 0_N = h(b) \in h(B)$. This verifies $h(A) \subseteq h(B)$. ☐

Problem 4.3. Let R be a **commutative** ring, M an R-module, and $\varnothing \neq S \subseteq R$ such that S is multiplicatively closed (i.e., $st \in S$ for all s, $t \in S$).

(1) Denote $T := \bigcup_{s \in S} (0 :_M s)$. Prove that T is an R-submodule of M.
(2) Consider the quotient R-module M/T. Prove that $\bigcup_{s \in S} (0 :_{M/T} s) = \{0_{M/T}\}$.

Proof. (1) Since $s0_M = 0_M$ for any $s \in S$, we see $0_M \in (0 :_M s) \subseteq T$. Next, let $x, y \in T$ and $r \in R$. Then $x \in (0 :_M s)$ and $y \in (0 :_M t)$ for some $s, t \in S$. Thus

$$(st)(x-y) = (st)x - (st)y = t(sx) - s(ty) = t0_M - s0_M = 0_M - 0_M = 0_M.$$

Thus $x - y \in (0 :_M st) \subseteq T$, since $st \in S$. Also, from $s(rx) = r(sx) = r0_M = 0_M$, it follows that $rx \in (0 :_M s) \subseteq T$. This proves that T is an R-submodule of M.

(2) It is clear that $0_{M/T} \in \bigcup_{s\in S}(0 :_{M/T} s)$, hence $\bigcup_{s\in S}(0 :_{M/T} s) \supseteq \{0_{M/T}\}$. To prove the other inclusion, let $\xi \in \bigcup_{s\in S}(0 :_{M/T} s)$. This means $\xi \in M/T$ and $\xi \in (0 :_{M/T} u)$ for some $u \in S$. Since $\xi \in M/T$, we may write $\xi = z + T$ for some $z \in M$. Consequently,

$$uz + T = u(z+T) = u\xi = 0_{M/T} = 0_M + T,$$

which implies $uz \in T = \bigcup_{s\in S}(0 :_M s)$. So there exists $v \in S$ such that $uz \in (0 :_M v)$. Thus $(vu)z = v(uz) = 0_M$, showing $z \in (0 :_M vu) \subseteq \bigcup_{s\in S}(0 :_M s) = T$, since $vu \in S$. Now we see

$$\xi = z + T = 0_M + T = 0_{M/T}.$$

Thus $\bigcup_{s\in S}(0 :_{M/T} s) \subseteq \{0_{M/T}\}$. Therefore, $\bigcup_{s\in S}(0 :_{M/T} s) = \{0_{M/T}\}$. □

Problem 4.4. Let M be an R-module, and N_1, N_2 be R-submodules of M. Consider the R-homomorphism $h : M \to M/N_1 \times M/N_2$ defined by $h(m) = (m + N_1, m + N_2)$, $\forall m \in M$.

(1) Fill in the blank: $\mathrm{Ker}(h) = N_1 \boxed{?} N_2$. Justify you claim.

(2) Prove that h is onto $M/N_1 \times M/N_2$ if $N_1 + N_2 = M$. `(Also see Problem 4.5.)`

(3) Assume $N_1 + N_2 = M$. Fill in the blank: $M/\big(N_1 \boxed{?} N_2\big) \cong \frac{M}{N_1} \times \frac{M}{N_2}$. Justify.

Proof. (1) We claim that $\mathrm{Ker}(h) = N_1 \cap N_2$. Indeed, we have

$$\begin{aligned} w \in \mathrm{Ker}(h) &\iff (w + N_1, w + N_2) = h(w) = (0 + N_1, 0 + N_2) \\ &\iff w + N_1 = 0 + N_1 \text{ and } w + N_2 = 0 + N_2 \\ &\iff w - 0 \in N_1 \text{ and } w - 0 \in N_2 \\ &\iff w \in N_1 \text{ and } w \in N_2 \iff w \in N_1 \cap N_2. \end{aligned}$$

(2) Let $(x + N_1, y + N_2) \in M/N_1 \times M/N_2$, with $x, y \in M$. As $M = N_1 + N_2$, we see $x = a + b$ and $y = c + d$ for some $a, c \in N_1$ and $b, d \in N_2$. Let $z = b + c \in M$. We have

$$z + N_1 = x + N_1 \qquad \text{and} \qquad z + N_2 = y + N_2,$$

since $z - x = (b + c) - (a + b) = c - a \in N_1$ and $z - y = (b + c) - (c + d) = b - d \in N_2$. Thus

$$h(z) = (z + N_1, z + N_2) = (x + N_1, y + N_2).$$

This shows that h is onto $M/N_1 \times M/N_2$, as required.

(3) By the fundamental theorem of homomorphisms, $M/(N_1 \cap N_2) \cong M/N_1 \times M/N_2$. □

Problem 4.5 (Extra Credit, 1 point)**.** Let R, M, N_1, N_2 and $h : M \to M/N_1 \times M/N_2$ be as in Problem 4.4 above. **Prove or disprove** the converse of Problem 4.4(2): the homomorphism h is onto $M/N_1 \times M/N_2$ **only if** $N_1 + N_2 = M$.

Proof. We prove the claim: Let $x \in M$ be arbitrary. As h is onto, there exists $v \in M$ such that $(x + N_1, 0 + N_2) = h(v)$, which means $(x + N_1, 0 + N_2) = (v + N_1, v + N_2)$. Thus

$$x + N_1 = v + N_1 \text{ and } 0 + N_2 = v + N_2, \quad \text{which imply} \quad x - v \in N_1 \text{ and } v \in N_2.$$

Then $x = (x - v) + v \in N_1 + N_2$. This shows $M \subseteq N_1 + N_2$. Hence $M = N_1 + N_2$. □

Modules, basic notions: Problems 1.1, 1.2, 1.3, 1.4, 1.5, 2.1, 2.2.

Submodules, properties: Problems 2.1, 2.2, 2.3, 2.4, 2.5, 3.1, 3.5.

Homomorphisms: Problems 3.2, 3.3, 3.4, 4.1, 4.2, 4.4.

Quotient modules: Problems 4.3, 4.4, 4.5.

Lecture notes and textbooks: All we have covered in class.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Module. Let R be a ring and $(M, +)$ an abelian group. We say M is an R-module if there is a scalar multiplication $rx \in M$, defined for all $r \in R$ and $x \in M$, such that

$$r(x+y) = rx + ry, \quad (r+s)x = rx + sx, \quad (rs)x = r(sx), \quad \text{and} \quad 1_R x = x \ (\text{if } 1_R \in R)$$

for all r, $s \in R$ and all x, $y \in M$. If M is an R-module, then

$$0_R x = 0_M = r0_M \quad \text{and} \quad (-r)x = -(rx) = r(-x) \quad \text{for all} \quad r \in R,\, x \in M.$$

Notations. Let R be a ring, M an R-module, $\varnothing \neq A \subseteq R$ and $\varnothing \neq X \subseteq M$.

- $\mathrm{Ann}_R(X) := \{r \in R \mid rx = 0_M \text{ for all } x \in X\}$.
- $(0 :_M A) := \{x \in M \mid ax = 0_M \text{ for all } a \in A\}$.

Submodule. Let R be a ring, M an R-module and $N \subseteq M$. Then N is an R-submodule of M, denoted $N \leqslant M$, if and only if $0_M \in N$, $x - y \in N$ and $rx \in N$ for all x, $y \in N$, $r \in R$.

- The R-submodule generated by $X \subseteq M$ is $\Big\{ \sum_{\text{finite}} (r_x x + n_x x) \mid x \in X,\, r_x \in R,\, n_x \in \mathbb{Z} \Big\}$.
- For any family $\{N_i\}_{i\in\Lambda}$ of R-submodules of M, $\sum_{i\in\Lambda} N_i = \Big\{ \sum_{\text{finite}} y_i \mid y_i \in N_i,\, i \in \Lambda \Big\}$.

Homomorphism. Let M, N be R-modules (with R a ring). A function $h\colon M \to N$ is said to be an R-homomorphism (or an R-linear map) iff $h(x+y) = h(x)+h(y)$ and $h(rx) = rh(x)$ for all x, $y \in M$ and all $r \in R$. For any R-linear map $h\colon M \to N$, we have

- $h(0_M) = 0_N$, $h(-x) = -h(x)$ and $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(h(x))$ for all $x \in M$.
- $\mathrm{Ker}(h) := \{x \in M \mid h(x) = 0_N\} \leqslant M$ and $\mathrm{Im}(h) := \{h(x) \mid x \in M\} \leqslant N$.
- Denote by $\mathrm{Hom}_R(M, N)$ the set of all R-homomorphisms from M to N.
- For φ, $\psi \in \mathrm{Hom}_R(M, N)$ and $r \in R$, define $\varphi + \psi\colon M \to N$ and $r * \varphi\colon M \to N$ by
$$(\varphi+\psi)(x) := \varphi(x) + \psi(x) \text{ and } (r*\varphi)(x) := r\varphi(x) \quad \text{for all} \quad x \in M.$$
- If R is commutative, then $\mathrm{Hom}_R(M, N)$ is an R-module under the above operations.

Quotient module. Let R be a ring, $N \leqslant M$ be R-modules. Then the quotient R-module of M modulo N is the quotient (abelian) group $M/N := \{x + N \mid x \in M\}$ together with

$$r(x+N) = rx + N, \quad \forall r \in R,\, \forall x \in M.$$

Note that, for x, $y \in M$, $x + N = y + N \iff x - y \in N$.

Isomorphism theorems. Let R be a ring, M, N be R-modules, and $h \in \mathrm{Hom}_R(M, N)$. Let H and $K \leqslant L$ be R-submodules of M. Then

$$\tfrac{M}{\mathrm{Ker}(h)} \cong \mathrm{Im}(h), \qquad \tfrac{H}{(H\cap K)} \cong \tfrac{H+K}{K} \qquad \text{and} \qquad \tfrac{M/K}{L/K} \cong \tfrac{M}{L}.$$

Direct product, external direct sum. Let $\{M_i\}_{i\in\Lambda}$ be a family of R-modules. The Cartesian product, denote $\prod_{i\in\Lambda} M_i$, is an R-module under component-wise operations.

- The direct product of $\{M_i\}_{i\in\Lambda}$ is exactly the above R-module structure on $\prod_{i\in\Lambda} M_i$.
- The external direct sum, denoted $\bigoplus_{i\in\Lambda} M_i$, consists of the elements of $\prod_{i\in\Lambda} M_i$ whose components are almost all zero. Hence $\bigoplus_{i\in\Lambda} M_i$ is an R-submodule of $\prod_{i\in\Lambda} M_i$.
- In case $\Lambda = \{1, \ldots, n\}$, we have $\prod_{i=1}^n M_i = \{(x_1, \ldots, x_n) \mid x_i \in M_i\} = \bigoplus_{i=1}^n M_i$.

Note: The above list is not intended to be complete.

Solutions

have been withdrawn

from the site

PROBLEMS HINTS SOLUTIONS

Problem 5.1. Determine whether $\mathbb{Q}$, which is a $\mathbb{Z}$-module, is free over $\mathbb{Z}$. Justify.

Solution. We claim that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module. By way of contradiction, suppose $\mathbb{Q}$ is free over $\mathbb{Z}$ with a basis B. Then $|B| = \infty$, because $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module by Problem 2.4(2). Choose any two distinct elements $b_1, b_2 \in B$. Write $b_i = \frac{m_i}{n_i}$ with $m_i \in \mathbb{Z}$ and $0 \neq n_i \in \mathbb{Z}$ for $i = 1, 2$. In fact, we must have $m_i \neq 0$ since $b_i \neq 0$ for $i = 1, 2$. `(If` $0 \in B$`, then` B `must be linearly dependent over` $\mathbb{Z}$`.)` Therefore, we get

$$(n_1 m_2)b_1 + (-n_2 m_1)b_2 = 0 \qquad \text{with} \qquad n_1 m_2 \text{ and } -n_2 m_1 \in \mathbb{Z} \setminus \{0\}.$$

This is a contradiction (to the assumption that B is a basis).

Problem 5.2. Let F, M and N be R-modules, where R is a ring with unity, and let $g \in \mathrm{Hom}_R(M, N)$. Further assume that F is free over R and g is onto N (i.e., $g(M) = N$). Prove that, for every $f \in \mathrm{Hom}_R(F, N)$, there exists $h \in \mathrm{Hom}_R(F, M)$ such that $f = g \circ h$.

Proof. Since F is a free R-module, it admits a basis, say B. Let $f \in \mathrm{Hom}_R(M, N)$. For every $x \in B$, we have $f(x) \in N = g(M)$ and thus $f(x) = g(m_x)$ for some $m_x \in M$. `(Such` $m_x \in M$ `is not uniquely determined by` x `in general, but we can choose one such` m_x `for each` $x \in B$`.)` Stated differently, we have that

$$\text{for every } x \in B, \text{ there exists } m_x \in M \quad \text{such that} \quad g(m_x) = f(x).$$

Define a function $\theta\colon B \to M$ by $\theta(x) = m_x$ for all $x \in B$. By the universal property of free modules, there exists $h \in \mathrm{Hom}_R(F, M)$ such that $h|_B = \theta$. Thus, for all $x \in B$,

$$(g \circ h)(x) = g\big(h(x)\big) = g\big(\theta(x)\big) = g(m_x) = f(x).$$

This shows that $f|_B = (g \circ h)|_B$. By Problem 3.2, we establish that $f = g \circ h$. ☐

Problem 5.3. Let M and N be R-modules, where R is a ring, and let $h \in \mathrm{Hom}_R(M, N)$. Let $X \subseteq M$ and $K \subseteq N$. **Prove or disprove** each of the following:

(1) If K is an R-submodule of N, then $h^{-1}(K)$ is an R-submodule of M.
(2) If M is generated by X over R, then $\mathrm{Im}(h)$ is generated by $h(X)$ over R.

Proof. (1) We prove the statement. Let $K \leqslant N$ be an arbitrary R-submodule of N. Clearly, $h^{-1}(K) \subseteq M$. Since $h(0_M) = 0_N \in K$, we see that $0_M \in h^{-1}(K)$ and hence $h^{-1}(K) \neq \varnothing$. Next, let $a, b \in h^{-1}(K)$ and let $r \in R$. This implies $h(a) \in K$ and $h(b) \in K$. Hence

$$h(a - b) = h(a) - h(b) \in K \qquad \text{and} \qquad h(ra) = rh(a) \in K,$$

which imply $a - b \in h^{-1}(K)$ and $ra \in h^{-1}(K)$. This proves that $h^{-1}(K) \leqslant M$.

(2) We prove the statement. Let H be the R-submodule generated by $h(X)$. As $X \subseteq M$, we see $h(X) \subseteq h(M) \leqslant N$. Thus $H \leqslant h(M)$. In order to show $h(M) \leqslant H$, let K be any R-submodule of N such that $h(X) \subseteq K$. Then $X \subseteq h^{-1}\big(h(X)\big) \subseteq h^{-1}(K) \leqslant M$. Since X generates M, this forces $h^{-1}(K) = M$, which then implies that $h(M) = h\big(h^{-1}(K)\big) \subseteq K$. In other words, every R-submodule K (of N) that contains $h(X)$ must contain $h(M)$. Therefore, the R-submodule generated by $h(X)$ must contain $h(M)$, i.e., $H \geqslant h(M)$. This completes the proof that $H = h(M)$, meaning that $\mathrm{Im}(h)$ is generated by $h(X)$ over R. `(This could also be proved in a 'more elementary' fashion: For every` $n \in \mathrm{Im}(h)$`, there exists`

$m \in M$ such that $n = h(m)$. Moreover, as $M = (X)$, write $m = \sum_{x \in X}(r_x x + n_x x)$, with $r_x \in R$ and $n_x \in \mathbb{Z}$ (almost all zero). Consequently,

$$h(m) = h\left(\sum_{x \in X}(r_x x + n_x x)\right) = \sum_{x \in X}\big(r_x h(x) + n_x h(x)\big) \in \big(h(X)\big).$$

This verifies that $\operatorname{Im}(h)$ is generated by $h(X)$.) □

Problem 5.4. Let R be a ring with unity and M an R-module. Let $n \in \mathbb{N}$, and consider the R-module $R^n = R \times \cdots \times R = R \oplus \cdots \oplus R = \{(r_1, \ldots, r_n) \mid r_i \in R\}$. Show that the following statements are equivalent:

(1) There exist $x_1, \ldots, x_n \in M$ such that they generate M (i.e., $M = \sum_{i=1}^n Rx_i$).
(2) There exists $\varphi \in \operatorname{Hom}_R(R^n, M)$ such that $\operatorname{Im}(\varphi) = M$ (i.e., φ is onto M).

Proof. Note that R^n is free over R, with a standard basis $B = \{e_1, \ldots, e_n\}$.

(1) $\implies$ (2): Assume that there exist $x_1, \ldots, x_n \in M$ such that they generate M over R. By a theorem proved in class (a.k.a. the universal property of free modules),

$$\exists\, \varphi \in \operatorname{Hom}_R(R^n, M) \qquad \text{such that} \qquad \varphi(e_i) = x_i \quad \text{for all} \quad i = 1, \ldots, n.$$

Thus $\{x_1, \ldots, x_n\} \subseteq \operatorname{Im}(\varphi)$. Note that $\operatorname{Im}(\varphi)$ is an R-submodule of M. Therefore, the hypothesis that $\{x_1, \ldots, x_n\}$ generates M implies $M \leqslant \operatorname{Im}(\varphi)$. This proves $M = \operatorname{Im}(\varphi)$. (Or, since R^n is generated by B, we apply Problem 5.3(2) to conclude that $\operatorname{Im}(\varphi)$ is generated by $\varphi(B)$, which then forces $\operatorname{Im}(\varphi) = M$.)

(2) $\implies$ (1): Assume that there exists $\varphi \in \operatorname{Hom}_R(R^n, M)$ such that $\operatorname{Im}(\varphi) = M$. Denote $x_i = \varphi(e_i)$ for all $i = 1, \ldots, n$. Note that R^n is generated by $\{e_1, \ldots, e_n\}$. Hence, by Problem 5.3(2), we see that $\operatorname{Im}(\varphi)$ is generated by $\varphi(B)$. In other words, M is generated by $x_1, \ldots, x_n$. (For a 'more elementary' fashion, let $m \in M$. As φ is onto, there exists $(r_1, \ldots, r_n) \in R^n$ such that $\varphi(r_1, \ldots, r_n) = m$. Consequently,

$$m = \varphi(r_1, \ldots, r_n) = \varphi\Big(\sum_{i=1}^n r_i e_i\Big) = \sum_{i=1}^n r_i \varphi(e_i) = \sum_{i=1}^n r_i x_i \in \sum_{i=1}^n Rx_i.$$

This shows that $M \subseteq \sum_{i=1}^n Rx_i$. Therefore, $M = \sum_{i=1}^n Rx_i$.) □

Problem 5.5 (Extra Credit, 1 point)**.** Let R, F, M, N and $g \in \operatorname{Hom}_(M, N)$ be as in Problem 5.2 but **without** the hypothesis that F is free over R. **Disprove** the following statement: For every $f \in \operatorname{Hom}_R(F, N)$, there exists $h \in \operatorname{Hom}_R(F, M)$ such that $f = g \circ h$.

Solution. We disprove the claim with a counterexample as follows: Let

$$R = \mathbb{Z}, \quad M = R, \quad N = \mathbb{Z}_2 \quad \text{and} \quad F = \mathbb{Z}_2.$$

Define $g\colon M \to N$ by $g(n) = \overline{n}$ for all $n \in \mathbb{Z}$. It is clear that $g \in \operatorname{Hom}_R(M, N)$ and that g is onto N. Let $f\colon F \to N$ be the identity map (i.e., $f = \operatorname{Id}_{\mathbb{Z}_2}$), which is in $\operatorname{Hom}_R(F, N)$. However, $f \neq g \circ h$ for all $h \in \operatorname{Hom}_R(F, M)$, because $f \neq 0$ while $\operatorname{Hom}_R(F, M)$ consists of the zero map only. (For all $h \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z})$ and all $\overline{n} \in \mathbb{Z}_2$, we have $2h(\overline{n}) = h(\overline{2n}) = h(\overline{0}) = 0$, which forces $h(\overline{n}) = 0$.) (Note that $F = \mathbb{Z}_2$ is not free over $\mathbb{Z}$.)

Problem 6.1. Let R be a ring. Let M be an R-module such that $M = M_1 \boxplus M_2$ (meaning that $M_i \leqslant M$, $M = M_1 + M_2$ and $M_1 \cap M_2 = \{0_M\}$). Also, let $N = N_1 \oplus N_2$, where N_1, N_2 are R-modules. Denote $N_1' = \{(n_1, 0_{N_2}) \mid n_1 \in N_1\}$ and $N_2' = \{(0_{N_1}, n_2) \mid n_2 \in N_2\}$.

(1) Prove that $M \cong M_1 \oplus M_2$ (which simply says that $M_1 \boxplus M_2 \cong M_1 \oplus M_2$).
(2) True or false: (a) $N_1 \leqslant N$. (b) $N_2 \leqslant N$. (c) $N_1' \leqslant N$. (d) $N_2' \leqslant N$.
(3) Prove the one that is true: $N_1 \oplus N_2 = N_1 \boxplus N_2$ or $N_1 \oplus N_2 = N_1' \boxplus N_2'$.
(4) True or false: (a) $N_1 \cong N_1'$. (b) $N_2 \cong N_2'$.

Proof/solution. (1) Define $h : M_1 \oplus M_2 \to M$ by $h(m_1, m_2) = m_1 + m_2$ for all $m_i \in M_i$. This function is well-defined and it is routine to check that $h \in \operatorname{Hom}_R(M_1 \oplus M_2, M)$. It is also routine to verify that h is an isomorphism, but we include its verification as follows: If $(m_1, m_2) \in \operatorname{Ker}(h)$ then $m_1 + m_2 = 0_M$ which implies $m_1 = -m_2 \in M_1 \cap M_2 = \{0_M\}$ and hence $m_1 = m_2 = 0_M$. This shows that $\operatorname{Ker}(h) = \{(0_M, 0_M)\}$ and hence h is injective. Next, let $m \in M$. Then $m \in M_1 + M_2$ so $m = x_1 + x_2$ for some $x_i \in M_i$, which implies $m = h(x_1, x_2)$ where $(m_1, m_2) \in M_1 \oplus M_2$. Hence h is an isomorphism and $M_1 \oplus M_2 \cong M$.

(2) Here (a) and (b) are false, while (c) and (d) are true.

(3) We prove $N_1 \oplus N_2 = N_1' \boxplus N_2'$ as follows: For every $(n_1, n_2) \in N_1 \oplus N_2$ with $n_i \in N_i$, we have $(n_1, n_2) = (n_1, 0_{N_2}) + (0_{N_1}, n_2) \in N_1' + N_2'$, which shows that $N_1 \oplus N_2 \subseteq N_1' + N_2'$ and hence $N_1 \oplus N_2 = N_1' + N_2'$. Also, it is clear that $N_1' \cap N_2 = \{(0_{N_1}, 0_{N_2})\} = \{0_{N_1 \oplus N_2}\}$. This completes the proof that $N_1 \oplus N_2 = N_1' \boxplus N_2'$. (The other statement, $N_1 \oplus N_2 = N_1 \boxplus N_2$, is false, because N_1 and N_2 are not R-submodules of $N_1 \oplus N_2$.)

(4) Both (a) and (b) are true. It is straightforward to construct isomorphisms $N_i \cong N_i'$.

(The point of this exercise is to see that, although 'internal direct sum' and 'external direct sum' are different from a technical point of view, they can be identified with each other up to isomorphism.) □

Problem 6.2. Let M and N be R-modules (where R is a ring). Assume that there exist $g \in \operatorname{Hom}_R(M, N)$ and $h \in \operatorname{Hom}_R(N, M)$ such that $g \circ h = \operatorname{Id}_N$, the identity map on N.

(1) Prove (from scratch) that $M = \operatorname{Ker}(g) \boxplus \operatorname{Im}(h)$.
(2) **Prove or disprove**: $N = \operatorname{Ker}(h) \boxplus \operatorname{Im}(g)$.
(3) True or false: $\operatorname{Im}(h) \cong N$.
(4) True or false: $M \cong \operatorname{Ker}(g) \oplus N$.

Proof/solution. (1) It is clear that $\operatorname{Ker}(g) \leqslant M$ and $\operatorname{Im}(h) \leqslant M$. For every $x \in M$, since $g \circ h = \operatorname{Id}_N$, we see that $g\Big(x - h\big(g(x)\big)\Big) = 0_N$ hence $x - h\big(g(x)\big) \in \operatorname{Ker}(g)$, which yields $x = \Big(x - h\big(g(x)\big)\Big) + h\big(g(x)\big) \in \operatorname{Ker}(g) + \operatorname{Im}(h)$. This shows $M \subseteq \operatorname{Ker}(g) + \operatorname{Im}(h)$ and hence $M = \operatorname{Ker}(g) + \operatorname{Im}(h)$. To show that $\operatorname{Ker}(g) \cap \operatorname{Im}(h) = \{0_M\}$, let $m \in \operatorname{Ker}(g) \cap \operatorname{Im}(h)$, so that $g(m) = 0_N$ and $m = h(n)$ for some $n \in N$, which implies that $m = h(n) = h\Big(g\big(h(n)\big)\Big) = h\big(g(m)\big) = h(0_N) = 0_M$. Consequently, $\operatorname{Ker}(g) \cap \operatorname{Im}(h) \subseteq \{0_M\}$ and hence $\operatorname{Ker}(g) \cap \operatorname{Im}(h) = \{0_M\}$. Therefore, $M = \operatorname{Ker}(g) \boxplus \operatorname{Im}(h)$.

(2) We prove the statement as follows: From the assumption that $g \circ h = \operatorname{Id}_N$, which is bijective, we see that h is injective and g is surjective, i.e., $\operatorname{Ker}(h) = \{0_N\}$ and $\operatorname{Im}(g) = N$. Now it is clear that $N = \operatorname{Ker}(h) \boxplus \operatorname{Im}(g)$.

(3) This is true. Since $h : N \to M$ is injective, it is clear that $N \cong \operatorname{Im}(h)$.

(4) This is true. We have $M = \operatorname{Ker}(g) \boxplus \operatorname{Im}(h) \cong \operatorname{Ker}(g) \oplus \operatorname{Im}(h) \cong \operatorname{Ker}(g) \oplus N$. □

Problem 6.3. Let S be a ring and let R be a subring of S. (If $1_R \in R$, then further assume that $1_S \in S$ and $1_R = 1_S$.) Let M be an S-module, so that there is a scalar multiplication denoted by sx for all $s \in S$ and $x \in M$. (Again, a module means a left module.)

(1) Prove that M is an R-module under the existing (and obvious) scalar multiplication.
(2) **Prove or disprove**: Every S-submodule of M is an R-submodule of M.
(3) **Prove or disprove**: Every R-submodule of M is an S-submodule of M.

Proof. (1) `(To some degree, this is trivial.)` Since M is an S-module, we have

$$ax \in M, \quad (ab)x = a(bx), \quad (a+b)x = ax + bx \quad \text{and} \quad a(x+y) = ax + ay$$

for all a, $b \in S$, hence for all a, $b \in R$, and for all x, $y \in M$. In the case where $1_R \in R$, we have $1_R = 1_S$ and hence $1_R x = 1_S x = x$ for all $x \in M$. Thus M is an R-module.

(2) We prove the statement: Let N be any S-submodule of M. (So $N \neq \varnothing$ and, for all $x, y \in N$ and $s \in S$, we have $x - y \in N$ and $sx \in N$.) Thus $N \neq \varnothing$ and, for all $x, y \in N$ and $r \in R$, we have $x - y \in N$ and $rx \in N$. Thus N is an R-submodule of M.

(3) We disprove the statement as follows: Let $R = \mathbb{Z}$, $S = \mathbb{Q} = M$. Let $N = \mathbb{Z}$. Then N is an R-submodule of M but N is not an S-submodule of M. □

Problem 6.4. Let R, S and M be as in Problem 6.3, so that M is also an R-module.

(1) **Prove**: If M is Noetherian over R then it is Noetherian over S.
(2) **Disprove**: If M is Noetherian over S then it is Noetherian over R.
(3) **Prove or disprove**: If M is Artinian over R then it is Artinian over S.
(4) **Prove or disprove**: If M is Artinian over S then it is Artinian over R.

Proof. (1) We prove the statement as follows: Let $N_1 \leqslant N_2 \leqslant \cdots \leqslant N_i \leqslant N_{i+1} \leqslant \cdots$ be any ascending chain of S-submodules of M. This is also an ascending chain of R-submodules of M in light of Problem 6.3(2). Since M is Noetherian as an R-module, there exists $k \in \mathbb{N}$ such that $N_k = N_{k+i}$ for all $i \geqslant 0$. This proves that M is a Noetherian S-module.

(2) We disprove the statement: Let $R = \mathbb{Q}$, $S = \mathbb{R} = M$. Then M is Noetherian as an S-module (as S is a field and $M = S$); but M is not Noetherian as an R-module (since $M = \mathbb{R}$ is an infinite dimensional vector space over $R = \mathbb{Q}$.

(3) This can be prove in a very similar fashion as (1). (By Problem 6.3(2), every descending chain of S-submodules of M is also a descending chain of R-submodules of M, which then must stabilize because M is Artinian as an R-module. This proves that M is an Artinian S-module.)

(4) This can be disproved by the same example ($R = \mathbb{Q}$, $S = \mathbb{R} = M$) as in (2) above. □

Problem 6.5 (Extra Credit, 1 point)**.** Let R be a ring and let M be an R-module. Let H, K_1 and K_2 be R-submodules of M and further assume that $H \boxplus K_1 = M = H \boxplus K_2$. **Prove or disprove** each of the following: (1) $K_1 = K_2$. (2) $K_1 \cong K_2$ as R-modules.

Solution. (1) The claim can be disproved as follows: Let $R = \mathbb{Q}$, $M = \{(a,\, b) \,|\, a, b \in \mathbb{Q}\}$, $H = \{(a,\, a) \,|\, a \in \mathbb{Q}\}$, $K_1 = \{(a,\, 0) \,|\, a \in \mathbb{Q}\}$ and $K_2 = \{(0,\, a) \,|\, a \in \mathbb{Q}\}$. It is easy to see that $H \boxplus K_1 = M = H \boxplus K_2$ but $K_1 \neq K_2$.

(2) We prove the claim as follows: By an isomorphism theorem, we have

$$M/H = (H + K_i)/H \cong K_i/(H \cap K_i) = K_i/\{0\} \cong K_i, \quad \text{for each } i = 1,\, 2.$$

In short, we have $K_1 \cong M/H \cong K_2$, which implies $K_1 \cong K_2$. □

Problem 7.1. Let R be a ring, let M, N be R-modules, and let $h \in \operatorname{Hom}_R(M, N)$. Also, let A and B be R-submodules of M such that $A \leqslant B$ (that is, $A \leqslant B \leqslant M$). Prove the statement: If $h(A) = h(B)$ and $A \cap \operatorname{Ker}(h) = B \cap \operatorname{Ker}(h)$ then $A = B$.

Proof. It suffices to show $B \subseteq A$. Let $b \in B$, so that $h(b) \in h(B) = h(A)$. Thus there exists $a \in A$ such that $h(b) = h(a)$. Consequently, $h(b - a) = 0_N$ and hence $b - a \in \operatorname{Ker}(h)$. Moreover, as $b - a \in B$, we have $b - a \in \operatorname{Ker}(h) \cap B = \operatorname{Ker}(h) \cap A \subseteq A$. Therefore, $b = (b - a) + a \in A + A = A$. This completes the proof that $B \subseteq A$ and, hence $B = A$. (Or, by Problem 4.2, we see that $B \subseteq A + \operatorname{Ker}(h)$. Then, by Problem 3.1(2), we have $B = \big(A + \operatorname{Ker}(h)\big) \cap B \overset{3.1(2)}{=} A + \big(\operatorname{Ker}(h) \cap B\big) = A + \big(\operatorname{Ker}(h) \cap A\big) \subseteq A$.) □

Problem 7.2. Let R be a ring, let M be an R-module. Prove that, for any R-submodules N_1 and N_2 of M, the following statements are equivalent:

(1) $N_1 + N_2$ is a Noetherian (respectively, Artinian) R-module.
(2) Both N_1 and N_2 are Noetherian (respectively, Artinian) R-modules.

Proof. (1) ⇒ (2): If $N_1 + N_2$ is Noetherian/Artinian over R, then all its R-submodules are Noetherian/Artinian, therefore both N_1 and N_2 are Noetherian/Artinian R-modules.

(2) ⇒ (1): Assume that both N_1 and N_2 are Noetherian/Artinian over R. Note that $\frac{N_1+N_2}{N_1} \cong \frac{N_2}{N_1 \cap N_2}$, which is Noetherian/Artinian since N_2 is Noetherian/Artinian. Now that both N_1 and $\frac{N_1+N_2}{N_1}$ are Noetherian/Artinian, the module $N_1 + N_2$ is Noetherian/Artinian. (Alternatively, consider $h \in \operatorname{Hom}_R(N_1 \oplus N_2, N_1 + N_2)$ defined by $h(n_1, n_2) = n_1 + n_2$ for all $n_i \in N_i$. It is routine to see that h is surjective. Now, as $N_1 \oplus N_2$ is Noetherian/Artinian, $N_1 + N_2$ is Noetherian/Artinian.) □

Problem 7.3. Let R be a ring and $\{0_M\} \gneqq K \gneqq M$ be R-modules. Assume that there exist *simple* R-submodules N_1 and N_2 of M such that $M = N_1 + N_2$. (An R-module S is called simple if $S \neq \{0_S\}$ and the only R-submodules of S are $\{0_S\}$ and S.)

(1) Prove $M = N_1 \boxplus N_2$.
(2) Prove that there exists $i \in \{1, 2\}$ such that $M = N_i \boxplus K$.
(3) Prove that K is a simple R-module.

Proof. (1) Suppose $N_1 \cap N_2 \neq \{0_M\}$, so that $\{0_M\} \subsetneq N_1 \cap N_2 \subseteq N_i$ for $i = 1, 2$. Because of the simplicity of N_i, we must have $N_1 = N_1 \cap N_2 = N_2$. This implies

$$M = N_1 + N_2 = N_1.$$

In particular, M is simple. But this contradicts the assumption $\{0_M\} \subsetneq K \subsetneq M$. This proves $N_1 \cap N_2 = \{0_M\}$. Now, as $M = N_1 + N_2$, we see $M = N_1 \boxplus N_2$.

(2) First, we show $N_i \cap K = \{0_M\}$ for some $i \in \{1, 2\}$. Indeed, if $N_i \cap K \neq \{0_M\}$ for all $i \in \{1, 2\}$, then $N_i \cap K = N_i$ and hence $N_i \subseteq K$ for all $i \in \{1, 2\}$, which implies $M = N_1 + N_2 \subseteq K$, which is a contradiction. Say $N_1 \cap K = \{0_M\}$.

It remains to prove $N_1 + K = M$. Choose any $0_M \neq k \in K$. As $k \in M$, we may write $k = n_1 + n_2$ with $n_i \in N_i$. If $n_2 = 0_M$, then $N_1 \ni n_1 = k \in K$, which contradicts the fact that $N_1 \cap K = \{0_M\}$. Therefore, $0_M \neq n_2$. Since $N_2 \ni n_2 = k - n_1 \in N_1 + K$, we see $0_M \neq n_2 \in N_2 \cap (N_1 + K)$, showing $\{0_M\} \neq N_2 \cap (N_1 + K) \subseteq N_2$. Thus $N_2 \cap (N_1 + K) = N_2$, which implies $N_2 \subseteq N_1 + K$. Now that $N_1 \subseteq N_1 + K$ and $N_2 \subseteq N_1 + K$, we conclude $N_1 + N_2 \subseteq N_1 + K$. Consequently, $M = N_1 + N_2 \subseteq N_1 + K \subseteq M$, proving $N_1 + K = M$.

(3) Say $M = N_1 \boxplus K$, as in (2) above. Consequently,

$$K \cong \frac{K}{\{0\}} = \frac{K}{N_1 \cap K} \cong \frac{N_1+K}{N_1} \cong \frac{N_1+N_2}{N_1} \cong \frac{N_2}{N_1 \cap N_2} = \frac{N_2}{\{0\}} \cong N_2.$$

`(Or, alternatively, use Problem 6.5(2) to deduce that` $K \cong N_2$`.)` Now, since N_2 is simple and $K \cong N_2$, we conclude that K is simple. □

Problem 7.4. Let R be a commutative Noetherian ring with $1_R \in R$ and let M be a non-zero R-module. Prove that there exists $x \in M \setminus \{0_M\}$ such that $\mathrm{Ann}_R(x)$ is a prime ideal of R. `(For an ideal` $P \subsetneqq R$`, with` R `commutative with` 1_R`, we say that` P `is a prime ideal if, for all` $a, b \in R$`,` $ab \in P$ `implies` $a \in P$ `or` $b \in P$`.)`

Proof. Let us define $\Omega = \left\{ \mathrm{Ann}_R(m) \,\middle|\, m \in M \setminus \{0_M\} \right\}$, which is not empty since $M \neq \{0_M\}$. Since R is Noetherian and since every element of Ω is an ideal of R (cf. Problem 1.2(1)), there exists $P \in \Omega$ such that P is maximal in Ω. As $P \in \Omega$, we may write $P = \mathrm{Ann}_R(x)$ for some $x \in M \setminus \{0_M\}$.

It suffices to show that $P = \mathrm{Ann}_R(x)$ is a prime ideal of R. It is clear that $P \subsetneqq R$ (cf. Problem 1.3). To prove that P is prime, let $a, b \in R$ such that $ab \in P$ and $a \notin P$. `(It suffices to show` $b \in P$`.)` The assumption that $ab \in \mathrm{Ann}_R(x) \not\ni a$ simply says

$$(ab)x = 0_M \neq ax \qquad \text{and hence} \qquad \mathrm{Ann}_R(ax) \in \Omega.$$

Also note that $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(ax)$ (cf. Problem 1.2(2)). Now, given that $\mathrm{Ann}_R(x)$ is maximal in Ω, we must have

$$\mathrm{Ann}_R(x) = \mathrm{Ann}_R(ax).$$

Finally, since $b(ax) = (ba)x = (ab)x = 0_M$, we see $b \in \mathrm{Ann}_R(ax) = \mathrm{Ann}_R(x) = P$. This verifies that $\mathrm{Ann}_R(x)$ is a prime ideal of R, completing the proof. □

Problem 7.5 (Extra Credit, 1 point)**.** Let M and F be R-modules (with R a ring with unity) such that F is free over R. Let $h \in \mathrm{Hom}_R(M, F)$ such that h is onto F (i.e., $h(M) = F$). **Prove or disprove**: $M \cong \mathrm{Ker}(h) \oplus F$.

Proof. We prove the claim as follows: By Problem 5.2, there exists $g \in \mathrm{Hom}_R(F, M)$ such that $h \circ g = \mathrm{Id}_F$. Then, by Problem 6.2, we see $M = \mathrm{Ker}(h) \boxplus \mathrm{Im}(g) \cong \mathrm{Ker}(h) \oplus F$. □

PROBLEMS HINTS SOLUTIONS

Problem 8.1. Let R be a ring and M a non-zero Artinian R-module. Prove that there exists $L \leqslant M$ such that L is simple.

Proof. Consider the set Ω consisting of all non-zero R-submodules of M, that is

$$\Omega = \{N \mid N \text{ is an } R\text{-submodule of } M \text{ and } N \neq \{0_M\}\}.$$

Note that $M \in \Omega$, hence $\Omega \neq \varnothing$. Because M is Artinian, there exists $L \in \Omega$ such that L is minimal in Ω. In particular, L is a non-zero R-module. To show that L is an simple R-module, let B be any R-submodule of L. If $B \neq \{0_M\}$, then $B \in \Omega$ while $B \leqslant L$, which forces $B = L$ since L is minimal in Ω. So L is simple. □

Problem 8.2. Let R be a ring, M be an R-module, and $2 \leqslant n \in \mathbb{N}$. Also, let N and N_i, where $i = 1, \ldots, n$, be R-submodules of M such that $N = \sum_{i=1}^n N_i$. Prove that the following statements are equivalent to each other:

(1) $N = \boxplus_{i=1}^n N_i$ (i.e., $N = \sum_{i=1}^n N_i$ and $\left(\sum_{i\neq j} N_i\right) \cap N_j = \{0_N\}$ for all $j = 1, \ldots, n$).
(2) $\left(\sum_{i=1}^{j-1} N_i\right) \cap N_j = \{0_N\}$ for all $j = 2, \ldots, n$.
(3) Every element $x \in N$ can be uniquely expressed as $x = \sum_{i=1}^n x_i$ with $x_i \in N_i$.
(4) The element $0_N \in N$ can be uniquely expressed as $0_N = \sum_{i=1}^n x_i$ with $x_i \in N_i$.

Proof. (1) $\implies$ (2): This is because $\{0_N\} \subseteq \left(\sum_{i=1}^{j-1} N_i\right) \cap N_j \subseteq \left(\sum_{i\neq j} N_i\right) \cap N_j = \{0_N\}$.

(2) $\implies$ (3): Let $x \in N$ and suppose that $x = \sum_{i=1}^n x_i = \sum_{i=1}^n x_i'$, where $x_i, x_i' \in N_i$. This implies that $\sum_{i=1}^n (x_i - x_i') = 0_N$, with $x_i - x_i' \in N_i$. Then, we have

$$N_n \ni x_n - x_n' = -\sum_{i=1}^{n-1}(x_i - x_i') \in \sum_{i=1}^{n-1} N_i,$$

which implies that $x_n - x_n' \in \left(\sum_{i=1}^{n-1} N_i\right) \cap N_n = \{0_N\}$. Hence $x_n - x_n' = 0_N$ and $x_n = x_n'$. Then, we have $\sum_{i=1}^{n-1}(x_i - x_i') = 0_N$, which implies

$$N_{n-1} \ni x_{n-1} - x_{n-1}' = -\sum_{i=1}^{n-2}(x_i - x_i') \in \sum_{i=1}^{n-2} N_i.$$

Hence $x_{n-1} - x_{n-1}' \in \left(\sum_{i=1}^{n-2} N_i\right) \cap N_{n-1} = \{0_N\}$, which implies $x_{n-1} - x_{n-1}' = 0_N$ and $x_{n-1} = x_{n-1}'$. So, inductively, we see $x_i = x_i'$ for all $i = n, n-1, \ldots, 2, 1$.

(3) $\implies$ (4): This is trivial.

(4) $\implies$ (1): As $N = \sum_{i=1}^n N_i$ is given, it remains to show $\left(\sum_{i\neq j} N_i\right) \cap N_j = \{0_N\}$ for all $j = 1, \ldots, n$. Let $j \in \{1, \ldots, n\}$ and let $y \in \left(\sum_{i\neq j} N_i\right) \cap N_j$. Then $N_j \ni y = \sum_{i\neq j} y_i$ with $y_i \in N_i$, which forces

$$y_1 + \cdots + y_{j-1} - y + y_{j+1} \cdots y_n = 0 = 0_{N_1} + \cdots + 0_{N_{j-1}} + 0_{N_j} + 0_{N_{j+1}} + \cdots + 0_{N_n}.$$

By uniqueness, we conclude $-y = 0_{N_j}$ so $y = 0_{N_j} = 0_N$. Thus $\left(\sum_{i\neq j} N_i\right) \cap N_j \subseteq \{0_N\}$ and hence $\left(\sum_{i\neq j} N_i\right) \cap N_j = \{0_N\}$. This completes the proof. □

Problem 8.3. Let R be a ring, N be an R-module, and $2 \leqslant n \in \mathbb{N}$. Let N_i, where $i = 1, \ldots, n$, be R-submodules of N. Also, let $K_i \leqslant M_i$, where $i = 1, \ldots, n$, be R-modules.

(1) Prove that if $N = \boxplus_{i=1}^n N_i$ then $N \cong \oplus_{i=1}^n N_i$.
(2) True or false: $\oplus_{i=1}^n K_i \leqslant \oplus_{i=1}^n M_i$.
(3) **Prove or disprove**: $(\oplus_{i=1}^n M_i) / (\oplus_{i=1}^n K_i) \cong \oplus_{i=1}^n (M_i/K_i)$.

Proof. (1) Assume that $N = \boxplus_{i=1}^n N_i$. Define $h : \oplus_{i=1}^n N_i \to N$ by $h(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$ for all $(x_1, \ldots, x_n) \in N_1 \oplus \cdots \oplus N_n$. It is routine to verify that $h \in \mathrm{Hom}_R(\oplus_{i=1}^n N_i, N)$ and that h is an isomorphism. Therefore, $N \cong \oplus_{i=1}^n N_i$. `(See Problem 6.1(1).)`

(2) True.

(3) We prove the claim as follows. Define a function $\varphi : \oplus_{i=1}^n M_i \to \oplus_{i=1}^n (M_i/K_i)$ by $\varphi(m_1, \ldots, m_n) = (m_1 + K_1, \ldots, m_n + K_n)$ for all $(m_1, \ldots, m_n) \in M_1 \oplus \cdots \oplus M_n$. It is routine to verify that $\varphi \in \mathrm{Hom}_R(\oplus_{i=1}^n M_i, \oplus_{i=1}^n (M_i/K_i))$. Moreover, it is routine to see that $\mathrm{Ker}(\varphi) = \oplus_{i=1}^n K_i$ and $\mathrm{Im}(\varphi) = \oplus_{i=1}^n (M_i/K_i)$ `(details skipped)`. By the fundamental theorem of homomorphisms, we see that $(\oplus_{i=1}^n M_i) / (\oplus_{i=1}^n K_i) \cong \oplus_{i=1}^n (M_i/K_i)$. □

Problem 8.4. Consider the free $\mathbb{Z}$-module $\mathbb{Z}^2 = \{\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) \mid r, s \in \mathbb{Z}\}$. (We write the elements of $\mathbb{Z}^2$ as column vectors.) Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{smallmatrix}\right)$ and $x_i = \left(\begin{smallmatrix} a_{1i} \\ a_{2i} \end{smallmatrix}\right) \in \mathbb{Z}^2$ be the i-th column of A. Let $y_i \in \mathbb{Z}^2$ be the i-th column of $A \left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right)$. Suppose $\left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1} A \left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{smallmatrix}\right)$.

(1) Compute $\left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1}$ and $\left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right)^{-1}$ explicitly. It suffices to write down your answers.
(2) Express each x_i as a linear combination of y_1, y_2, y_3 explicitly. No need to justify.
(3) True or false: $\sum_{i=1}^3 \mathbb{Z}x_i = \sum_{i=1}^3 \mathbb{Z}y_i$. No justification is necessary.
(4) **Prove** that $z_1 = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ and $z_2 = \left(\begin{smallmatrix} 5 \\ 8 \end{smallmatrix}\right)$ form a basis of $\mathbb{Z}^2$.
(5) Express each y_i as a linear combination of z_1 and z_2. No need to justify.

Solution/Proof. (1) We have that $\left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 8 & -5 \\ -3 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -4 & 4 & -1 \\ 3 & -3 & 1 \\ -1 & 2 & -1 \end{smallmatrix}\right)$.

(2) We have $x_1 = -4y_1 + 3y_2 - y_3$, $x_2 = 4y_1 - 3y_2 + 2y_3$ and $x_1 = -y_1 + y_2 - y_3$.

(3) True. (As $(y_1, y_2, y_3) = (x_1, x_2, x_3) \left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right)$, we see $(x_1, x_2, x_3) = (y_1, y_2, y_3) \left(\begin{smallmatrix} -4 & 4 & -1 \\ 3 & -3 & 1 \\ -1 & 2 & -1 \end{smallmatrix}\right)$. So each y_i is a linear combination of x_1, x_2, x_3; and each x_i is a linear combination of y_1, y_2, y_3.)

(4) We first show that z_1 and z_2 are linearly independent as follows: For all $c_1, c_2 \in \mathbb{Z}$,

$$c_1 z_1 + c_2 z = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right) \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right) \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right).$$

Next, we show that z_1 and z_2 generate $\mathbb{Z}^2$: For every $\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) \in \mathbb{Z}^2$, we see

$$\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right) \left(\begin{smallmatrix} 8 & -5 \\ -3 & 2 \end{smallmatrix}\right) \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = (z_1, z_2) \left(\begin{smallmatrix} 8m-5n \\ -3m+2n \end{smallmatrix}\right) = (8m - 5n)z_1 + (-3m + 2n)z_2.$$

As z_1 and z_2 are linearly independent generators of $\mathbb{Z}^2$, they form a basis of $\mathbb{Z}^2$.

(5) We have $y_1 = d_1 z_1$, $y_2 = d_2 z_2$ and $y_3 = 0z_1 + 0z_2$, as $(y_1, y_2, y_3) = (z_1, z_2) \left(\begin{smallmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{smallmatrix}\right)$.

Problem 8.5 (Extra Credit, 1 point)**.** Let R be a ring. **Prove or disprove**: If M is a non-zero Noetherian R-module then there exists $K \leqslant M$ such that M/K is simple.

Proof. Consider the set Ω_1 consisting of all R-submodules of M except M, that is

$$\Omega_1 = \{N \mid N \text{ is an } R\text{-submodule of } M \text{ and } N \neq M\}.$$

Note that $\{0_M\} \in \Omega_1$, hence $\Omega_1 \neq \varnothing$. Because M is Noetherian, there exists $K \in \Omega_1$ such that K is maximal in Ω_1. Since $K \lneqq M$, we see that M/K is a non-zero R-module.

To show that M/K is simple, let A be any R-submodule of M/K such that $A \lneqq M/K$. `(It suffices to show` $A = \{0_{M/K}\}$`.)` Then $A = L/K$ for some R-submodule L of M such that $K \leqslant L$. Since $L/K \lneqq M/K$, we see $L \lneqq M$ and hence $L \in \Omega_1$. In light of the fact that $K \leqslant L \in \Omega_1$ and the fact that K is maximal in Ω_1, we see that $K = L$. Thus $A = K/K = \{0_{M/K}\}$. This finishes the proof that M/K is simple, as required. □

Materials covered earlier: Homework Sets 1, 2, 3, 4; Exam I.

General problems about modules: Problems 5.1, 5.3, 5.4, 6.3, 7.3, 8.1, 8.2, 8.4.

Free modules: Problems 5.2, 5.5, 7.3.

Direct sums (internal or external): Problems 6.1, 6.2, 6.5.

Noetherian modules, Artinian modules: Problems 6.4, 7.1, 7.2, 7.4, 7.5.

Simple modules: Problems 7.2, 8.3, 8.5.

Modules over a PID: Problem 8.4.

Lecture notes and textbooks: All we have covered.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Internal direct sum. Let R be a ring, M an R-module, N an R-submodule of M, and $\{N_i\}_{i\in\Delta}$ is family of R-submodules of M. We say that N is an internal direct sum of $\{N_i\}_{i\in\Delta}$, denoted $N = \boxplus_{i\in\Delta} N_i$, if $N = \sum_{i\in\Delta} N_i$ and $N_j \cap (\sum_{i\in\Delta\setminus\{j\}} N_i) = \{0_M\}$ for all $j \in \Delta$.

- $N = \boxplus_{i\in\Delta} N_i$ if and only if $N = \sum_{i\in\Delta} N_i$ and every element $x \in N$ is uniquely expressed as $x = \sum_{i\in\Delta} n_i$ with $x_i \in N_i$ and $x_i = 0_M$ for almost all $i \in \Delta$.
- If $N = \boxplus_{i\in\Delta} N_i$, then $N \cong \oplus_{i\in\Delta} N_i$.

Free module. Let R be a ring with unity and F an R-module. We say that F is a free (over R) if there exists $B \subseteq F$ such that B generates F and B is linear independent over R, in which case we say that B is a basis of F.

- F is a free R-module with a basis $B \iff$ every $x \in F$ can be uniquely written as $x = \sum_{b\in B} r_b b$ with $r_b \in R$ and $r_b = 0_R$ for almost all $b \in B$.
- F is a free R-module $\iff F \cong R^{\oplus B}$ for some set B.
- F is a free R-module with a basis $B \iff F = \boxplus_{b\in B} Rb$ and $\mathrm{Ann}(b) = \{0_R\}$, $\forall\, b \in B$.
- F is a free R-module with a basis $B \iff$ for any R-module M and any map $\theta\colon B \to M$, there is a unique R-linear map $h \in \mathrm{Hom}_R(F, M)$ such that $h|_B = \theta$.
- If R is commutative and F is free, then all bases of F have the same cardinality, which is called the rank of F.

Noetherian module. We say that an R-module M is Noetherian (over R) if every ascending chain of R-submodules of M eventually stabilizes. The following are equivalent:

- M is Noetherian as an R-module.
- All R-submodules of M are finitely generated.
- Every non-empty set of R-submodules of M has a maximal member/object.
- N and M/N are Noetherian over R for every R-submodule N of M.
- N and M/N are Noetherian over R for some R-submodule N of M.

We say R is a (left) Noetherian ring if R is Noetherian as a (left) R-module.

- [Hilbert basis theorem] If R is Noetherian, so is $R[x]$ and so is $R[x_1, \ldots, x_n]$.

Artinian module. We say that an R-module M is Artinian (over R) if every descending chain of R-submodules of M eventually stabilizes. The following are equivalent:

- M is Artinian as an R-module.
- Every non-empty set of R-submodules of M has a minimal member/object.
- N and M/N are Artinian over R for every (or for some) R-submodule N of M.

We say R is a (left) Artinian ring if R is Artinian as a (left) R-module.

Simple module. Let R be a ring and let M, N be R-modules. We say that M is a simple R-module if $M \neq \{0_M\}$ and the only R-submodules of M are $\{0_M\}$ and M.

- Assume that M is simple. Then every $\varphi \in \mathrm{Hom}_R(M, N)$ is either injective or the zero map. Similarly, every $\psi \in \mathrm{Hom}_R(N, M)$ is either the zero map or surjective.

Modules over a PID. Let R be a PID and let F be a free R-module of finite rank n.

- Every R-submodule of F is free (over R) of rank $\leqslant n$.

Note: The above list is not intended to be complete.

You must solve a problem **completely and correctly** in order to get the extra credit. You may attempt a problem for as many times as you wish by 04/25.

The points you get here will be added to the total score from the homework assignments.

Problem E-1 (3 points)**.** Let R be a commutative ring with unity and $S = R[x]$. For all ideals I and J of R, prove $I \cap J = \big(xIS + (1-x)JS\big) \cap R$.

Problem E-2 (3 points)**.** Let R be a commuative ring with unity and I, J ideals of R. **Prove or disprove**: If $R/I \cong R/J$ as R-modules then $I = J$.

Problem E-3 (3 points)**.** Is there a field that is free over $\mathbb{Z}$? Explain.

PROBLEMS HINTS SOLUTIONS