◇ ◇ ◇ ◇ MATH 8221: ABSTRACT ALGEBRA II ◇ ◇ ◇ ◇
HOMEWORK SETS AND EXAMS

Yongwei Yao

2025 SPRING SEMESTER
GEORGIA STATE UNIVERSITY

Contents

Note. Each homework set contains four (4) regular problems. When solving the problems, please make sure that your arguments are rigorous and complete.

There is also a set of problems for extra credits; see the last page of this file.

In this course, a ring may not be commutative and may not have unity. By default, a module over a ring R means a left R-module.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

Problem 1.1. Let R be a ring (not necessarily with 1), $r \in R$, M an R-module (i.e., a left R-module), and $x \in M$.

(1) Prove that $\operatorname{Ann}_R(x)$ is a left ideal of R.
(2) Prove that $\operatorname{Ann}_R(M)$ is an ideal (i.e., a two-sided ideal) of R.
(3) Prove the one that (always) holds: $\operatorname{Ann}_R(x) \subseteq \operatorname{Ann}_R(M)$ or $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(x)$.
(4) **Disprove** $\operatorname{Ann}_R(x) \subseteq \operatorname{Ann}_R(rx)$ with a concrete counterexample.

Problem 1.2. Let R be a commutative ring (not necessarily with 1), $r \in R$, M an R-module (i.e., a left R-module), and $x \in M$.

(1) Prove that $\operatorname{Ann}_R(x)$ is an ideal of R.
(2) Prove $\operatorname{Ann}_R(x) \subseteq \operatorname{Ann}_R(rx)$. (Compare with Problem 1.1(4) above.)

Problem 1.3. Let R be a ring with 1 (i.e., $1_R \in R$), M a (left) R-module, and $x \in M$.

(1) Prove the following: $x = 0_M \iff \operatorname{Ann}_R(x) = R$.
(2) Prove the following: $M = \{0_M\} \iff \operatorname{Ann}_R(M) = R$. (Compare with Problem 1.5.)

Problem 1.4. Let R be a ring, M an R-module, and x, $y \in M$.

(1) Show $\operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y) \subseteq \operatorname{Ann}_R(x + y)$.
(2) **Prove or disprove**: $\operatorname{Ann}_R(x) = \operatorname{Ann}_R(-x)$.
(3) **Prove or disprove**: $\operatorname{Ann}_R(x) \cap \operatorname{Ann}_R(y) = \operatorname{Ann}_R(x + y)$.

Problem 1.5 (Extra Credit, 1 point)**.** Let M be a (left) module over a ring R (not necessarily with 1). **Prove or disprove**: $M = \{0_M\} \iff \operatorname{Ann}_R(M) = R$. (Compare with Problem 1.3.)

PROBLEMS HINTS SOLUTIONS

Problem 2.1. Let $R = M_2(\mathbb{Z})$, the ring of all 2×2 matrices over $\mathbb{Z}$; and let $M = R$, which is naturally a (left) R-module. Also consider

$$x = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right) \in M, \quad N_1 = \{\left(\begin{smallmatrix} m & 0 \\ n & 0 \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \subseteq M \quad \text{and} \quad N_2 = \{\left(\begin{smallmatrix} m & n \\ 0 & 0 \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \subseteq M.$$

(1) Determine/describe $\operatorname{Ann}_R(x)$ and Rx explicitly. No need to justify.
(2) Out of N_1 and N_2, which one, if any, is an R-submodule of M? No need to justify.
(3) In case N_i is an R-submodule of M, find $\operatorname{Ann}_R(N_i)$ explicitly. No need to justify.
(4) If N_i is not an R-submodule of M, explain why it is not an R-submodule of M.

Problem 2.2. Let R be a ring (that may not be commutative or have unity), $a \in R$, and M an R-module. We define/denote $(0 :_M a)$ to be $\{x \in M \mid ax = 0_M\}$.

(1) True or false: $(0 :_M a) = M \iff a \in \operatorname{Ann}_R(M)$. No justification is necessary.
(2) **Disprove** with a counterexample: $(0 :_M a)$ is (always) an R-submodule of M.
(3) Prove that, if R is commutative, then $(0 :_M a)$ is an R-submodule of M.

Problem 2.3. Let R be a ring, M be an R-module, and N_1, N_2 be R-submodules of M.

(1) True or false: both $\operatorname{Ann}_R(N_1)$ and $\operatorname{Ann}_R(N_2)$ are (2-sided) ideals of R.
(2) Prove $\operatorname{Ann}_R(N_1 + N_2) = \operatorname{Ann}_R(N_1) \cap \operatorname{Ann}_R(N_2)$.
(3) Prove $\operatorname{Ann}_R(N_1 \cap N_2) \supseteq \operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2)$.

Problem 2.4. (1) Prove that $\mathbb{C}$ is a finitely generated $\mathbb{R}$-module.
(2) Prove that $\mathbb{Q}$ is *not* a finitely generated $\mathbb{Z}$-module.

Problem 2.5 (Extra Credit, 1 point). Let R, M, N_1 and N_2 be as in Problem 2.3. **Prove or disprove**: $\operatorname{Ann}_R(N_1 \cap N_2) = \operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2)$. (Compare with Problem 2.3(3).)

PROBLEMS HINTS SOLUTIONS

Problem 3.1. Let R be a ring and M an R-module. Let A, B, C be R-submodules of M.

(1) Prove $(A+B)\cap C \supseteq (A\cap C)+(B\cap C)$.

(2) Assume $A \subseteq C$ hence $A \cap C = A$. Prove $(A+B)\cap C = (A\cap C)+(B\cap C)$.

Problem 3.2. Let R be a ring (not necessarily with unity), M and N be R-modules, $X \subseteq M$ such that X generates M, and f, $g \in \operatorname{Hom}_R(M,N)$. **Prove or disprove**: If $f|_X = g|_X$ then $f = g$. (You may assume $X \neq \varnothing$, as this is clear when $X = \varnothing$.)

Problem 3.3. Let R be a **commutative** ring and let M, N be R-modules. Note that $\operatorname{Hom}_R(M,N)$ is an R-module `(such that, for all` $r \in R$ `and` $h \in \operatorname{Hom}_R(M,N)$`, we define` $r*h \in \operatorname{Hom}_R(M,N)$ `by` $(r*h)(m) = rh(m)$ `for all` $m \in M$`)`.

(1) **Prove ot disprove**: $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(\operatorname{Hom}_R(M,N))$.

(2) **Prove ot disprove**: $\operatorname{Ann}_R(N) \subseteq \operatorname{Ann}_R(\operatorname{Hom}_R(M,N))$.

Problem 3.4. Let R be a (not necessarily commutative) ring and let N be an R-module. We already know that $(\operatorname{Hom}_R(R,N), +)$ is an abelian group. For every $r \in R$ and for every $h \in \operatorname{Hom}_R(R,N)$, we define $r*h : R \to N$ as $(r*h)(x) = h(xr)$ for all $x \in R$.

(1) Show that $r*h \in \operatorname{Hom}_R(R,N)$ for all $r \in R$ and for all $h \in \operatorname{Hom}_R(R,N)$.

(2) Prove that $\operatorname{Hom}_R(R,N)$ is an R-module (under the scalar multiplication $r*h$).

Problem 3.5 (Extra Credit, 1 point)**.** Let R and M be as in Problem 3.1. **Prove or disprove**: $(A+B)\cap C = (A\cap C)+(B\cap C)$ for all R-submodules A, B, C of M.

PROBLEMS HINTS SOLUTIONS

Problem 4.1. Let R be a ring with unity and let M be an R-module. For every $x \in M$, define $h_x : R \to M$ by $h_x(s) = sx$, $\forall s \in R$. By Problem 3.4, $\operatorname{Hom}_R(R, M)$ is an R-module.

(1) For every $x \in M$, prove that $h_x \in \operatorname{Hom}_R(R, M)$.

Define $\varphi\colon M \to \operatorname{Hom}_R(R, M)$ by $\varphi(x) = h_x$, $\forall x \in M$. Complete the following as well:

(2) **Prove or disprove**: $\varphi \in \operatorname{Hom}_R\big(M, \operatorname{Hom}_R(R, M)\big)$, i.e., φ is R-linear.
(3) **Prove or disprove**: φ is an injective (i.e., 1-1) function.
(4) **Prove or disprove**: φ is a surjective (i.e., onto) function.
(5) **Prove or disprove**: φ is an R-linear isomorphism, so that $M \cong \operatorname{Hom}_R(R, M)$.

Problem 4.2. Let R be a ring and $h \in \operatorname{Hom}_R(M, N)$, where M and N are R-modules. Let A, B be subsets of M. Prove that $h(A) \subseteq h(B) \iff A \subseteq B + \operatorname{Ker}(h)$. Here, for any $X, Y \subseteq M$, define $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$.

Problem 4.3. Let R be a **commutative** ring, M an R-module, and $\varnothing \neq S \subseteq R$ such that S is multiplicatively closed (i.e., $st \in S$ for all $s, t \in S$).

(1) Denote $T := \bigcup_{s \in S}(0 :_M s)$. Prove that T is an R-submodule of M.
(2) Consider the quotient R-module M/T. Prove that $\bigcup_{s \in S}(0 :_{M/T} s) = \{0_{M/T}\}$.

Problem 4.4. Let M be an R-module, and N_1, N_2 be R-submodules of M. Consider the R-homomorphism $h : M \to M/N_1 \times M/N_2$ defined by $h(m) = (m + N_1, m + N_2)$, $\forall m \in M$.

(1) Fill in the blank: $\operatorname{Ker}(h) = N_1 \boxed{?} N_2$. Justify you claim.
(2) Prove that h is onto $M/N_1 \times M/N_2$ if $N_1 + N_2 = M$. `(Also see Problem 4.5.)`
(3) Assume $N_1 + N_2 = M$. Fill in the blank: $M/\big(N_1 \boxed{?} N_2\big) \cong \frac{M}{N_1} \times \frac{M}{N_2}$. Justify.

Problem 4.5 (Extra Credit, 1 point)**.** Let R, M, N_1, N_2 and $h : M \to M/N_1 \times M/N_2$ be as in Problem 4.4 above. **Prove or disprove** the converse of Problem 4.4(2): the homomorphism h is onto $M/N_1 \times M/N_2$ **only if** $N_1 + N_2 = M$.

PROBLEMS HINTS SOLUTIONS

Modules, basic notions: Problems 1.1, 1.2, 1.3, 1.4, 1.5, 2.1, 2.2.

Submodules, properties: Problems 2.1, 2.2, 2.3, 2.4, 2.5, 3.1, 3.5.

Homomorphisms: Problems 3.2, 3.3, 3.4, 4.1, 4.2, 4.4.

Quotient modules: Problems 4.3, 4.4, 4.5.

Lecture notes and textbooks: All we have covered in class.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Module. Let R be a ring and $(M, +)$ an abelian group. We say M is an R-module if there is a scalar multiplication $rx \in M$, defined for all $r \in R$ and $x \in M$, such that

$$r(x+y) = rx + ry, \quad (r+s)x = rx + sx, \quad (rs)x = r(sx), \quad \text{and} \quad 1_R x = x \text{ (if } 1_R \in R)$$

for all r, $s \in R$ and all x, $y \in M$. If M is an R-module, then

$$0_R x = 0_M = r0_M \quad \text{and} \quad (-r)x = -(rx) = r(-x) \quad \text{for all} \quad r \in R,\, x \in M.$$

Notations. Let R be a ring, M an R-module, $\varnothing \neq A \subseteq R$ and $\varnothing \neq X \subseteq M$.

- $\operatorname{Ann}_R(X) := \{r \in R \mid rx = 0_M \text{ for all } x \in X\}$.
- $(0 :_M A) := \{x \in M \mid ax = 0_M \text{ for all } a \in A\}$.

Submodule. Let R be a ring, M an R-module and $N \subseteq M$. Then N is an R-submodule of M, denoted $N \leqslant M$, if and only if $0_M \in N$, $x - y \in N$ and $rx \in N$ for all x, $y \in N$, $r \in R$.

- The R-submodule generated by $X \subseteq M$ is $\Big\{ \sum\limits_{\text{finite}} (r_x x + n_x x) \,\big|\, x \in X,\, r_x \in R,\, n_x \in \mathbb{Z} \Big\}$.
- For any family $\{N_i\}_{i\in\Lambda}$ of R-submodules of M, $\sum_{i\in\Lambda} N_i = \Big\{ \sum\limits_{\text{finite}} y_i \,\big|\, y_i \in N_i,\, i \in \Lambda \Big\}$.

Homomorphism. Let M, N be R-modules (with R a ring). A function $h\colon M \to N$ is said to be an R-homomorphism (or an R-linear map) iff $h(x+y) = h(x)+h(y)$ and $h(rx) = rh(x)$ for all x, $y \in M$ and all $r \in R$. For any R-linear map $h\colon M \to N$, we have

- $h(0_M) = 0_N$, $h(-x) = -h(x)$ and $\operatorname{Ann}_R(x) \subseteq \operatorname{Ann}_R(h(x))$ for all $x \in M$.
- $\operatorname{Ker}(h) := \{x \in M \mid h(x) = 0_N\} \leqslant M$ and $\operatorname{Im}(h) := \{h(x) \mid x \in M\} \leqslant N$.
- Denote by $\operatorname{Hom}_R(M, N)$ the set of all R-homomorphisms from M to N.
- For φ, $\psi \in \operatorname{Hom}_R(M, N)$ and $r \in R$, define $\varphi + \psi\colon M \to N$ and $r * \varphi\colon M \to N$ by

$$(\varphi + \psi)(x) := \varphi(x) + \psi(x) \text{ and } (r * \varphi)(x) := r\varphi(x) \quad \text{for all} \quad x \in M.$$

- If R is commutative, then $\operatorname{Hom}_R(M, N)$ is an R-module under the above operations.

Quotient module. Let R be a ring, $N \leqslant M$ be R-modules. Then the quotient R-module of M modulo N is the quotient (abelian) group $M/N := \{x + N \mid x \in M\}$ together with

$$r(x+N) = rx + N, \quad \forall r \in R, \forall x \in M.$$

Note that, for x, $y \in M$, $x + N = y + N \iff x - y \in N$.

Isomorphism theorems. Let R be a ring, M, N be R-modules, and $h \in \operatorname{Hom}_R(M, N)$. Let H and $K \leqslant L$ be R-submodules of M. Then

$$\frac{M}{\operatorname{Ker}(h)} \cong \operatorname{Im}(h), \qquad \frac{H}{(H\cap K)} \cong \frac{H+K}{K} \qquad \text{and} \qquad \frac{M/K}{L/K} \cong \frac{M}{L}.$$

Direct product, external direct sum. Let $\{M_i\}_{i\in\Lambda}$ be a family of R-modules. The Cartesian product, denote $\prod_{i\in\Lambda} M_i$, is an R-module under component-wise operations.

- The direct product of $\{M_i\}_{i\in\Lambda}$ is exactly the above R-module structure on $\prod_{i\in\Lambda} M_i$.
- The external direct sum, denoted $\bigoplus_{i\in\Lambda} M_i$, consists of the elements of $\prod_{i\in\Lambda} M_i$ whose components are almost all zero. Hence $\bigoplus_{i\in\Lambda} M_i$ is an R-submodule of $\prod_{i\in\Lambda} M_i$.
- In case $\Lambda = \{1, \ldots, n\}$, we have $\prod_{i=1}^n M_i = \{(x_1, \ldots, x_n) \mid x_i \in M_i\} = \bigoplus_{i=1}^n M_i$.

Note: The above list is not intended to be complete.

Problems

have been withdrawn

from the site

PROBLEMS HINTS SOLUTIONS

Problem 5.1. Determine whether $\mathbb{Q}$, which is a $\mathbb{Z}$-module, is free over $\mathbb{Z}$. Justify.

Problem 5.2. Let F, M and N be R-modules, where R is a ring with unity, and let $g \in \operatorname{Hom}_R(M, N)$. Further assume that F is free over R and g is onto N (i.e., $g(M) = N$). Prove that, for every $f \in \operatorname{Hom}_R(F, N)$, there exists $h \in \operatorname{Hom}_R(F, M)$ such that $f = g \circ h$.

Problem 5.3. Let M and N be R-modules, where R is a ring, and let $h \in \operatorname{Hom}_R(M, N)$. Let $X \subseteq M$ and $K \subseteq N$. **Prove or disprove** each of the following:

(1) If K is an R-submodule of N, then $h^{-1}(K)$ is an R-submodule of M.
(2) If M is generated by X over R, then $\operatorname{Im}(h)$ is generated by $h(X)$ over R.

Problem 5.4. Let R be a ring with unity and M an R-module. Let $n \in \mathbb{N}$, and consider the R-module $R^n = R \times \cdots \times R = R \oplus \cdots \oplus R = \{(r_1, \ldots, r_n) \mid r_i \in R\}$. Show that the following statements are equivalent:

(1) There exist $x_1, \ldots, x_n \in M$ such that they generate M (i.e., $M = \sum_{i=1}^n Rx_i$).
(2) There exists $\varphi \in \operatorname{Hom}_R(R^n, M)$ such that $\operatorname{Im}(\varphi) = M$ (i.e., φ is onto M).

Problem 5.5 (Extra Credit, 1 point)**.** Let R, F, M, N and $g \in \operatorname{Hom}_(M, N)$ be as in Problem 5.2 but **without** the hypothesis that F is free over R. **Disprove** the following statement: For every $f \in \operatorname{Hom}_R(F, N)$, there exists $h \in \operatorname{Hom}_R(F, M)$ such that $f = g \circ h$.

PROBLEMS HINTS SOLUTIONS

Problem 6.1. Let R be a ring. Let M be an R-module such that $M = M_1 \boxplus M_2$ (meaning that $M_i \leqslant M$, $M = M_1 + M_2$ and $M_1 \cap M_2 = \{0_M\}$). Also, let $N = N_1 \oplus N_2$, where N_1, N_2 are R-modules. Denote $N_1' = \{(n_1, 0_{N_2}) \mid n_1 \in N_1\}$ and $N_2' = \{(0_{N_1}, n_2) \mid n_2 \in N_2\}$.

(1) Prove that $M \cong M_1 \oplus M_2$ (which simply says that $M_1 \boxplus M_2 \cong M_1 \oplus M_2$).
(2) True or false: (a) $N_1 \leqslant N$. (b) $N_2 \leqslant N$. (c) $N_1' \leqslant N$. (d) $N_2' \leqslant N$.
(3) Prove the one that is true: $N_1 \oplus N_2 = N_1 \boxplus N_2$ or $N_1 \oplus N_2 = N_1' \boxplus N_2'$.
(4) True or false: (a) $N_1 \cong N_1'$. (b) $N_2 \cong N_2'$.

Problem 6.2. Let M and N be R-modules (where R is a ring). Assume that there exist $g \in \mathrm{Hom}_R(M, N)$ and $h \in \mathrm{Hom}_R(N, M)$ such that $g \circ h = \mathrm{Id}_N$, the identity map on N.

(1) Prove (from scratch) that $M = \mathrm{Ker}(g) \boxplus \mathrm{Im}(h)$.
(2) **Prove or disprove**: $N = \mathrm{Ker}(h) \boxplus \mathrm{Im}(g)$.
(3) True or false: $\mathrm{Im}(h) \cong N$.
(4) True or false: $M \cong \mathrm{Ker}(g) \oplus N$.

Problem 6.3. Let S be a ring and let R be a subring of S. (If $1_R \in R$, then further assume that $1_S \in S$ and $1_R = 1_S$.) Let M be an S-module, so that there is a scalar multiplication denoted by sx for all $s \in S$ and $x \in M$. (Again, a module means a left module.)

(1) Prove that M is an R-module under the existing (and obvious) scalar multiplication.
(2) **Prove or disprove**: Every S-submodule of M is an R-submodule of M.
(3) **Prove or disprove**: Every R-submodule of M is an S-submodule of M.

Problem 6.4. Let R, S and M be as in Problem 6.3, so that M is also an R-module.

(1) **Prove**: If M is Noetherian over R then it is Noetherian over S.
(2) **Disprove**: If M is Noetherian over S then it is Noetherian over R.
(3) **Prove or disprove**: If M is Artinian over R then it is Artinian over S.
(4) **Prove or disprove**: If M is Artinian over S then it is Artinian over R.

Problem 6.5 (Extra Credit, 1 point)**.** Let R be a ring and let M be an R-module. Let H, K_1 and K_2 be R-submodules of M and further assume that $H \boxplus K_1 = M = H \boxplus K_2$. **Prove or disprove** each of the following: (1) $K_1 = K_2$. (2) $K_1 \cong K_2$ as R-modules.

PROBLEMS HINTS SOLUTIONS

Problem 7.1. Let R be a ring, let M, N be R-modules, and let $h \in \mathrm{Hom}_R(M, N)$. Also, let A and B be R-submodules of M such that $A \leqslant B$ (that is, $A \leqslant B \leqslant M$). Prove the statement: If $h(A) = h(B)$ and $A \cap \mathrm{Ker}(h) = B \cap \mathrm{Ker}(h)$ then $A = B$.

Problem 7.2. Let R be a ring, let M be an R-module. Prove that, for any R-submodules N_1 and N_2 of M, the following statements are equivalent:

(1) $N_1 + N_2$ is a Noetherian (respectively, Artinian) R-module.
(2) Both N_1 and N_2 are Noetherian (respectively, Artinian) R-modules.

Problem 7.3. Let R be a ring and $\{0_M\} \lneqq K \lneqq M$ be R-modules. Assume that there exist *simple* R-submodules N_1 and N_2 of M such that $M = N_1 + N_2$. `(An` R`-module` S `is called simple if` $S \neq \{0_S\}$ `and the only` R`-submodules of` S `are` $\{0_S\}$ `and` S`.)`

(1) Prove $M = N_1 \boxplus N_2$.
(2) Prove that there exists $i \in \{1, 2\}$ such that $M = N_i \boxplus K$.
(3) Prove that K is a simple R-module.

Problem 7.4. Let R be a commutative Noetherian ring with $1_R \in R$ and let M be a non-zero R-module. Prove that there exists $x \in M \setminus \{0_M\}$ such that $\mathrm{Ann}_R(x)$ is a prime ideal of R. `(For an ideal` $P \lneqq R$`, with` R `commutative with` 1_R`, we say that` P `is a prime ideal if, for all` $a, b \in R$`,` $ab \in P$ `implies` $a \in P$ `or` $b \in P$`.)`

Problem 7.5 (Extra Credit, 1 point)**.** Let M and F be R-modules (with R a ring with unity) such that F is free over R. Let $h \in \mathrm{Hom}_R(M, F)$ such that h is onto F (i.e., $h(M) = F$). **Prove or disprove**: $M \cong \mathrm{Ker}(h) \oplus F$.

Problem 8.1. Let R be a ring and M a non-zero Artinian R-module. Prove that there exists $L \leqslant M$ such that L is simple.

Problem 8.2. Let R be a ring, M be an R-module, and $2 \leqslant n \in \mathbb{N}$. Also, let N and N_i, where $i = 1, \ldots, n$, be R-submodules of M such that $N = \sum_{i=1}^{n} N_i$. Prove that the following statements are equivalent to each other:

(1) $N = \boxplus_{i=1}^{n} N_i$ (i.e., $N = \sum_{i=1}^{n} N_i$ and $\left(\sum_{i \neq j} N_i\right) \cap N_j = \{0_N\}$ for all $j = 1, \ldots, n$).
(2) $\left(\sum_{i=1}^{j-1} N_i\right) \cap N_j = \{0_N\}$ for all $j = 2, \ldots, n$.
(3) Every element $x \in N$ can be uniquely expressed as $x = \sum_{i=1}^{n} x_i$ with $x_i \in N_i$.
(4) The element $0_N \in N$ can be uniquely expressed as $0_N = \sum_{i=1}^{n} x_i$ with $x_i \in N_i$.

Problem 8.3. Let R be a ring, N be an R-module, and $2 \leqslant n \in \mathbb{N}$. Let N_i, where $i = 1, \ldots, n$, be R-submodules of N. Also, let $K_i \leqslant M_i$, where $i = 1, \ldots, n$, be R-modules.

(1) Prove that if $N = \boxplus_{i=1}^{n} N_i$ then $N \cong \oplus_{i=1}^{n} N_i$.
(2) True or false: $\oplus_{i=1}^{n} K_i \leqslant \oplus_{i=1}^{n} M_i$.
(3) **Prove or disprove**: $(\oplus_{i=1}^{n} M_i) \,/\, (\oplus_{i=1}^{n} K_i) \cong \oplus_{i=1}^{n} (M_i/K_i)$.

Problem 8.4. Consider the free $\mathbb{Z}$-module $\mathbb{Z}^2 = \{\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) \mid r, s \in \mathbb{Z}\}$. (We write the elements of $\mathbb{Z}^2$ as column vectors.) Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{smallmatrix}\right)$ and $x_i = \left(\begin{smallmatrix} a_{1i} \\ a_{2i} \end{smallmatrix}\right) \in \mathbb{Z}^2$ be the i-th column of A. Let $y_i \in \mathbb{Z}^2$ be the i-th column of $A \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{pmatrix}$. Suppose $\left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1} A \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{pmatrix} = \left(\begin{smallmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{smallmatrix}\right)$.

(1) Compute $\left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1}$ and $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{pmatrix}^{-1}$ explicitly. It suffices to write down your answers.
(2) Express each x_i as a linear combination of y_1, y_2, y_3 explicitly. No need to justify.
(3) True or false: $\sum_{i=1}^{3} \mathbb{Z}x_i = \sum_{i=1}^{3} \mathbb{Z}y_i$. No justification is necessary.
(4) **Prove** that $z_1 = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ and $z_2 = \left(\begin{smallmatrix} 5 \\ 8 \end{smallmatrix}\right)$ form a basis of $\mathbb{Z}^2$.
(5) Express each y_i as a linear combination of z_1 and z_2. No need to justify.

Problem 8.5 (Extra Credit, 1 point)**.** Let R be a ring. **Prove or disprove**: If M is a non-zero Noetherian R-module then there exists $K \leqslant M$ such that M/K is simple.

PROBLEMS HINTS SOLUTIONS

Materials covered earlier: Homework Sets 1, 2, 3, 4; Exam I.

General problems about modules: Problems 5.1, 5.3, 5.4, 6.3, 7.3, 8.1, 8.2, 8.4.

Free modules: Problems 5.2, 5.5, 7.3.

Direct sums (internal or external): Problems 6.1, 6.2, 6.5.

Noetherian modules, Artinian modules: Problems 6.4, 7.1, 7.2, 7.4, 7.5.

Simple modules: Problems 7.2, 8.3, 8.5.

Modules over a PID: Problem 8.4.

Lecture notes and textbooks: All we have covered.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Internal direct sum. Let R be a ring, M an R-module, N an R-submodule of M, and $\{N_i\}_{i\in\Delta}$ is family of R-submodules of M. We say that N is an internal direct sum of $\{N_i\}_{i\in\Delta}$, denoted $N = \boxplus_{i\in\Delta} N_i$, if $N = \sum_{i\in\Delta} N_i$ and $N_j \cap (\sum_{i\in\Delta\setminus\{j\}} N_i) = \{0_M\}$ for all $j \in \Delta$.

- $N = \boxplus_{i\in\Delta} N_i$ if and only if $N = \sum_{i\in\Delta} N_i$ and every element $x \in N$ is uniquely expressed as $x = \sum_{i\in\Delta} n_i$ with $x_i \in N_i$ and $x_i = 0_M$ for almost all $i \in \Delta$.
- If $N = \boxplus_{i\in\Delta} N_i$, then $N \cong \oplus_{i\in\Delta} N_i$.

Free module. Let R be a ring with unity and F an R-module. We say that F is a free (over R) if there exists $B \subseteq F$ such that B generates F and B is linear independent over R, in which case we say that B is a basis of F.

- F is a free R-module with a basis $B \iff$ every $x \in F$ can be uniquely written as $x = \sum_{b\in B} r_b b$ with $r_b \in R$ and $r_b = 0_R$ for almost all $b \in B$.
- F is a free R-module $\iff F \cong R^{\oplus B}$ for some set B.
- F is a free R-module with a basis $B \iff F = \boxplus_{b\in B} Rb$ and $\operatorname{Ann}(b) = \{0_R\}, \forall\, b \in B$.
- F is a free R-module with a basis $B \iff$ for any R-module M and any map $\theta\colon B \to M$, there is a unique R-linear map $h \in \operatorname{Hom}_R(F, M)$ such that $h|_B = \theta$.
- If R is commutative and F is free, then all bases of F have the same cardinality, which is called the rank of F.

Noetherian module. We say that an R-module M is Noetherian (over R) if every ascending chain of R-submodules of M eventually stabilizes. The following are equivalent:

- M is Noetherian as an R-module.
- All R-submodules of M are finitely generated.
- Every non-empty set of R-submodules of M has a maximal member/object.
- N and M/N are Noetherian over R for every R-submodule N of M.
- N and M/N are Noetherian over R for some R-submodule N of M.

We say R is a (left) Noetherian ring if R is Noetherian as a (left) R-module.

- [Hilbert basis theorem] If R is Noetherian, so is $R[x]$ and so is $R[x_1, \ldots, x_n]$.

Artinian module. We say that an R-module M is Artinian (over R) if every descending chain of R-submodules of M eventually stabilizes. The following are equivalent:

- M is Artinian as an R-module.
- Every non-empty set of R-submodules of M has a minimal member/object.
- N and M/N are Artinian over R for every (or for some) R-submodule N of M.

We say R is a (left) Artinian ring if R is Artinian as a (left) R-module.

Simple module. Let R be a ring and let M, N be R-modules. We say that M is a simple R-module if $M \neq \{0_M\}$ and the only R-submodules of M are $\{0_M\}$ and M.

- Assume that M is simple. Then every $\varphi \in \operatorname{Hom}_R(M, N)$ is either injective or the zero map. Similarly, every $\psi \in \operatorname{Hom}_R(N, M)$ is either the zero map or surjective.

Modules over a PID. Let R be a PID and let F be a free R-module of finite rank n.

- Every R-submodule of F is free (over R) of rank $\leqslant n$.

Note: The above list is not intended to be complete.

You must solve a problem **completely and correctly** in order to get the extra credit. You may attempt a problem for as many times as you wish by 04/25.

The points you get here will be added to the total score from the homework assignments.

Problem E-1 (3 points)**.** Let R be a commutative ring with unity and $S = R[x]$. For all ideals I and J of R, prove $I \cap J = \big(xIS + (1-x)JS\big) \cap R$.

Problem E-2 (3 points)**.** Let R be a commuative ring with unity and I, J ideals of R. **Prove or disprove**: If $R/I \cong R/J$ as R-modules then $I = J$.

Problem E-3 (3 points)**.** Is there a field that is free over $\mathbb{Z}$? Explain.

PROBLEMS HINTS SOLUTIONS