◇ ◇ ◇ ◇ MATH 8221: ABSTRACT ALGEBRA II ◇ ◇ ◇ ◇
HOMEWORK SETS AND EXAMS

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2025 SPRING SEMESTER
GEORGIA STATE UNIVERSITY

Contents

Note. Each homework set contains four (4) regular problems. When solving the problems, please make sure that your arguments are rigorous and complete.

There is also a set of problems for extra credits; see the last page of this file.

In this course, a ring may not be commutative and may not have unity. By default, a module over a ring R means a left R-module.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

$M \oplus N$ ...... $h \in \mathrm{Hom}_R(M,N) \implies M/\operatorname{Ker}(h) \cong \operatorname{Im}(h)$ ...... $R^{\oplus X} \twoheadrightarrow M$ ...... $M \cong \frac{R}{(d_1)} \oplus \cdots \oplus \frac{R}{(d_r)} \oplus R^m$ ...... $M \otimes_R N$

Problem 1.1. Let R be a ring (not necessarily with 1), $r \in R$, M an R-module (i.e., a left R-module), and $x \in M$.

(1) Prove that $\mathrm{Ann}_R(x)$ is a left ideal of R.
(2) Prove that $\mathrm{Ann}_R(M)$ is an ideal (i.e., a two-sided ideal) of R.
(3) Prove the one that (always) holds: $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(M)$ or $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(x)$.
(4) **Disprove** $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(rx)$ with a concrete counterexample.

Hint. (1) & (2) Use the definitions or criteria of left ideals and (two-sided) ideals. Recall that $\mathrm{Ann}_R(x) = \{s \in R \mid sx = 0_R\}$ and $\mathrm{Ann}_R(M) = \{s \in R \mid sm = 0_R \text{ for all } m \in M\}$.

(3) Decide which inclusion holds; and then prove it. How to show $X \subseteq Y$ in set theory?

(4) The ring in your counterexample must be non-commutative in light of Problem 1.2(2). One such ring is $R = M_2(\mathbb{Z})$ of 2×2 matrices over $\mathbb{Z}$; and R is (naturally) a left R-module.

Problem 1.2. Let R be a commutative ring (not necessarily with 1), $r \in R$, M an R-module (i.e., a left R-module), and $x \in M$.

(1) Prove that $\mathrm{Ann}_R(x)$ is an ideal of R.
(2) Prove $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(rx)$. (Compare with Problem 1.1(4) above.)

Hint. Note that R is commutative. Compare with Problem 1.1.

Problem 1.3. Let R be a ring with 1 (i.e., $1_R \in R$), M a (left) R-module, and $x \in M$.

(1) Prove the following: $x = 0_M \iff \mathrm{Ann}_R(x) = R$.
(2) Prove the following: $M = \{0_M\} \iff \mathrm{Ann}_R(M) = R$. (Compare with Problem 1.5.)

Hint. Note that $1_R \in R$, so that $1_R \cdot y = y$ for all $y \in M$.

Problem 1.4. Let R be a ring, M an R-module, and x, $y \in M$.

(1) Show $\mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y) \subseteq \mathrm{Ann}_R(x+y)$.
(2) **Prove or disprove**: $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(-x)$.
(3) **Prove or disprove**: $\mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y) = \mathrm{Ann}_R(x+y)$.

Hint. (2) For any $r \in R$, does the implication $r \in \mathrm{Ann}_R(x) \iff r \in \mathrm{Ann}_R(-x)$ hold?

(3) For any $r \in R$, is it true that $r \in \mathrm{Ann}_R(x) \cap \mathrm{Ann}_R(y)$ if and only if $r \in \mathrm{Ann}_R(x+y)$?

Study each equivalence in (2) and (3) carefully and determine whether it holds. You may just as well consider the case $R = \mathbb{Z} = M$. Either give a rigorous proof or give a concrete counterexample. Actually, part (2) could help you with part (3).

Problem 1.5 (Extra Credit, 1 point)**.** Let M be a (left) module over a ring R (not necessarily with 1). **Prove or disprove**: $M = \{0_M\} \iff \mathrm{Ann}_R(M) = R$. (Compare with Problem 1.3.)

Hint. No partial credit. Either give a rigorous proof or give a concrete counterexample.

Problem 2.1. Let $R = M_2(\mathbb{Z})$, the ring of all 2×2 matrices over $\mathbb{Z}$; and let $M = R$, which is naturally a (left) R-module. Also consider

$$x = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right) \in M, \quad N_1 = \{\left(\begin{smallmatrix} m & 0 \\ n & 0 \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \subseteq M \quad \text{and} \quad N_2 = \{\left(\begin{smallmatrix} m & n \\ 0 & 0 \end{smallmatrix}\right) \mid m, n \in \mathbb{Z}\} \subseteq M.$$

(1) Determine/describe $\operatorname{Ann}_R(x)$ and Rx explicitly. No need to justify.
(2) Out of N_1 and N_2, which one, if any, is an R-submodule of M? No need to justify.
(3) In case N_i is an R-submodule of M, find $\operatorname{Ann}_R(N_i)$ explicitly. No need to justify.
(4) If N_i is not an R-submodule of M, explain why it is not an R-submodule of M.

Hint. All should be straightforward from definition/criterion. Just be careful.

Problem 2.2. Let R be a ring (that may not be commutative or have unity), $a \in R$, and M an R-module. We define/denote $(0 :_M a)$ to be $\{x \in M \mid ax = 0_M\}$.

(1) True or false: $(0 :_M a) = M \iff a \in \operatorname{Ann}_R(M)$. No justification is necessary.
(2) **Disprove** with a counterexample: $(0 :_M a)$ is (always) an R-submodule of M.
(3) Prove that, if R is commutative, then $(0 :_M a)$ is an R-submodule of M.

Hint. (2) Find a concrete example of R, a concrete $a \in R$ and a concrete R-module M such that $(0 :_M a)$ is not an R-submodule of M. In light of (3), the R in (2) cannot be commutative. You may let $M = R$.

Problem 2.3. Let R be a ring, M be an R-module, and N_1, N_2 be R-submodules of M.

(1) True or false: both $\operatorname{Ann}_R(N_1)$ and $\operatorname{Ann}_R(N_2)$ are (2-sided) ideals of R.
(2) Prove $\operatorname{Ann}_R(N_1 + N_2) = \operatorname{Ann}_R(N_1) \cap \operatorname{Ann}_R(N_2)$.
(3) Prove $\operatorname{Ann}_R(N_1 \cap N_2) \supseteq \operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2)$.

Hint. (1) Note that each N_i is an R-module on its own. Then see Problem 1.1.
(2) Note that $N_1 + N_2 = \{x_1 + x_2 \mid x_1 \in N_1,\ x_2 \in N_2\}$.
(3) Note that, for any ideals I and J of R, $I + J = \{i + j \mid i \in I,\ j \in J\}$.

Problem 2.4. (1) Prove that $\mathbb{C}$ is a finitely generated $\mathbb{R}$-module.
(2) Prove that $\mathbb{Q}$ is *not* a finitely generated $\mathbb{Z}$-module.

Hint. (2) Suppose, on the contrary, that $\mathbb{Q}$ is a finitely generated module over $\mathbb{Z}$, say by $\{x_1 = \frac{a_1}{b_1}, \ldots, x_g = \frac{a_g}{b_g}\}$, with a_i, $b_i \in \mathbb{Z}$, $b_i \neq 0$ and $g \in \mathbb{N}$. Then, as $\mathbb{Z}$ has unity, we should have $\mathbb{Q} = \{\sum_{i=1}^{g} n_i x_i \mid n_i \in \mathbb{Z}\}$. That is, every rational number is a linear combination of these (fixed) $x_1, \ldots, x_g$. Is this possible? Why or why not? (For example, if q is any $\mathbb{Z}$-linear combination of $\frac{13}{4}$, $\frac{8}{5}$ and $\frac{-17}{6}$, then $60q \in \mathbb{Z}$. Thus $\frac{1}{61}$ cannot be a $\mathbb{Z}$-linear combination of $\frac{13}{4}$, $\frac{8}{5}$ and $\frac{-17}{6}$ since $60 \cdot \frac{1}{61} \notin \mathbb{Z}$. Mimic this argument for general $x_1 = \frac{a_1}{b_1}, \ldots, x_g = \frac{a_g}{b_g}$.)

Problem 2.5 (Extra Credit, 1 point)**.** Let R, M, N_1 and N_2 be as in Problem 2.3. **Prove or disprove**: $\operatorname{Ann}_R(N_1 \cap N_2) = \operatorname{Ann}_R(N_1) + \operatorname{Ann}_R(N_2)$. (Compare with Problem 2.3(3).)

Hint. No partial credit. Either give a rigorous proof or find a concrete counterexample.

Problem 3.1. Let R be a ring and M an R-module. Let A, B, C be R-submodules of M.

(1) Prove $(A+B)\cap C \supseteq (A\cap C)+(B\cap C)$.

(2) Assume $A\subseteq C$ hence $A\cap C = A$. Prove $(A+B)\cap C = (A\cap C)+(B\cap C)$.

Hint. (1) This is straightforward: Let $x\in(A\cap C)+(B\cap C)$. Show $x\in(A+B)\cap C$.

(2) This is also straightforward: Let $y\in(A+B)\cap C$. Show $y\in(A\cap C)+(B\cap C)$.

Problem 3.2. Let R be a ring (not necessarily with unity), M and N be R-modules, $X\subseteq M$ such that X generates M, and f, $g\in \operatorname{Hom}_R(M,N)$. **Prove or disprove**: If $f|_X = g|_X$ then $f=g$. (You may assume $X\neq\varnothing$, as this is clear when $X=\varnothing$.)

Hint. Provide a rigorous proof or a concrete counterexample. Note that, here, $f|_X$ stands for the restriction of f to X. Also, as R does not necessarily have unity, how should the assumption $M=(X)$ be interpreted?

Problem 3.3. Let R be a **commutative** ring and let M, N be R-modules. Note that $\operatorname{Hom}_R(M,N)$ is an R-module `(such that, for all` $r\in R$ `and` $h\in\operatorname{Hom}_R(M,N)$`, we define` $r*h\in\operatorname{Hom}_R(M,N)$ `by` $(r*h)(m)=rh(m)$ `for all` $m\in M$`)`.

(1) **Prove ot disprove**: $\operatorname{Ann}_R(M)\subseteq\operatorname{Ann}_R(\operatorname{Hom}_R(M,N))$.

(2) **Prove ot disprove**: $\operatorname{Ann}_R(N)\subseteq\operatorname{Ann}_R(\operatorname{Hom}_R(M,N))$.

Hint. Note that the zero element of $\operatorname{Hom}_R(M,N)$ is the zero map $0_H : M\to N$ determined by $0_H(x)=0_N$ for all $x\in M$. (Here 0_H is short for $0_{\operatorname{Hom}_R(M,N)}$.)

Problem 3.4. Let R be a (not necessarily commutative) ring and let N be an R-module. We already know that $(\operatorname{Hom}_R(R,N),+)$ is an abelian group. For every $r\in R$ and for every $h\in\operatorname{Hom}_R(R,N)$, we define $r*h: R\to N$ as $(r*h)(x)=h(xr)$ for all $x\in R$.

(1) Show that $r*h\in\operatorname{Hom}_R(R,N)$ for all $r\in R$ and for all $h\in\operatorname{Hom}_R(R,N)$.

(2) Prove that $\operatorname{Hom}_R(R,N)$ is an R-module (under the scalar multiplication $r*h$).

Hint. All is straightforward, as long as you know what you are doing at each step. Given g, $h\in\operatorname{Hom}_R(M,N)$, recall that $g+h$ is defined via $(g+h)(x)=g(x)+h(x)$, $\forall x\in M$.

(2) You need to show $(rs)*h = r*(s*h)$, $(r+s)*h=(r*h)+(s*h)$ as well as $r*(g+h)=(r*g)+(r*h)$ for all r, $s\in R$ and for all g, $h\in\operatorname{Hom}_R(R,N)$. In the case where $1_R\in R$, remember to show that $1_R*h=h$. For example, to prove $r*(g+h)=(r*g)+(r*h)$, you need to pick an arbitrary $x\in R$ and verify that $[r*(g+h)](x)=[(r*g)+(r*h)](x)$.

Be super careful here, especially with $(rs)*h = r*(s*h)$.

Problem 3.5 (Extra Credit, 1 point)**.** Let R and M be as in Problem 3.1. **Prove or disprove**: $(A+B)\cap C=(A\cap C)+(B\cap C)$ for all R-submodules A, B, C of M.

Hint. No partial credit. Either give a rigorous proof or give a concrete counterexample.

PROBLEMS HINTS SOLUTIONS

Problem 4.1. Let R be a ring with unity and let M be an R-module. For every $x \in M$, define $h_x : R \to M$ by $h_x(s) = sx$, $\forall s \in R$. By Problem 3.4, $\operatorname{Hom}_R(R, M)$ is an R-module.

(1) For every $x \in M$, prove that $h_x \in \operatorname{Hom}_R(R, M)$.

Define $\varphi\colon M \to \operatorname{Hom}_R(R, M)$ by $\varphi(x) = h_x$, $\forall x \in M$. Complete the following as well:

(2) **Prove or disprove**: $\varphi \in \operatorname{Hom}_R\big(M, \operatorname{Hom}_R(R, M)\big)$, i.e., φ is R-linear.
(3) **Prove or disprove**: φ is an injective (i.e., 1-1) function.
(4) **Prove or disprove**: φ is a surjective (i.e., onto) function.
(5) **Prove or disprove**: φ is an R-linear isomorphism, so that $M \cong \operatorname{Hom}_R(R, M)$.

Hint. By Problem 3.4, for every $r \in R$ and $h \in \operatorname{Hom}_R(R, M)$, the scalar multiplication $r * h$ is given by $(r * h)(x) = h(xr)$ for all $x \in R$.

(2) Examine $\varphi(x + y) \overset{?}{=} \varphi(x) + \varphi(y)$ and $\varphi(rx) \overset{?}{=} r * \varphi(x)$ for all $r \in R$ and $x, y \in M$.

(3) & (4) For all $x, y \in M$ such that $\varphi(x) = \varphi(y)$, try to determine $x \overset{?}{=} y$. Note that $\varphi(x)$ and $\varphi(y)$ are functions from R to M. Finally, for any $g \in \operatorname{Hom}_R(R, M)$, note that $g(s) = g(s1_R) = sg(1_R)$ for all $s \in R$. See whether you can fill in the blank: $g = \varphi(?)$.

Problem 4.2. Let R be a ring and $h \in \operatorname{Hom}_R(M, N)$, where M and N are R-modules. Let A, B be subsets of M. Prove that $h(A) \subseteq h(B) \iff A \subseteq B + \operatorname{Ker}(h)$. Here, for any $X, Y \subseteq M$, define $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$.

Hint. All is straightforward. How to show $\iff$? How to show $X \subseteq Y$ in set theory?

Problem 4.3. Let R be a **commutative** ring, M an R-module, and $\varnothing \neq S \subseteq R$ such that S is multiplicatively closed (i.e., $st \in S$ for all $s, t \in S$).

(1) Denote $T := \bigcup_{s \in S} (0 :_M s)$. Prove that T is an R-submodule of M.
(2) Consider the quotient R-module M/T. Prove that $\bigcup_{s \in S} (0 :_{M/T} s) = \{0_{M/T}\}$.

Hint. Use the relevant definitions and criteria. What is the meaning of $x, y \in \bigcup_{s \in S} (0 :_M s)$ in set theory? What is $0_{M/T}$?

Problem 4.4. Let M be an R-module, and N_1, N_2 be R-submodules of M. Consider the R-homomorphism $h : M \to M/N_1 \times M/N_2$ defined by $h(m) = (m + N_1, m + N_2)$, $\forall m \in M$.

(1) Fill in the blank: $\operatorname{Ker}(h) = N_1 \boxed{?} N_2$. Justify you claim.
(2) Prove that h is onto $M/N_1 \times M/N_2$ if $N_1 + N_2 = M$. `(Also see Problem 4.5.)`
(3) Assume $N_1 + N_2 = M$. Fill in the blank: $M/\big(N_1 \boxed{?} N_2\big) \cong \frac{M}{N_1} \times \frac{M}{N_2}$. Justify.

Hint. (1) Fill in the blank with one of the following: $+$, $\times$, $\oplus$, $\cap$, $\cup$. Justify your claim.

(2) Let $(x + N_1, y + N_2)$ be an arbitrary element of $M/N_1 \times M/N_2$, with $x, y \in M$. Try to find/construct an pre-image of $(x + N_1, y + N_2)$ in M, by taking advantage of the assumption $M = N_1 + N_2$. Or, you could study $(0 + N_1, y + N_2)$ and $(x + N_1, 0 + N_2)$ separately.

(3) The justification should be short and easy, in light of a theorem (which one?).

Problem 4.5 (Extra Credit, 1 point)**.** Let R, M, N_1, N_2 and $h : M \to M/N_1 \times M/N_2$ be as in Problem 4.4 above. **Prove or disprove** the converse of Problem 4.4(2): the homomorphism h is onto $M/N_1 \times M/N_2$ **only if** $N_1 + N_2 = M$.

Hint. No partial credit. Either give a rigorous proof or give a concrete counterexample.

Modules, basic notions: Problems 1.1, 1.2, 1.3, 1.4, 1.5, 2.1, 2.2.

Submodules, properties: Problems 2.1, 2.2, 2.3, 2.4, 2.5, 3.1, 3.5.

Homomorphisms: Problems 3.2, 3.3, 3.4, 4.1, 4.2, 4.4.

Quotient modules: Problems 4.3, 4.4, 4.5.

Lecture notes and textbooks: All we have covered in class.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Module. Let R be a ring and $(M, +)$ an abelian group. We say M is an R-module if there is a scalar multiplication $rx \in M$, defined for all $r \in R$ and $x \in M$, such that

$$r(x+y) = rx + ry, \quad (r+s)x = rx + sx, \quad (rs)x = r(sx), \quad \text{and} \quad 1_R x = x \text{ (if } 1_R \in R)$$

for all r, $s \in R$ and all x, $y \in M$. If M is an R-module, then

$$0_R x = 0_M = r0_M \quad \text{and} \quad (-r)x = -(rx) = r(-x) \quad \text{for all} \quad r \in R,\, x \in M.$$

Notations. Let R be a ring, M an R-module, $\varnothing \neq A \subseteq R$ and $\varnothing \neq X \subseteq M$.

- $\mathrm{Ann}_R(X) := \{r \in R \mid rx = 0_M \text{ for all } x \in X\}$.
- $(0 :_M A) := \{x \in M \mid ax = 0_M \text{ for all } a \in A\}$.

Submodule. Let R be a ring, M an R-module and $N \subseteq M$. Then N is an R-submodule of M, denoted $N \leqslant M$, if and only if $0_M \in N$, $x - y \in N$ and $rx \in N$ for all x, $y \in N$, $r \in R$.

- The R-submodule generated by $X \subseteq M$ is $\Big\{ \sum_{\text{finite}} (r_x x + n_x x) \,\big|\, x \in X,\, r_x \in R,\, n_x \in \mathbb{Z} \Big\}$.
- For any family $\{N_i\}_{i\in\Lambda}$ of R-submodules of M, $\sum_{i\in\Lambda} N_i = \Big\{ \sum_{\text{finite}} y_i \,\big|\, y_i \in N_i,\, i \in \Lambda \Big\}$.

Homomorphism. Let M, N be R-modules (with R a ring). A function $h\colon M \to N$ is said to be an R-homomorphism (or an R-linear map) iff $h(x+y) = h(x) + h(y)$ and $h(rx) = rh(x)$ for all x, $y \in M$ and all $r \in R$. For any R-linear map $h\colon M \to N$, we have

- $h(0_M) = 0_N$, $h(-x) = -h(x)$ and $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(h(x))$ for all $x \in M$.
- $\mathrm{Ker}(h) := \{x \in M \mid h(x) = 0_N\} \leqslant M$ and $\mathrm{Im}(h) := \{h(x) \mid x \in M\} \leqslant N$.
- Denote by $\mathrm{Hom}_R(M, N)$ the set of all R-homomorphisms from M to N.
- For φ, $\psi \in \mathrm{Hom}_R(M, N)$ and $r \in R$, define $\varphi + \psi\colon M \to N$ and $r * \varphi\colon M \to N$ by

$$(\varphi + \psi)(x) := \varphi(x) + \psi(x) \text{ and } (r * \varphi)(x) := r\varphi(x) \quad \text{for all} \quad x \in M.$$

- If R is commutative, then $\mathrm{Hom}_R(M, N)$ is an R-module under the above operations.

Quotient module. Let R be a ring, $N \leqslant M$ be R-modules. Then the quotient R-module of M modulo N is the quotient (abelian) group $M/N := \{x + N \mid x \in M\}$ together with

$$r(x+N) = rx + N, \quad \forall r \in R,\, \forall x \in M.$$

Note that, for x, $y \in M$, $x + N = y + N \iff x - y \in N$.

Isomorphism theorems. Let R be a ring, M, N be R-modules, and $h \in \mathrm{Hom}_R(M, N)$. Let H and $K \leqslant L$ be R-submodules of M. Then

$$\frac{M}{\mathrm{Ker}(h)} \cong \mathrm{Im}(h), \qquad \frac{H}{(H \cap K)} \cong \frac{H+K}{K} \qquad \text{and} \qquad \frac{M/K}{L/K} \cong \frac{M}{L}.$$

Direct product, external direct sum. Let $\{M_i\}_{i\in\Lambda}$ be a family of R-modules. The Cartesian product, denote $\prod_{i\in\Lambda} M_i$, is an R-module under component-wise operations.

- The direct product of $\{M_i\}_{i\in\Lambda}$ is exactly the above R-module structure on $\prod_{i\in\Lambda} M_i$.
- The external direct sum, denoted $\bigoplus_{i\in\Lambda} M_i$, consists of the elements of $\prod_{i\in\Lambda} M_i$ whose components are almost all zero. Hence $\bigoplus_{i\in\Lambda} M_i$ is an R-submodule of $\prod_{i\in\Lambda} M_i$.
- In case $\Lambda = \{1, \ldots, n\}$, we have $\prod_{i=1}^n M_i = \{(x_1, \ldots, x_n) \mid x_i \in M_i\} = \bigoplus_{i=1}^n M_i$.

Note: The above list is not intended to be complete.

Hints

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PROBLEMS HINTS SOLUTIONS

Problem 5.1. Determine whether $\mathbb{Q}$, which is a $\mathbb{Z}$-module, is free over $\mathbb{Z}$. Justify.

Hint. Either find a $\mathbb{Z}$-basis for $\mathbb{Q}$ or explain why $\mathbb{Q}$ can not admit a $\mathbb{Z}$-basis. (Suppose that $\mathbb{Q}$ is a free $\mathbb{Z}$-module, say with a basis B. What can you say about B?)

Problem 5.2. Let F, M and N be R-modules, where R is a ring with unity, and let $g \in \operatorname{Hom}_R(M, N)$. Further assume that F is free over R and g is onto N (i.e., $g(M) = N$). Prove that, for every $f \in \operatorname{Hom}_R(F, N)$, there exists $h \in \operatorname{Hom}_R(F, M)$ such that $f = g \circ h$.

Hint. Say B is a basis of the free R-module F (so $B \subseteq F$ in particular). By Problem 3.2, it suffices to find $h \in \operatorname{Hom}_R(F, M)$ such that $f|_B = (g \circ h)|_B$. Construct one such h.

Let us temporarily consider $f : F \to N$ and $g : M \to N$ as functions (i.e., forget that they are actually R-linear homomorphisms) with g onto N. Can you define/construct a map $\theta : B \to M$ such that $f|_B = g \circ \theta$? How/why?

Using the fact that B is a basis of F, can you construct $h \in \operatorname{Hom}_R(F, M)$ as required?

Problem 5.3. Let M and N be R-modules, where R is a ring, and let $h \in \operatorname{Hom}_R(M, N)$. Let $X \subseteq M$ and $K \subseteq N$. **Prove or disprove** each of the following:

(1) If K is an R-submodule of N, then $h^{-1}(K)$ is an R-submodule of M.
(2) If M is generated by X over R, then $\operatorname{Im}(h)$ is generated by $h(X)$ over R.

Hint. All is straightforward. Here R does not necessarily have unity.

Problem 5.4. Let R be a ring with unity and M an R-module. Let $n \in \mathbb{N}$, and consider the R-module $R^n = R \times \cdots \times R = R \oplus \cdots \oplus R = \{(r_1, \ldots, r_n) \mid r_i \in R\}$. Show that the following statements are equivalent:

(1) There exist $x_1, \ldots, x_n \in M$ such that they generate M (i.e., $M = \sum_{i=1}^n Rx_i$).
(2) There exists $\varphi \in \operatorname{Hom}_R(R^n, M)$ such that $\operatorname{Im}(\varphi) = M$ (i.e., φ is onto M).

Hint. Note that R^n is free over R. For example, $\{e_1, \ldots, e_n\}$ is a basis of R^n over R, where $e_i = (0, \ldots, 0, 1_R, 0, \ldots, 0) \in R^n$ with the only 1_R at the i-th position.

(1) $\Longrightarrow$ (2): Use a theorem to construct an R-homomorphism φ. Make sure it is onto.
(2) $\Longrightarrow$ (1): Given an onto R-linear map φ, try to find $x_1, \ldots, x_n \in M$ as required.

Problem 5.5 (Extra Credit, 1 point)**.** Let R, F, M, N and $g \in \operatorname{Hom}_(M, N)$ be as in Problem 5.2 but **without** the hypothesis that F is free over R. **Disprove** the following statement: For every $f \in \operatorname{Hom}_R(F, N)$, there exists $h \in \operatorname{Hom}_R(F, M)$ such that $f = g \circ h$.

Hint. No partial credit. Provide a concrete counterexample with reasoning and explanation.

Problem 6.1. Let R be a ring. Let M be an R-module such that $M = M_1 \boxplus M_2$ (meaning that $M_i \leqslant M$, $M = M_1 + M_2$ and $M_1 \cap M_2 = \{0_M\}$). Also, let $N = N_1 \oplus N_2$, where N_1, N_2 are R-modules. Denote $N_1' = \{(n_1, 0_{N_2}) \mid n_1 \in N_1\}$ and $N_2' = \{(0_{N_1}, n_2) \mid n_2 \in N_2\}$.

(1) Prove that $M \cong M_1 \oplus M_2$ (which simply says that $M_1 \boxplus M_2 \cong M_1 \oplus M_2$).
(2) True or false: (a) $N_1 \leqslant N$. (b) $N_2 \leqslant N$. (c) $N_1' \leqslant N$. (d) $N_2' \leqslant N$.
(3) Prove the one that is true: $N_1 \oplus N_2 = N_1 \boxplus N_2$ or $N_1 \oplus N_2 = N_1' \boxplus N_2'$.
(4) True or false: (a) $N_1 \cong N_1'$. (b) $N_2 \cong N_2'$.

Hint. All is straightforward. For (1), it might be (slightly) easier to construct an isomorphism from $M_1 \oplus M_2$ to M. For (2)–(4), interpret N, N_i and N_i' precisely as what they are.

Problem 6.2. Let M and N be R-modules (where R is a ring). Assume that there exist $g \in \mathrm{Hom}_R(M, N)$ and $h \in \mathrm{Hom}_R(N, M)$ such that $g \circ h = \mathrm{Id}_N$, the identity map on N.

(1) Prove (from scratch) that $M = \mathrm{Ker}(g) \boxplus \mathrm{Im}(h)$.
(2) **Prove or disprove**: $N = \mathrm{Ker}(h) \boxplus \mathrm{Im}(g)$.
(3) True or false: $\mathrm{Im}(h) \cong N$.
(4) True or false: $M \cong \mathrm{Ker}(g) \oplus N$.

Hint. Note that $\boxplus$ stands for internal direct sum. Does (1) look familiar? Even though (1) has been (essentially) done in a previous problem, here you need to prove it from scratch (i.e., without quoting that previous problem).

For (2), you are welcome to use results (from set theory) about functions.

Problem 6.3. Let S be a ring and let R be a subring of S. (If $1_R \in R$, then further assume that $1_S \in S$ and $1_R = 1_S$.) Let M be an S-module, so that there is a scalar multiplication denoted by sx for all $s \in S$ and $x \in M$. (Again, a module means a left module.)

(1) Prove that M is an R-module under the existing (and obvious) scalar multiplication.
(2) **Prove or disprove**: Every S-submodule of M is an R-submodule of M.
(3) **Prove or disprove**: Every R-submodule of M is an S-submodule of M.

Hint. (1) For every $r \in R$ and $x \in M$, we see that $r \in S$ and, hence, rx is defined. You need to prove that, under this scalar multiplication, M is an R-module. (If you think that this is trivial, then you are right!)

(2) and/or (3): The following rings might help: $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Problem 6.4. Let R, S and M be as in Problem 6.3, so that M is also an R-module.

(1) **Prove**: If M is Noetherian over R then it is Noetherian over S.
(2) **Disprove**: If M is Noetherian over S then it is Noetherian over R.
(3) **Prove or disprove**: If M is Artinian over R then it is Artinian over S.
(4) **Prove or disprove**: If M is Artinian over S then it is Artinian over R.

Hint. Problem 6.3 (2)&(3) should help, if solved correctly. The following rings might be useful in constructing counterexample(s): $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Problem 6.5 (Extra Credit, 1 point)**.** Let R be a ring and let M be an R-module. Let H, K_1 and K_2 be R-submodules of M and further assume that $H \boxplus K_1 = M = H \boxplus K_2$. **Prove or disprove** each of the following: (1) $K_1 = K_2$. (2) $K_1 \cong K_2$ as R-modules.

Hint. No partial credit. Either give a rigorous proof or give a concrete counterexample.

Problem 7.1. Let R be a ring, let M, N be R-modules, and let $h \in \mathrm{Hom}_R(M, N)$. Also, let A and B be R-submodules of M such that $A \leqslant B$ (that is, $A \leqslant B \leqslant M$). Prove the statement: If $h(A) = h(B)$ and $A \cap \mathrm{Ker}(h) = B \cap \mathrm{Ker}(h)$ then $A = B$.

Hint. Straightforward (and very short). `(This has been used in class.)`

Problem 7.2. Let R be a ring, let M be an R-module. Prove that, for any R-submodules N_1 and N_2 of M, the following statements are equivalent:

(1) $N_1 + N_2$ is a Noetherian (respectively, Artinian) R-module.
(2) Both N_1 and N_2 are Noetherian (respectively, Artinian) R-modules.

Hint. For an R-module K, note that K is Noetherian/Artinian if and only if, for some (equivalently, for all) R-submodule H of K, both H and K/H are Noetherian/Artinian. Also, (one of) the isomorphism theorems might be helpful. `(If right results are used, the proof could be very short.)`

Problem 7.3. Let R be a ring and $\{0_M\} \subsetneqq K \subsetneqq M$ be R-modules. Assume that there exist *simple* R-submodules N_1 and N_2 of M such that $M = N_1 + N_2$. (An R-module S is called simple if $S \neq \{0_S\}$ and the only R-submodules of S are $\{0_S\}$ and S.)

(1) Prove $M = N_1 \boxplus N_2$.
(2) Prove that there exists $i \in \{1, 2\}$ such that $M = N_i \boxplus K$.
(3) Prove that K is a simple R-module.

Hint. (1) If $N_1 \cap N_2 \neq \{0_M\}$, what will happen to $N_1 \cap N_2$ and M? Be thorough and rigorous.

(2) To prove $N_i \cap K = \{0_M\}$ for some $i \in \{1, 2\}$, study what happens otherwise. Say $N_1 \cap K = \{0_M\}$. To verify $N_1 + K = M$, you might want to show $N_2 \subseteq N_1 + K$ by showing $N_2 \cap (N_1 + K) \neq \{0_M\}$: Choose any $0_M \neq k \in K$ and write $k = n_1 + n_2$ with $n_i \in N_i$. Is it possible that $n_2 = 0_M$? Why? Once you know $N_2 \cap (N_1 + K) \neq \{0_M\}$, what must $N_2 \cap (N_1 + K)$ be, given that N_2 is simple?

(3) Say $M = N_1 \boxplus K$, by (2). Try showing $K \cong N_2$. `(Does this look familiar?)`

Problem 7.4. Let R be a commutative Noetherian ring with $1_R \in R$ and let M be a non-zero R-module. Prove that there exists $x \in M \setminus \{0_M\}$ such that $\mathrm{Ann}_R(x)$ is a prime ideal of R. (For an ideal $P \subsetneqq R$, with R commutative with 1_R, we say that P is a prime ideal if, for all $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.)

Hint. Let $\Omega = \{\mathrm{Ann}_R(m) \,|\, m \in M \setminus \{0_M\}\}$. Does Ω have a maximal element? Why? Pick any maximal element, say P, of Ω. Try to prove that P is a prime ideal of R.

Problem 7.5 (Extra Credit, 1 point)**.** Let M and F be R-modules (with R a ring with unity) such that F is free over R. Let $h \in \mathrm{Hom}_R(M, F)$ such that h is onto F (i.e., $h(M) = F$). **Prove or disprove**: $M \cong \mathrm{Ker}(h) \oplus F$.

Hint. No partial credit. Either give a rigorous proof or give a concrete counterexample.

Problem 8.1. Let R be a ring and M a non-zero Artinian R-module. Prove that there exists $L \leqslant M$ such that L is simple.

Hint. Consider the set Ω consisting of all non-zero R-submodules of M. (Make sure to show $\Omega \neq \varnothing$.) Use the assumption on M.

Problem 8.2. Let R be a ring, M be an R-module, and $2 \leqslant n \in \mathbb{N}$. Also, let N and N_i, where $i = 1, \ldots, n$, be R-submodules of M such that $N = \sum_{i=1}^{n} N_i$. Prove that the following statements are equivalent to each other:

(1) $N = \boxplus_{i=1}^{n} N_i$ (i.e., $N = \sum_{i=1}^{n} N_i$ and $\left(\sum_{i \neq j} N_i\right) \cap N_j = \{0_N\}$ for all $j = 1, \ldots, n$).
(2) $\left(\sum_{i=1}^{j-1} N_i\right) \cap N_j = \{0_N\}$ for all $j = 2, \ldots, n$.
(3) Every element $x \in N$ can be uniquely expressed as $x = \sum_{i=1}^{n} x_i$ with $x_i \in N_i$.
(4) The element $0_N \in N$ can be uniquely expressed as $0_N = \sum_{i=1}^{n} x_i$ with $x_i \in N_i$.

Hint. All should be straightforward, just do it step by step. `(This problem, essentially, has nothing to do with` M`.)` Note that, when (4) is assumed, the unique expression $0_N = \sum_{i=1}^{n} x_i$, with $x_i \in N_i$, must be $0_N = \sum_{i=1}^{n} (?)$.

Problem 8.3. Let R be a ring, N be an R-module, and $2 \leqslant n \in \mathbb{N}$. Let N_i, where $i = 1, \ldots, n$, be R-submodules of N. Also, let $K_i \leqslant M_i$, where $i = 1, \ldots, n$, be R-modules.

(1) Prove that if $N = \boxplus_{i=1}^{n} N_i$ then $N \cong \oplus_{i=1}^{n} N_i$.
(2) True or false: $\oplus_{i=1}^{n} K_i \leqslant \oplus_{i=1}^{n} M_i$.
(3) **Prove or disprove**: $(\oplus_{i=1}^{n} M_i) \,/\, (\oplus_{i=1}^{n} K_i) \cong \oplus_{i=1}^{n} (M_i/K_i)$.

Hint. (1) It might be (slightly) easier to construction maps from $\oplus_{i=1}^{n} N_i$ to N.

(3) You may prove it from scratch by constructing an isomorphism (make sure it is well-defined). Or you can construct a homomorphism and then use the fundamental theorem.

Problem 8.4. Consider the free $\mathbb{Z}$-module $\mathbb{Z}^2 = \{\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) \mid r, s \in \mathbb{Z}\}$. (We write the elements of $\mathbb{Z}^2$ as column vectors.) Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{smallmatrix}\right)$ and $x_i = \left(\begin{smallmatrix} a_{1i} \\ a_{2i} \end{smallmatrix}\right) \in \mathbb{Z}^2$ be the i-th column of A. Let $y_i \in \mathbb{Z}^2$ be the i-th column of $A\left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right)$. Suppose $\left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1} A \left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{smallmatrix}\right)$.

(1) Compute $\left(\begin{smallmatrix} 2 & 5 \\ 3 & 8 \end{smallmatrix}\right)^{-1}$ and $\left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right)^{-1}$ explicitly. It suffices to write down your answers.
(2) Express each x_i as a linear combination of y_1, y_2, y_3 explicitly. No need to justify.
(3) True or false: $\sum_{i=1}^{3} \mathbb{Z}x_i = \sum_{i=1}^{3} \mathbb{Z}y_i$. No justification is necessary.
(4) **Prove** that $z_1 = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ and $z_2 = \left(\begin{smallmatrix} 5 \\ 8 \end{smallmatrix}\right)$ form a basis of $\mathbb{Z}^2$.
(5) Express each y_i as a linear combination of z_1 and z_2. No need to justify.

Hint. All are straightforward via matrix manipulations. For example, by means of block matrices, we can write $(y_1, y_2, y_3) = (x_1, x_2, x_3)\left(\begin{smallmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 0 \end{smallmatrix}\right)$. Then $(x_1, x_2, x_3) = (y_1, y_2, y_3)\boxed{?}$

Problem 8.5 (Extra Credit, 1 point)**.** Let R be a ring. **Prove or disprove**: If M is a non-zero Noetherian R-module then there exists $K \leqslant M$ such that M/K is simple.

Hint. No partial credit. Either give a rigorous proof or give a concrete counterexample.

PROBLEMS HINTS SOLUTIONS

Materials covered earlier: Homework Sets 1, 2, 3, 4; Exam I.

General problems about modules: Problems 5.1, 5.3, 5.4, 6.3, 7.3, 8.1, 8.2, 8.4.

Free modules: Problems 5.2, 5.5, 7.3.

Direct sums (internal or external): Problems 6.1, 6.2, 6.5.

Noetherian modules, Artinian modules: Problems 6.4, 7.1, 7.2, 7.4, 7.5.

Simple modules: Problems 7.2, 8.3, 8.5.

Modules over a PID: Problem 8.4.

Lecture notes and textbooks: All we have covered.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Internal direct sum. Let R be a ring, M an R-module, N an R-submodule of M, and $\{N_i\}_{i\in\Delta}$ is family of R-submodules of M. We say that N is an internal direct sum of $\{N_i\}_{i\in\Delta}$, denoted $N = \boxplus_{i\in\Delta} N_i$, if $N = \sum_{i\in\Delta} N_i$ and $N_j \cap (\sum_{i\in\Delta\setminus\{j\}} N_i) = \{0_M\}$ for all $j \in \Delta$.

- $N = \boxplus_{i\in\Delta} N_i$ if and only if $N = \sum_{i\in\Delta} N_i$ and every element $x \in N$ is uniquely expressed as $x = \sum_{i\in\Delta} n_i$ with $x_i \in N_i$ and $x_i = 0_M$ for almost all $i \in \Delta$.
- If $N = \boxplus_{i\in\Delta} N_i$, then $N \cong \oplus_{i\in\Delta} N_i$.

Free module. Let R be a ring with unity and F an R-module. We say that F is a free (over R) if there exists $B \subseteq F$ such that B generates F and B is linear independent over R, in which case we say that B is a basis of F.

- F is a free R-module with a basis $B \iff$ every $x \in F$ can be uniquely written as $x = \sum_{b\in B} r_b b$ with $r_b \in R$ and $r_b = 0_R$ for almost all $b \in B$.
- F is a free R-module $\iff F \cong R^{\oplus B}$ for some set B.
- F is a free R-module with a basis $B \iff F = \boxplus_{b\in B} Rb$ and $\operatorname{Ann}(b) = \{0_R\}$, $\forall\, b \in B$.
- F is a free R-module with a basis $B \iff$ for any R-module M and any map $\theta\colon B \to M$, there is a unique R-linear map $h \in \operatorname{Hom}_R(F, M)$ such that $h|_B = \theta$.
- If R is commutative and F is free, then all bases of F have the same cardinality, which is called the rank of F.

Noetherian module. We say that an R-module M is Noetherian (over R) if every ascending chain of R-submodules of M eventually stabilizes. The following are equivalent:

- M is Noetherian as an R-module.
- All R-submodules of M are finitely generated.
- Every non-empty set of R-submodules of M has a maximal member/object.
- N and M/N are Noetherian over R for every R-submodule N of M.
- N and M/N are Noetherian over R for some R-submodule N of M.

We say R is a (left) Noetherian ring if R is Noetherian as a (left) R-module.

- [Hilbert basis theorem] If R is Noetherian, so is $R[x]$ and so is $R[x_1, \ldots, x_n]$.

Artinian module. We say that an R-module M is Artinian (over R) if every descending chain of R-submodules of M eventually stabilizes. The following are equivalent:

- M is Artinian as an R-module.
- Every non-empty set of R-submodules of M has a minimal member/object.
- N and M/N are Artinian over R for every (or for some) R-submodule N of M.

We say R is a (left) Artinian ring if R is Artinian as a (left) R-module.

Simple module. Let R be a ring and let M, N be R-modules. We say that M is a simple R-module if $M \neq \{0_M\}$ and the only R-submodules of M are $\{0_M\}$ and M.

- Assume that M is simple. Then every $\varphi \in \operatorname{Hom}_R(M, N)$ is either injective or the zero map. Similarly, every $\psi \in \operatorname{Hom}_R(N, M)$ is either the zero map or surjective.

Modules over a PID. Let R be a PID and let F be a free R-module of finite rank n.

- Every R-submodule of F is free (over R) of rank $\leqslant n$.

Note: The above list is not intended to be complete.

You must solve a problem **completely and correctly** in order to get the extra credit. You may attempt a problem for as many times as you wish by 04/25.

The points you get here will be added to the total score from the homework assignments.

Problem E-1 (3 points)**.** Let R be a commutative ring with unity and $S = R[x]$. For all ideals I and J of R, prove $I \cap J = \big(xIS + (1-x)JS\big) \cap R$.

Hint. Note that IS consists of polynomials with coefficients in I; similarly for JS. This problem is more ring-theoretical than module-theoretical.

Problem E-2 (3 points)**.** Let R be a commuative ring with unity and I, J ideals of R. **Prove or disprove**: If $R/I \cong R/J$ as R-modules then $I = J$.

Problem E-3 (3 points)**.** Is there a field that is free over $\mathbb{Z}$? Explain.

PROBLEMS HINTS SOLUTIONS