◇ ◇ ◇ ◇ **MATH 4441/6441: MODERN ALGEBRA I** ◇ ◇ ◇ ◇
HOMEWORK SETS AND EXAMS

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CONTENTS

There are four (4) problems in each homework set. Math 6441 students need to do all 4 problems while Math 4441 students need to do any three (3) problems out the four. If a Math 4441 student submits all 4 problems, then one of the lowest score(s) is dropped. There is a bonus point for Math 4441 students solving all 4 problems correctly/perfectly.

When solving homework problems, make sure that your arguments and computations are rigorous, accurate, and complete. Present your step-by-step work in your solutions/proofs.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

$(G,*) \ldots H \leqslant G \ldots |G| = [G:H]\cdot|H| \ldots a^{|G|} = e \ldots \varphi\colon G \to G',\ \varphi(ab) = \varphi(a)\varphi(b) \ldots N \trianglelefteq G \ldots G/N \ldots G/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi)$

Problem 1.1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6, 7\}$ and $C = \{3, 6, 7, 8\}$ be sets.

(1) Compute $(A \setminus B) \cap C$ and $A \setminus (B \cap C)$. Are they equal?
(2) Compute $(A \cap B) \cup C$ and $A \cap (B \cup C)$. Are they equal?

Problem 1.2. Let A, B and C be sets. Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Problem 1.3. For each function f_i, determine whether it is injective but not surjective, surjective but not injective, bijective, or neither injective nor surjective. Explain why.

(1) $f_1 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $f_1(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$, where $\mathbb{R}_{\geqslant 0} = \{x \in \mathbb{R} \mid x \geqslant 0\} = [0, \infty)$.
(2) $f_2 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $f_2(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$.
(3) $f_3 \colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with $f_3(x) = x^4$ for all $x \in \mathbb{R}$.
(4) $f_4 \colon \mathbb{R} \to \mathbb{R}$ with $f_4(x) = 10^{x^2}$ for all $x \in \mathbb{R}$. (Here 10^{x^2} stands for $10^{(x^2)}$, not $(10^x)^2$.)

Problem 1.4. Let A, B and C be sets.

(1) Find a concrete example of A, B and C such that $(A \cup B) \cap C \subsetneq A \cup (B \cap C)$.
(2) Prove $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

PROBLEMS HINTS SOLUTIONS

Problem 2.1. Let $X = \{a, b\}$, $Y = \{1, 2\}$ and $Z = \{x, y, z\}$.

(1) Find all functions from X to Y.
(2) Find all injective functions from X to Y, if they exist.
(3) Write down all surjective functions from Y to Z, if they exist.
(4) Write down all **non**-injective functions from Y to Z, if they exist.

Problem 2.2. Let S_3 denote the set of all bijective functions from $X = \{1, 2, 3\}$ to itself. Let $\varphi \in S_3$ and $\psi \in S_3$ be defined as follows

$$\varphi\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3 \qquad \text{and} \qquad \psi\colon\ 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2.$$

(1) Determine $\varphi \circ \psi$ and $\psi \circ \varphi$ explicitly. Are they equal?
(2) Determine φ^{-1} and ψ^{-1} explicitly.
(3) Determine φ^2 and φ^3 explicitly. Is anyone of the two equal to I_X?
(4) Determine ψ^2 and ψ^3 explicitly. Is anyone of the two equal to I_X?

Problem 2.3. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions, in which X, Y and Z are non-empty sets.

(1) If both f and g are surjective (i.e., onto), prove that $g \circ f$ is surjective.
(2) **Disprove**: If $g \circ f$ is surjective (i.e., onto), then both f and g are surjective.

Problem 2.4. Let $f,\ f_1,\ f_2\colon X \to Y$ and $g,\ g_1,\ g_2\colon Y \to Z$ be functions, in which X, Y and Z are (non-empty) sets.

(1) Prove that if g is 1–1 (i.e., injective) and $g \circ f_1 = g \circ f_2$, then $f_1 = f_2$.
(2) **Disprove** the statement: If $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$.

Problem 3.1. Consider integers 24, 60, 67 and 97.

(1) List **all** (positive and negative) common divisors of 24 and 60. Determine $\gcd(24, 60)$.
(2) Express $\gcd(67, 97)$ as a linear combination of 67 and 97 (with integer coefficients).

Problem 3.2. Let $x = 3 - i$, $y = 4 + 2i$ and $z = -3 - \sqrt{3}i$.

(1) Compute $x + y$ and $x - y$.
(2) Compute xy and x/y.
(3) Write z in polar form $z = r(\cos\theta + i\sin\theta)$ with $0 \leqslant r \in \mathbb{R}$ and $0 \leqslant \theta < 2\pi$.
(4) Compute z^{33}. Is z^{33} in $\mathbb{R}$? Show your reasoning/computation.

Problem 3.3. Let $D = \{13^i \mid i \in \mathbb{Z}\}$, the set consisting of all powers of 13 (of all integer exponents). (For example, 13^{-18}, 13^0, $13^{451} \in D$.) For all m, $n \in D$, let $m * n = mn$, the (ordinary) product of m and n. Determine whether statements (1)–(4) are true or false **with justification**. Also answer (5).

(1) For all a, $b \in D$, it holds that $a * b \in D$.
(2) For all a, b, $c \in D$, it holds that $(a * b) * c = a * (b * c)$.
(3) There exists a (fixed) element $e \in D$ such that $e * a = a = a * e$ for all $a \in D$.
(4) For every $a \in D$, there exists $a' \in D$ such that $a' * a = e = a * a'$.
(5) $(D, *)$ is an abelian group a non-abelian group not a group (choose one)

Problem 3.4. Let a, b, $c \in \mathbb{Z}$, i.e., a, b, c are all integers.

(1) Give a concrete example of a, b, $c \in \mathbb{Z}$ such that $a \mid c$ and $b \mid c$, but $(ab) \nmid c$.
(2) Prove that if $\gcd(a, b) = 1$, $a \mid c$ and $b \mid c$ then $(ab) \mid c$.

PROBLEMS HINTS SOLUTIONS

Problem 4.1. For all x, $y \in \mathbb{Z}$, let $x*y = |x|+y$. (For example, $(-1)*(-2) = -1 = 1*(-2)$.) Determine whether (1)–(4) are true or false with justification. And then answer (5).

(1) For all a, $b \in \mathbb{Z}$, it holds that $a * b \in \mathbb{Z}$.
(2) For all a, b, $c \in \mathbb{Z}$, it holds that $(a * b) * c = a * (b * c)$.
(3) There exists a (fixed) element $e \in \mathbb{Z}$ such that $e * a = a$ for all $a \in \mathbb{Z}$.
(4) For every $a \in \mathbb{Z}$, there exists $a' \in \mathbb{Z}$ such that $a' * a = e$.
(5) $(\mathbb{Z}, *)$ is an abelian group a non-abelian group not a group (choose one)

Problem 4.2. Let G be a group of order 2 (meaning $|G| = 2$). Say $G = \{e, a\}$, in which e and a denote the two distinct elements of G with e being the identity element of G.

(1) Fill in each of the blanks with a or e: $ee = \boxed{}$, $ea = \boxed{}$, $ae = \boxed{}$ and $aa = \boxed{}$. Justify your claims rigorously.
(2) Determine whether G is abelian. Justify your claim rigorously.

Problem 4.3. Let $(G, *)$ be a group, and a, b, c, $d \in G$. Fill in each of the blanks (?) with an expression involving a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1} such that the equation holds. (Note that ab is short for $a * b$, and $c(?)db$ short for $c * (?) * d * b$, etcetera.)

(1) $a(?)dc = abc$.
(2) $(?)abd = dc^{-1}d$.
(3) $ba^{-1}(?)d^{-1}bc = abc$.

Problem 4.4. Let $D = \mathbb{Q} \setminus \{0\}$, the set of all non-zero rational numbers. For all x, $y \in D$, define $x*y = 4xy$, the ordinary product of 4, x and y. (For example, $(2)*(3) = 4(2)(3) = 24$.)

(1) Determine whether $(D, *)$ is a group.
(2) Justify your claim in (1) carefully.

PROBLEMS HINTS SOLUTIONS

Sets: Problems 1.1, 1.2, 1.4.

Functions: Problems 1.3, 2.1, 2.2, 2.3, 2.4.

About S_n (e.g., with $n = 3$): Problem 2.2.

Integers, complex numbers: Problems 3.1, 3.2, 3.4.

Definition of groups: Problems 3.3, 4.1, 4.4.

Properties of groups: Problems 4.2, 4.3.

Lecture notes and textbooks: All we have covered, **including properties of groups**.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Problems

have been withdrawn

from the site

PROBLEMS HINTS SOLUTIONS

Problem 5.1. Let G be a group and let a, b be (fixed) elements of G such that $ab^{-1} = b^{-1}a$. Prove the following equations.

(1) $ab = ba$.
(2) $a^{-1}b = ba^{-1}$.

Problem 5.2. Let G be a group of order 3. Say $G = \{e, a, b\}$, in which e, a and b are the three distinct elements of G with e the identity element. (Compare with Problem 4.2.)

(1) Fill in each of the blanks with a, b or e: $ab = \boxed{}$ and $ba = \boxed{}$. Explain why.
(2) Prove that G is abelian. That is, every group of order 3 is abelian.
(3) Fill in each of the blanks with a, b or e: $a^2 = \boxed{}$ and $b^2 = \boxed{}$. Prove your claims.

Problem 5.3. Let G be a group, a, $b \in G$ and m, $n \in \mathbb{Z}$.

(1) Prove that if $a^5 = b^5$ and $a^7 = b^7$ then $a = b$.
(2) Prove that if $a^m = b^m$, $a^n = b^n$ and $\gcd(m, n) = 1$ then $a = b$.

Problem 5.4. Let G be a group such that $(ab)^4 = a^4b^4$, $(ab)^5 = a^5b^5$ and $(ab)^6 = a^6b^6$ for all a, $b \in G$. Prove that G is abelian.

PROBLEMS HINTS SOLUTIONS

Problem 6.1. Let $A = \mathbb{C} \setminus \{0\}$. Consider the group $(A, \cdot)$ under the usual multiplication. Also consider φ defined by $1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 1$, which is in the group $(S_4, \circ)$.

(1) Determine the order of 3, considered as an element of $(A, \cdot)$.
(2) Determine the order of $\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})$ as an element of $(A, \cdot)$.
(3) Determine $\mathrm{o}(\varphi)$.
(4) Compute φ^{1234} in the format of $1 \mapsto \boxed{?}$, $2 \mapsto \boxed{?}$, $3 \mapsto \boxed{?}$, $4 \mapsto \boxed{?}$.

Problem 6.2. Let G be a group such that $x^2 = e$ for all $x \in G$.

(1) True or false: $x^{-1} = x$ for all $x \in G$. Justify.
(2) Prove that G is abelian.

Problem 6.3. Let G be an **abelian** group, $a \in G$ a **fixed** element of G, and n a **fixed** integer. Define $f\colon G \to G$ by $f(x) = x^n$ for all $x \in G$.

(1) Determine whether f is a group homomorphism, with justification.
(2) Prove that $f(a) = a$ **if** $a^{n-1} = e$.
(3) Prove that $f(a) = a$ **only if** $a^{n-1} = e$ (i.e., $a^{n-1} = e$ if $f(a) = a$.)

Problem 6.4. Let G be a group and let $a, g \in G$ be **fixed** elements. Define $h\colon G \to G$ by $h(x) = g^{-1}xg$ for all $x \in G$.

(1) Determine whether h is a group homomorphism, with justification.
(2) Prove that $h(a) = a$ **if** $ag = ga$.
(3) Prove that $h(a) = a$ **only if** $ag = ga$ (i.e., $ag = ga$ if $h(a) = a$.)

PROBLEMS HINTS SOLUTIONS

Problem 7.1. Let $\varphi\colon G \to G'$ be a group homomorphism, in which G and G' are groups. Let $a \in G$ such that $\mathrm{o}(a) < \infty$. (Denote the identity elements of G and G' by e and e' respectively.)

(1) Prove that $\mathrm{o}(\varphi(a)) < \infty$.
(2) Prove that $\mathrm{o}(\varphi(a)) \mid \mathrm{o}(a)$.

Problem 7.2. Consider the group $(S_3, \circ)$ under composition, which consists of the following

$$f_1\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3; \qquad f_2\colon\ 1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 2; \qquad f_3\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3;$$
$$f_4\colon\ 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1; \qquad f_5\colon\ 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2; \qquad f_6\colon\ 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1.$$

$H = \{x \in G \mid x^2 = f_1\}$ and $K = \{x^3 \mid x \in S_3\}$.

(1) Determine whether H is a subgroup of S_3.
(2) Determine whether K is a subgroup of S_3.

Problem 7.3. Let G be an abelian group, $H = \{x \in G \mid x^9 = e\}$. Prove $H \leqslant G$, that is, prove that H is a subgroup of G.

Problem 7.4. Let G be an abelian group, $H = \{a^4 \mid a \in G\}$ and $K = \{a^{52} \mid a \in G\}$.

(1) Prove $H \leqslant G$, that is, prove that H is a subgroup of G.
(2) Prove $K \subseteq H$, that is, prove that K is a subset of H.

PROBLEMS HINTS SOLUTIONS

Problem 8.1. Consider the group $(S_3, \circ)$ under composition, which consists of the following

$$f_1\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3; \quad f_2\colon\ 1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 2; \quad f_3\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3;$$
$$f_4\colon\ 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1; \quad f_5\colon\ 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2; \quad f_6\colon\ 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1.$$

Find as many (distinct) subgroups of S_3 as possible. You will receive 1 point per correct subgroup and -1 point per incorrect choice.

Problem 8.2. Consider the group $(G, \circ)$ under composition, which consists of the following

$$f_1\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3,\ 4 \mapsto 4; \qquad f_2\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 4,\ 4 \mapsto 3;$$
$$f_3\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3,\ 4 \mapsto 4; \qquad f_4\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 4,\ 4 \mapsto 3.$$

Find as many (distinct) subgroups of $(G, \circ)$ as possible. You will receive 1 point per correct subgroup and -1 point per incorrect choice. (Note that $(G, \circ)$ is a subgroup of $(S_4, \circ)$.)

Problem 8.3. Let G be a group of order 30, i.e., $|G| = 30$, and let $x, y \in G$.

(1) If $x \in G$ satisfies $x^{24} = e$ and $x^9 \neq e$, determine all the possible value(s) of $\mathrm{o}(x)$.
(2) If $y^{20} = e$, $y^8 \neq e$ and $y^{15} \neq e$, then determine all the possible value(s) of $\mathrm{o}(y)$.
(3) Is it possible to ever have $z \in G$ such that $z^{45} = e$ and $z^{105} \neq e$? Why or why not?

Problem 8.4. Prove that every group of order 4 is abelian as follows: Let G be any group of order 4, i.e., $|G| = 4$.

(1) Suppose there exists $a \in G$ such that $\mathrm{o}(a) = 4$. Prove that G is abelian.
(2) Suppose that no element of G has order 4. Prove $x^2 = e$ for all $x \in G$.
(3) Suppose that no element of G has order 4. Prove that G is abelian.

PROBLEMS HINTS SOLUTIONS

Materials covered earlier: Midterm I; Homework Sets 1, 2, 3, 4.

Basic properties, direct calculations: Problems 5.1, 5.2, 5.3, 6.1, 6.2, 7.4, 8.1, 8.3, 8.4, etcetera.

Abelian groups: Problems 5.2, 5.4, 6.2, 8.4.

Orders of elements: Problems 6.1, 7.1, 8.3, 8.4.

Homomorphisms: Problems 6.3, 6.4, 7.1.

Subgroups, the subgroup criterion: Problems 5.3, 7.2, 7.3, 7.4, 8.1, 8.2.

Lagrange's theorem: Problems 8.3, 8.4.

Lecture notes and textbooks: All we have covered.

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