◇ ◇ ◇ ◇ MATH 4441/6441: MODERN ALGEBRA I ◇ ◇ ◇ ◇
HOMEWORK SETS AND EXAMS

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Contents

There are four (4) problems in each homework set. Math 6441 students need to do all 4 problems while Math 4441 students need to do any three (3) problems out the four. If a Math 4441 student submits all 4 problems, then one of the lowest score(s) is dropped. There is a bonus point for Math 4441 students solving all 4 problems correctly/perfectly.

When solving homework problems, make sure that your arguments and computations are rigorous, accurate, and complete. Present your step-by-step work in your solutions/proofs.

There are three (3) PDF files for the homework sets and exams, one with the problems only, one with hints, and one with solutions. Links are available below.

PROBLEMS HINTS SOLUTIONS

Problem 1.1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6, 7\}$ and $C = \{3, 6, 7, 8\}$ be sets.

(1) Compute $(A \setminus B) \cap C$ and $A \setminus (B \cap C)$. Are they equal?
(2) Compute $(A \cap B) \cup C$ and $A \cap (B \cup C)$. Are they equal?

Hint. All should be straightforward. Determine each set by listing its elements explicitly. (Note that $A \setminus B$ may be also denoted by $A - B$.)

Problem 1.2. Let A, B and C be sets. Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Hint. How to show two sets are equal?

Problem 1.3. For each function f_i, determine whether it is injective but not surjective, surjective but not injective, bijective, or neither injective nor surjective. Explain why.

(1) $f_1 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $f_1(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$, where $\mathbb{R}_{\geqslant 0} = \{x \in \mathbb{R} \mid x \geqslant 0\} = [0, \infty)$.
(2) $f_2 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $f_2(x) = x^2$ for all $x \in \mathbb{R}_{\geqslant 0}$.
(3) $f_3 \colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ with $f_3(x) = x^4$ for all $x \in \mathbb{R}$.
(4) $f_4 \colon \mathbb{R} \to \mathbb{R}$ with $f_4(x) = 10^{x^2}$ for all $x \in \mathbb{R}$. (Here 10^{x^2} stands for $10^{(x^2)}$, not $(10^x)^2$.)

Hint. Your solution may follow this example: Let $f_5 \colon \mathbb{R} \to \mathbb{R}$ with $f_5(x) = |x|$ for all $x \in \mathbb{R}$. Then f_5 is neither injective nor surjective. It is not injective because $f_5(1) = f_5(-1)$ while $1 \neq -1$. It is not surjective because there is no $x \in \mathbb{R}$ such that $f_5(x) = -2$

Problem 1.4. Let A, B and C be sets.

(1) Find a concrete example of A, B and C such that $(A \cup B) \cap C \subsetneq A \cup (B \cap C)$.
(2) Prove $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

Hint. (1) For example, you may try letting $A = \{1, 2\}$, $B = \{\dots\}$ and $C = \{\dots\}$. You may even start with $A = \{1\}$. (Note that $X \subsetneq Y$ means $X \subseteq Y$ and $X \neq Y$.)

(2) How to show that a set is a subset of another set? Let $x \in (A \cup B) \cap C$ be an (arbitrary) element. Try to show $x \in A \cup (B \cap C)$.

PROBLEMS HINTS SOLUTIONS

Problem 2.1. Let $X = \{a, b\}$, $Y = \{1, 2\}$ and $Z = \{x, y, z\}$.

(1) Find all functions from X to Y.
(2) Find all injective functions from X to Y, if they exist.
(3) Write down all surjective functions from Y to Z, if they exist.
(4) Write down all **non**-injective functions from Y to Z, if they exist.

Hint. (1) Describe the functions from X to Y in the following format:

$$f_1\colon\ a \mapsto \boxed{?},\, b \mapsto \boxed{?}; \qquad f_2\colon\ a \mapsto \boxed{?},\, b \mapsto \boxed{?}; \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

You need to exhaust all functions and write them down explicitly.

(4) This should be straightforward. Don't miss anyone.

Problem 2.2. Let S_3 denote the set of all bijective functions from $X = \{1, 2, 3\}$ to itself. Let $\varphi \in S_3$ and $\psi \in S_3$ be defined as follows

$$\varphi\colon\ 1 \mapsto 2,\, 2 \mapsto 1,\, 3 \mapsto 3 \qquad \text{and} \qquad \psi\colon\ 1 \mapsto 3,\, 2 \mapsto 1,\, 3 \mapsto 2.$$

(1) Determine $\varphi \circ \psi$ and $\psi \circ \varphi$ explicitly. Are they equal?
(2) Determine φ^{-1} and ψ^{-1} explicitly.
(3) Determine φ^2 and φ^3 explicitly. Is anyone of the two equal to I_X?
(4) Determine ψ^2 and ψ^3 explicitly. Is anyone of the two equal to I_X?

Hint. You may determine/describe a function in S_3 in the format of $1 \mapsto \boxed{?}$, $2 \mapsto \boxed{?}$, $3 \mapsto \boxed{?}$. Recall that φ^2 simply denotes $\varphi \circ \varphi$ while ψ^3 simply denotes $\psi \circ \psi \circ \psi$. Also, I_X stands for the identity function on X.

Problem 2.3. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions, in which X, Y and Z are non-empty sets.

(1) If both f and g are surjective (i.e., onto), prove that $g \circ f$ is surjective.
(2) **Disprove**: If $g \circ f$ is surjective (i.e., onto), then both f and g are surjective.

Hint. (1) Let $z \in Z$ (be an arbitrary element). Try to show the existence of (some element) $x \in X$ such that $g \circ f(x) = z$.

(2) Find explicit/concrete X, Y, Z, f and g such that the statement fails. It would suffice to consider finite sets.

Problem 2.4. Let f, f_1, $f_2\colon X \to Y$ and g, g_1, $g_2\colon Y \to Z$ be functions, in which X, Y and Z are (non-empty) sets.

(1) Prove that if g is 1–1 (i.e., injective) and $g \circ f_1 = g \circ f_2$, then $f_1 = f_2$.
(2) **Disprove** the statement: If $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$.

Hint. (1) Let $x \in X$ (be an arbitrary element). You need to show $f_1(x) = f_2(x)$. Make use of the assumption that g is 1–1 and $g \circ f_1 = g \circ f_2$.

(2) Just disprove it by an explicit counterexample. It would suffice to consider finite sets.

Problem 3.1. Consider integers 24, 60, 67 and 97.

(1) List **all** (positive and negative) common divisors of 24 and 60. Determine $\gcd(24, 60)$.
(2) Express $\gcd(67, 97)$ as a linear combination of 67 and 97 (with integer coefficients).

Hint. (1) This is straightforward.

(2) Use the Euclidean Algorithm to find $\gcd(67, 97)$ and then find $r, s \in \mathbb{Z}$ such that $\gcd(67, 97) = 67r + 97s$.

Problem 3.2. Let $x = 3 - i$, $y = 4 + 2i$ and $z = -3 - \sqrt{3}i$.

(1) Compute $x + y$ and $x - y$.
(2) Compute xy and x/y.
(3) Write z in polar form $z = r(\cos\theta + i\sin\theta)$ with $0 \leqslant r \in \mathbb{R}$ and $0 \leqslant \theta < 2\pi$.
(4) Compute z^{33}. Is z^{33} in $\mathbb{R}$? Show your reasoning/computation.

Hint. (1)–(3) Straightforward. Be careful with the computations.

(4) Is there an efficient way to compute z^n in light of (3) and De Moivre's formula?

Problem 3.3. Let $D = \{13^i \mid i \in \mathbb{Z}\}$, the set consisting of all powers of 13 (of all integer exponents). (For example, 13^{-18}, 13^0, $13^{451} \in D$.) For all m, $n \in D$, let $m * n = mn$, the (ordinary) product of m and n. Determine whether statements (1)–(4) are true or false **with justification**. Also answer (5).

(1) For all a, $b \in D$, it holds that $a * b \in D$.
(2) For all a, b, $c \in D$, it holds that $(a * b) * c = a * (b * c)$.
(3) There exists a (fixed) element $e \in D$ such that $e * a = a = a * e$ for all $a \in D$.
(4) For every $a \in D$, there exists $a' \in D$ such that $a' * a = e = a * a'$.
(5) $(D, *)$ is   an abelian group   a non-abelian group   not a group   (choose one)

Hint. In (3), it suffices to write down the desired e explicitly if it exists in D. In (4), it suffices to write down the desired a' (depending on a) if it always exists in D for every $a \in D$ or, otherwise, find an concrete $a \in D$ for which there is no $a' \in D$ as claimed. The e in (4) should be the same e as found in (3), provided it exists.

Problem 3.4. Let a, b, $c \in \mathbb{Z}$, i.e., a, b, c are all integers.

(1) Give a concrete example of a, b, $c \in \mathbb{Z}$ such that $a \mid c$ and $b \mid c$, but $(ab) \nmid c$.
(2) Prove that if $\gcd(a, b) = 1$, $a \mid c$ and $b \mid c$ then $(ab) \mid c$.

Hint. (1) Find concrete integers a, b, c as required.

(2) Assume $\gcd(a, b) = 1$, $a \mid c$ and $b \mid c$. (You must prove $(ab) \mid c$.) Note that $\gcd(a, b) = 1$ is a linear combination of a and b. To be specific, there exist $r, s \in \mathbb{Z}$ such that $1 = ra + sb$. Multiplying both sides by c, we get $c = rac + sbc$. Now it suffices to prove that ab divides $rac + sbc$. To do this, make use of the assumption $a \mid c$ and $b \mid c$.

Problem 4.1. For all x, $y \in \mathbb{Z}$, let $x*y = |x|+y$. (For example, $(-1)*(-2) = -1 = 1*(-2)$.) Determine whether (1)–(4) are true or false with justification. And then answer (5).

(1) For all a, $b \in \mathbb{Z}$, it holds that $a * b \in \mathbb{Z}$.
(2) For all a, b, $c \in \mathbb{Z}$, it holds that $(a * b) * c = a * (b * c)$.
(3) There exists a (fixed) element $e \in \mathbb{Z}$ such that $e * a = a$ for all $a \in \mathbb{Z}$.
(4) For every $a \in \mathbb{Z}$, there exists $a' \in \mathbb{Z}$ such that $a' * a = e$.
(5) $(\mathbb{Z}, *)$ is an abelian group a non-abelian group not a group (choose one)

Hint. In (3), it suffices to write down the desired e explicitly if it exists in $\mathbb{Z}$. In (4), it suffices to write down the desired a' (depending on a) if it always exists in $\mathbb{Z}$ for all $a \in \mathbb{Z}$ or otherwise find a concrete $a \in \mathbb{Z}$ for which there is no $a' \in \mathbb{Z}$ as claimed. The e in (4) should be the same e as found in (3), provided it exists.

Problem 4.2. Let G be a group of order 2 (meaning $|G| = 2$). Say $G = \{e, a\}$, in which e and a denote the two distinct elements of G with e being the identity element of G.

(1) Fill in each of the blanks with a or e: $ee = \boxed{}$, $ea = \boxed{}$, $ae = \boxed{}$ and $aa = \boxed{}$. Justify your claims rigorously.
(2) Determine whether G is abelian. Justify your claim rigorously.

Hint. (1) To determine aa, note that aa is in G and $G = \{a, e\}$. So either $aa = a$ or $aa = e$. Is it ever possible that $aa = a$? why? (More explicitly, if $aa = a$ were true, one would get $aa = ea$. Could this be possible in a group?)

(2) Show $xy = yx$ for all x, $y \in G$ by exhausting all possible pairs: ee, ea, aa.

Problem 4.3. Let $(G, *)$ be a group, and a, b, c, $d \in G$. Fill in each of the blanks (?) with an expression involving a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1} such that the equation holds. (Note that ab is short for $a * b$, and $c(?)db$ short for $c * (?) * d * b$, etcetera.)

(1) $a(?)dc = abc$.
(2) $(?)abd = dc^{-1}d$.
(3) $ba^{-1}(?)d^{-1}bc = abc$.

Hint. Key words: definition of groups, associativity, cancellation, inverse, etcetera. For example, an answer to $b(?)c = dc$ is $b(b^{-1}d)c = dc$. (Indeed, $b(b^{-1}d)c = (bb^{-1})dc = dc$ in $(G, *)$.) **Warning**: notations such as bd/c and $\frac{d}{ca}$ are **not valid** here.

Problem 4.4. Let $D = \mathbb{Q} \setminus \{0\}$, the set of all non-zero rational numbers. For all x, $y \in D$, define $x*y = 4xy$, the ordinary product of 4, x and y. (For example, $(2)*(3) = 4(2)(3) = 24$.)

(1) Determine whether $(D, *)$ is a group.
(2) Justify your claim in (1) carefully.

Hint. To show $(D, *)$ is a group, you need to justify all the conditions in the definition of groups. However, to show $(D, *)$ is not a group, it suffices to point out one of the conditions in the definition that is not satisfied.

Sets: Problems 1.1, 1.2, 1.4.

Functions: Problems 1.3, 2.1, 2.2, 2.3, 2.4.

About S_n (e.g., with $n = 3$): Problem 2.2.

Integers, complex numbers: Problems 3.1, 3.2, 3.4.

Definition of groups: Problems 3.3, 4.1, 4.4.

Properties of groups: Problems 4.2, 4.3.

Lecture notes and textbooks: All we have covered, **including properties of groups**.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Hints

have been withdrawn

from the site

PROBLEMS HINTS SOLUTIONS

Problem 5.1. Let G be a group and let a, b be (fixed) elements of G such that $ab^{-1} = b^{-1}a$. Prove the following equations.

(1) $ab = ba$.
(2) $a^{-1}b = ba^{-1}$.

Hint. For example, from $ab^{-1} = b^{-1}a$, one sees $b(ab^{-1}) = b(b^{-1}a)$, etcetera. Keep "playing" with the equations (according to the rules). *Do the algebra.*

Problem 5.2. Let G be a group of order 3. Say $G = \{e, a, b\}$, in which e, a and b are the three distinct elements of G with e the identity element. (Compare with Problem 4.2.)

(1) Fill in each of the blanks with a, b or e: $ab = \square$ and $ba = \square$. Explain why.
(2) Prove that G is abelian. That is, every group of order 3 is abelian.
(3) Fill in each of the blanks with a, b or e: $a^2 = \square$ and $b^2 = \square$. Prove your claims.

Hint. (1) Determine whether any of the following is ever possible: $ab = a$, $ab = b$, $ba = a$ or $ba = b$. Why? If not, then what are ab and ba equal to in light of $ab \in G$ and $ba \in G$? (For example, $ab = a$ would imply $ab = ae$. Is this ever possible in G in light of cancellation?)

(2) Show $xy = yx$ for all $x, y \in G$ by exhausting all possible pairs: ee, ea, eb, aa, ab, bb.

(3) This is similar to part (1). (For example, examine the possibility of $a^2 = e$, $a^2 = a$ or $a^2 = b$.) Recall that a^2 stands for aa. Feel free to quote/use part (1).

Problem 5.3. Let G be a group, $a, b \in G$ and $m, n \in \mathbb{Z}$.

(1) Prove that if $a^5 = b^5$ and $a^7 = b^7$ then $a = b$.
(2) Prove that if $a^m = b^m$, $a^n = b^n$ and $\gcd(m, n) = 1$ then $a = b$.

Hint. Feel free to "play" with the equations (according to the rules). *Do the algebra.*

(1) From $a^5 = b^5$ and $a^7 = b^7$, one can deduce $a^2 = (a^7)(a^5)^{-1} = (b^7)(b^5)^{-1} = b^2$ (for example). Keep doing the algebra.

(2) Since $\gcd(m, n) = 1$, there exist $s, t \in \mathbb{Z}$ such that $ms + nt = 1$. Use this, together with assumption, to prove $a = b$.

Problem 5.4. Let G be a group such that $(ab)^4 = a^4b^4$, $(ab)^5 = a^5b^5$ and $(ab)^6 = a^6b^6$ for all $a, b \in G$. Prove that G is abelian.

Hint. This is, in spirit, similar to (essentially the same as) an example shown in class.

PROBLEMS HINTS SOLUTIONS

Problem 6.1. Let $A = \mathbb{C} \setminus \{0\}$. Consider the group $(A, \cdot)$ under the usual multiplication. Also consider φ defined by $1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 1$, which is in the group $(S_4, \circ)$.

(1) Determine the order of 3, considered as an element of $(A, \cdot)$.
(2) Determine the order of $\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})$ as an element of $(A, \cdot)$.
(3) Determine $\mathrm{o}(\varphi)$.
(4) Compute φ^{1234} in the format of $1 \mapsto \boxed{?}$, $2 \mapsto \boxed{?}$, $3 \mapsto \boxed{?}$, $4 \mapsto \boxed{?}$.

Hint. For any group $(G, *)$ and $x \in G$, recall that $\mathrm{o}(x)$ (if $\mathrm{o}(x) < \infty$) is defined as the least positive integer n such that $\underbrace{x * x * \cdots * x}_{n \text{ terms}}$ is the identity of $(G, *)$.

(2) Use De Moivre's formula when applicable.
(4) If $\mathrm{o}(\varphi) = k$ and $n = kq + r \in \mathbb{Z}$, then $\varphi^n = \varphi^{kq+r} = \cdots$.

Problem 6.2. Let G be a group such that $x^2 = e$ for all $x \in G$.

(1) True or false: $x^{-1} = x$ for all $x \in G$. Justify.
(2) Prove that G is abelian.

Hint. (2) Let $x, y \in G$. You need to show $xy = yx$. There are many ways to achieve this. For example, you might want to use (1) as a starting point. Other approaches are available too. *Do the **algebra** by 'playing' with symbols **according to the rules**.*

Problem 6.3. Let G be an **abelian** group, $a \in G$ a **fixed** element of G, and n a **fixed** integer. Define $f: G \to G$ by $f(x) = x^n$ for all $x \in G$.

(1) Determine whether f is a group homomorphism, with justification.
(2) Prove that $f(a) = a$ **if** $a^{n-1} = e$.
(3) Prove that $f(a) = a$ **only if** $a^{n-1} = e$ (i.e., $a^{n-1} = e$ if $f(a) = a$.)

Hint. Note that n is a fixed integer. By the construction of f, what are $f(x)$, $f(y)$ and $f(xy)$? Can you show/verify $f(xy) = f(x)f(y)$?

(2)&(3) Make sure you know what to assume and what to prove in each case.

Problem 6.4. Let G be a group and let $a, g \in G$ be **fixed** elements. Define $h: G \to G$ by $h(x) = g^{-1}xg$ for all $x \in G$.

(1) Determine whether h is a group homomorphism, with justification.
(2) Prove that $h(a) = a$ **if** $ag = ga$.
(3) Prove that $h(a) = a$ **only if** $ag = ga$ (i.e., $ag = ga$ if $h(a) = a$.)

Hint. This should be straightforward: By the construction of h, what are $h(x)$, $h(y)$ and $h(xy)$? For part (1), can you verify $h(xy) = h(x)h(y)$ or otherwise give a counterexample?

(2)&(3) Make sure you know what to assume and what to prove in each case.

Problem 7.1. Let $\varphi\colon G \to G'$ be a group homomorphism, in which G and G' are groups. Let $a \in G$ such that $\mathrm{o}(a) < \infty$. (Denote the identity elements of G and G' by e and e' respectively.)

(1) Prove that $\mathrm{o}(\varphi(a)) < \infty$.
(2) Prove that $\mathrm{o}(\varphi(a)) \mid \mathrm{o}(a)$.

Hint. Denote $\mathrm{o}(a) = k$. Consider $\varphi(a^k)$. Use what we have learned (concerning group homomorphism and order) in class.

Problem 7.2. Consider the group $(S_3, \circ)$ under composition, which consists of the following

$$f_1\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3; \quad f_2\colon\ 1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 2; \quad f_3\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3;$$
$$f_4\colon\ 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1; \quad f_5\colon\ 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2; \quad f_6\colon\ 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1.$$

$H = \{x \in G \mid x^2 = f_1\}$ and $K = \{x^3 \mid x \in S_3\}$.

(1) Determine whether H is a subgroup of S_3.
(2) Determine whether K is a subgroup of S_3.

Hint. Use the definition or the criteria for subgroup.

Problem 7.3. Let G be an abelian group, $H = \{x \in G \mid x^9 = e\}$. Prove $H \leqslant G$, that is, prove that H is a subgroup of G.

Hint. Just use the subgroup criterion to show H is a subgroup of G. There is no need to show $H \subseteq G$, as this is clear. For $x \in G$, note that the construction of H says that $x \in H$ if and only if $x^9 = e$. (To be specific, is it true that $e \in H$? why? Given x, $y \in H$, what can you say about x and y? Can you show $xy \in H$ and $x^{-1} \in H$? Show your work.)

Problem 7.4. Let G be an abelian group, $H = \{a^4 \mid a \in G\}$ and $K = \{a^{52} \mid a \in G\}$.

(1) Prove $H \leqslant G$, that is, prove that H is a subgroup of G.
(2) Prove $K \subseteq H$, that is, prove that K is a subset of H.

Hint. (1) Just use the subgroup criterion to show H is a subgroup of G. There is no need to show $H \subseteq G$, as this is clear. For $x \in G$, note that the construction of H says that $x \in H$ if and only if $x = a^4$ for some $a \in G$. (To be specific, is it true that $e \in H$? why? Given x, $y \in H$, what can you say about x and y? Can you show $xy \in H$ and $x^{-1} \in H$?)

(2) What is the conventional way of showing one set is a subset of another set? Let $z \in K$. Then what can be said about z? Try to show $z \in H$ via the defining property of H.

PROBLEMS HINTS SOLUTIONS

Problem 8.1. Consider the group $(S_3, \circ)$ under composition, which consists of the following

$$f_1\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3; \quad f_2\colon\ 1 \mapsto 1,\ 2 \mapsto 3,\ 3 \mapsto 2; \quad f_3\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3;$$
$$f_4\colon\ 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1; \quad f_5\colon\ 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2; \quad f_6\colon\ 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1.$$

Find as many (distinct) subgroups of S_3 as possible. You will receive 1 point per correct subgroup and -1 point per incorrect choice.

Hint. You may describe the subgroups in the format of $H_1 = \{f_?\}$, $H_2 = \{f_?, f_?\}$, $\ldots$, $H_n = \{f_?, \ldots, f_?\}$, etcetera. It is your responsibility to make sure that each H_i is a correct answer (i.e., make sure that each H_i is indeed a subgroup of S_3). Note that every element a in any group G generates a cyclic subgroup $[a]$ of G.

Problem 8.2. Consider the group $(G, \circ)$ under composition, which consists of the following

$$f_1\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 3,\ 4 \mapsto 4; \qquad f_2\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 4,\ 4 \mapsto 3;$$
$$f_3\colon\ 1 \mapsto 2,\ 2 \mapsto 1,\ 3 \mapsto 3,\ 4 \mapsto 4; \qquad f_4\colon\ 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 4,\ 4 \mapsto 3.$$

Find as many (distinct) subgroups of $(G, \circ)$ as possible. You will receive 1 point per correct subgroup and -1 point per incorrect choice. (Note that $(G, \circ)$ is a subgroup of $(S_4, \circ)$.)

Hint. You may describe the subgroups in the format of $H_1 = \{f_?\}$, $H_2 = \{f_?, f_?\}$, $\ldots$, $H_n = \{f_?, \ldots, f_?\}$, etcetera. It is your responsibility to make sure that each H_i is a correct answer (i.e., make sure that each H_i is indeed a subgroup of G). Note that every element a in any group G generates a cyclic subgroup $[a]$ of G.

Problem 8.3. Let G be a group of order 30, i.e., $|G| = 30$, and let $x, y \in G$.

(1) If $x \in G$ satisfies $x^{24} = e$ and $x^9 \neq e$, determine all the possible value(s) of $\mathrm{o}(x)$.
(2) If $y^{20} = e$, $y^8 \neq e$ and $y^{15} \neq e$, then determine all the possible value(s) of $\mathrm{o}(y)$.
(3) Is it possible to ever have $z \in G$ such that $z^{45} = e$ and $z^{105} \neq e$? Why or why not?

Hint. Use Lagrange's theorem. Also note that $a^n = e$ if and only if $\mathrm{o}(a)$ divides n.

Problem 8.4. Prove that every group of order 4 is abelian as follows: Let G be any group of order 4, i.e., $|G| = 4$.

(1) Suppose there exists $a \in G$ such that $\mathrm{o}(a) = 4$. Prove that G is abelian.
(2) Suppose that no elements of G have order 4. Prove $x^2 = e$ for all $x \in G$.
(3) Suppose that no elements of G have order 4. Prove that G is abelian.

Hint. (1) If there exists $a \in G$ such that $\mathrm{o}(a) = 4$, what can be said about G and $[a]$? Show G is abelian in this case.

(2) Let $x \in G$. If $\mathrm{o}(x) \neq 4$, what can you say about $\mathrm{o}(x)$ in light of Lagrange's theorem? Then study x^2 in each of the possible cases for $\mathrm{o}(x)$.

(3) Prove that G is abelian in this case. Use part (2) and a previous homework problem.

PROBLEMS HINTS SOLUTIONS

Materials covered earlier: Midterm I; Homework Sets 1, 2, 3, 4.

Basic properties, direct calculations: Problems 5.1, 5.2, 5.3, 6.1, 6.2, 7.4, 8.1, 8.3, 8.4, etcetera.

Abelian groups: Problems 5.2, 5.4, 6.2, 8.4.

Orders of elements: Problems 6.1, 7.1, 8.3, 8.4.

Homomorphisms: Problems 6.3, 6.4, 7.1.

Subgroups, the subgroup criterion: Problems 5.3, 7.2, 7.3, 7.4, 8.1, 8.2.

Lagrange's theorem: Problems 8.3, 8.4.

Lecture notes and textbooks: All we have covered.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.