◇ ◇ ◇ ◇ MATH 2420: DISCRETE MATHEMATICS ◇ ◇ ◇ ◇
QUIZZES AND EXAMS

Yongwei Yao

2024 FALL SEMESTER
GEORGIA STATE UNIVERSITY

Contents

Note. Quizzes are administered in class (face-2-face), on Mondays or Wednesdays. In the (very rare) occasion when a quiz is take-home, the quiz must be submitted by the next lecture day before the class starts.

When solving the problems, make sure that your arguments are rigorous, accurate, and complete. Present your step-by-step work in your solutions.

There are two (2) PDF files for the quizzes and exams, one with the problems, and one with solutions. Links are available below.

PROBLEMS SOLUTIONS

Problem 1.1. Let P and Q be statement variables. Consider statements $\sim(P \to Q)$ and $P \wedge \sim Q$. *[Note that $P \wedge \sim Q$ is short for $P \wedge (\sim Q)$, while $P \to \sim Q$ is short for $P \to (\sim Q)$.]*

(1) [3 points] Complete the following truth table

P	Q	$P \to Q$	$\sim(P \to Q)$	$\sim Q$	$P \wedge \sim Q$	$P \to \sim Q$
T	T	T	F	F	F	F
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

(2) [1 point] The statement $\sim(P \to Q)$ is equivalent to $P \wedge \sim Q$ .... $\boxed{\textbf{True}}$ **False**

(3) [1 point] The statement $\sim(P \to Q)$ is equivalent to $P \to \sim Q$ ...... **True** $\boxed{\textbf{False}}$

(4) [1 point] The statement $P \wedge \sim Q$ is a contradiction .... **True** $\boxed{\textbf{False}}$

(5) [1 point] The statement $P \to \sim Q$ is a tautology .... **True** $\boxed{\textbf{False}}$

Solution. (1) A completed truth table is provided above.

(2) Since $\sim(P \to Q)$ and $P \wedge \sim Q$ always have the same truth values (as indicated in the truth table above), they are equivalent.

(3) Since $\sim(P \to Q)$ and $P \to \sim Q$ do not have the same truth values (as indicated in the truth table above), they are not equivalent.

(4) The statement $P \wedge \sim Q$ is not a tautology, since it is not always true.

(5) The statement $P \to \sim Q$ is not a contradiction, since it is not always false.

Problem 1.2. Determine whether the following statements are true or false. *[Pay close attention to 'if', 'only if", and 'if and only if'.]*

(1) [1 point] If turtles can fly then $1 + 2 = 5$. .... $\boxed{\textbf{True}}$ **False**

(2) [1 point] Humans can walk only if turtles can fly .... **True** $\boxed{\textbf{False}}$

(3) [1 point] Turtles can fly if and only if eagles can fly .... **True** $\boxed{\textbf{False}}$

Solution. Answers are provided above, while brief explanations are given below.

(1) True, because the antecedent is false (regardless of the consequent).

(2) False, because this statement means "if turtles can not fly then humans can not walk" or, equivalently, "if humans can walk then turtles can fly".

(3) False, because 'turtles can fly' is false while 'eagles can fly' is true.

PROBLEMS SOLUTIONS

Problem 2.1. Determine whether the following statements are valid or invalid.

(1) [1 point] The following argument is .... $\boxed{\textbf{Valid}}$ **Invalid**
If Alice has a laptop, then Alice knows logic
Alice does not know logic
∴ Alice does not have a laptop — modus tollens

(2) [1 point] The following argument is .... **Valid** $\boxed{\textbf{Invalid}}$
If Alice has a laptop, then Alice knows logic
Alice knows logic
∴ Alice has a laptop — converse error

(3) [1 point] The following argument is .... **Valid** $\boxed{\textbf{Invalid}}$
If Alice has a laptop, then Alice knows logic
Alice does not have a laptop
∴ Alice does not know logic — inverse error

(4) [1 point] The following argument is .... $\boxed{\textbf{Valid}}$ **Invalid**
If Alice has a laptop, then Alice knows logic
Alice has a laptop
∴ Alice knows logic — modus ponens

Solution. Answers are circled above, with brief explanations.

Problem 2.2. For each of statement, determine true or false:

(1) [1 point] $\forall x \in \mathbb{R}$, if $x < 3$ then $x^2 < 9$ .... **True** $\boxed{\textbf{False}}$
(2) [1 point] $\forall x \in \mathbb{R}$, if $x \geqslant 3$ then $x^2 \geqslant 9$ .... $\boxed{\textbf{True}}$ **False**
(3) [1 point] $\exists x \in \mathbb{R}$ such that $x^2 = 7$ .... $\boxed{\textbf{True}}$ **False**
(4) [1 point] $\exists x \in \mathbb{Z}$ such that $x^2 = 7$ .... **True** $\boxed{\textbf{False}}$
(5) [1 point] $\forall x \in \mathbb{R}$, if $x \geqslant -3$ then $x^2 \geqslant 9$ .... **True** $\boxed{\textbf{False}}$
(6) [1 point] $\forall x \in \mathbb{Z}$, if $x < -3$ then $x^2 \geqslant 16$ .... $\boxed{\textbf{True}}$ **False**

Solution. We provide answers below, with some brief explanations.
(1) False, with $x = -4$ being a counterexample. (There exists $-4 < 3$ but $(-4)^2 \not< 9$.)
(2) True. (It is true that, for all real number x, if $x \geqslant 3$ then $x^2 \geqslant 9$.)
(3) True, by choosing $x = \pm\sqrt{7} \in \mathbb{R}$. (There exists $\sqrt{7} \in \mathbb{R}$ such that $\left(\sqrt{7}\right)^2 = 7$.)
(4) False, since there is no integer whose square is 7.
(5) False, with $x = -2$ being a counterexample. (Indeed, $-2 \geqslant -3$ but $(-2)^2 < 9$.)
(6) True. (Indeed, for all integer x, if $x < -3$ then $x \leqslant -4$ and hence $x^2 \geqslant 16$.)

PROBLEMS SOLUTIONS

Problem 3.1. For each of the statements, determine true or false:

(1) [1 point] $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{Z}$ such that $x + y = 5$ .... **True** $\boxed{\textbf{False}}$

(2) [1 point] $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{R}$ such that $x + y = 4$ .... $\boxed{\textbf{True}}$ **False**

(3) [1 point] $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x + y = 7$ .... $\boxed{\textbf{True}}$ **False**

(4) [1 point] $\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $x + y = 6$ .... **True** $\boxed{\textbf{False}}$

(5) [1 point] $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, if $x \neq y$ then $x^2 \neq y^2$ .... **True** $\boxed{\textbf{False}}$

(6) [1 point] $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, if $x = -y$ then $x^2 = y^2$ .... $\boxed{\textbf{True}}$ **False**

Solution. Answers are provided above, while brief explanations are given below.

(1) False: for $x = 0.5 \in \mathbb{R}$ such that for every $y \in \mathbb{Z}$, $x + y \neq 5$.
(2) True: for every $x \in \mathbb{Z}$, there exists $y = 4 - x \in \mathbb{R}$ such that $x + y = 4$.
(3) True: for every $x \in \mathbb{R}$, there exists $y = 7 - x \in \mathbb{R}$ such that $x + y = 7$.
(4) False: for every $x \in \mathbb{R}$, there exists $y = -x \in \mathbb{R}$ such that $x + y = 0 \neq 6$.
(5) False: there exist $x = -1 \in \mathbb{R}$ and $y = 1 \in \mathbb{R}$ such that $-1 \neq 1$ but $(-1)^2 = 1^2$.
(6) True: for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$, if $x = -y$ then $x^2 = (-y)^2 = y^2$.

Problem 3.2. For each statement (regardless of its truth value), re-write its negation. *[You are not supposed to add $\sim$ or the phrase "it is not the case that" before the given statements to form their negations. Instead, try to paraphrase their negations.]*

(1) [1 point] There exists a person in Atlanta whose height is $\leqslant 8$ feet.
(2) [1 point] All people in Atlanta have height $\leqslant 8$ feet.
(3) [1 point] $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x + y = 3$.
(4) [1 point] $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, if $x < y$ then $x^2 < y^2$.

Solution. Their negations are given below (regardless of their truth values).

(1) All people in Atlanta are taller than 8 feet, or, there is no person in Atlanta whose height is $\leqslant 8$ feet.

(2) There exists a person in Atlanta whose height is > 8 feet, or, there exists someone in Atlanta who is taller than 8 feet, or, someone in Atlanta is taller than 8 feet.

(3) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $x + y \neq 3$.

(4) $\exists x \in \mathbb{R}$ and $\exists y \in \mathbb{R}$ such that $x < y$ and $x^2 \geqslant y^2$. *[Note that the negation of "if $x < y$ then $x^2 < y^2$" is "$x < y$ and $x^2 \geqslant y^2$".]*

PROBLEMS SOLUTIONS

Truth table, conjunction, disjunction, negation, tautology, contradiction, (bi)conditional, etc.: Problems 1.1, 1.2.

Valid arguments, invalid arguments: Problem 2.1.

Statements with quantifiers ($\forall$, $\exists$), their negations: Problems 2.2, 3.1, 3.2.

Lecture notes and textbooks: All we have covered in class.

Problem. Determine the truth values of the following statements:

(1) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x - y^2 = 13$ . . . . **True** $\boxed{\textbf{False}}$

(2) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y^2 > 135$ . . . . $\boxed{\textbf{True}}$ **False**

Solution. (1) False: there exists $x = 12 \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ we have $12 - y^2 \neq 13$. (Put differently, for $x = 12$, there exists **NO** $y \in \mathbb{R}$ such that $12 - y^2 = 13$.)

(2) True: there exists $x = 135.1 \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ we have $135.1 + y^2 > 135$.

Problem. Let p, q and r be statement variables. Determine valid or invalid:

(1) $\begin{cases} p \to (q \vee r) \\ p \wedge (\sim q) \\ \therefore r \end{cases}$ . . . . $\boxed{\textbf{Valid}}$ **Invalid** (2) $\begin{cases} p \to q \\ p \vee r \\ \therefore q \end{cases}$ . . . . **Valid** $\boxed{\textbf{Invalid}}$

Solution. Answers are provided above, with brief explanations given below:

(1) Valid. From $p \wedge (\sim q)$, we can deduce both p and $\sim q$ via specialization. From $p \to (q \vee r)$ and p, we can deduce $q \vee r$ via modus ponens. Then, from $q \vee r$ and $\sim q$, we can conclude r by elimination.

(2) Invalid. From $p \vee r$, we can **NOT** necessarily deduce p. Thus we can not conclude q via modus ponens. In terms of truth table, when both p and q are false while r is true, we see that both premises $p \to q$ and $p \vee r$ are true, but the proposed conclusion q is false.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply rote-memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Solutions

have been withdrawn

from the site

PROBLEMS SOLUTIONS

Problem 4.1. Prove that, for all m, $n \in \mathbb{Z}$, if both m and n are odd then $m + n$ is even. (That is, prove that the sum of any two odd integers is even.)

Proof. Let m, $n \in \mathbb{Z}$; assume that both m and n are odd. Then

$$\begin{aligned} m &= 2k + 1 \quad \text{for some} \quad k \in \mathbb{Z}; \\ n &= 2l + 1 \quad \text{for some} \quad l \in \mathbb{Z}. \end{aligned}$$

Therefore, we see

$$\begin{aligned} m + n &= (2k + 1) + (2l + 1) \\ &= 2k + 2l + 2 \\ &= 2(k + l + 1). \end{aligned}$$

Note that $k + l + 1 \in \mathbb{Z}$, because k, l, $1 \in \mathbb{Z}$. *[This shows that $m + n$ equals 2 times an integer.]* Consequently, $m + n$ is even, since $m + n = 2(k + l + 1)$ with $k + l + 1 \in \mathbb{Z}$.

This proves that, for all m, $n \in \mathbb{Z}$, if both m and n are odd then $m + n$ is even. *[That is, the sum of any two odd integers is even.]* □

PROBLEMS SOLUTIONS

Problem 5.1 (5 points)**.** Prove that, for all a, b, $c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid (b + 3c)$.

Proof. Let a, b, $c \in \mathbb{Z}$; assume $a \mid b$ and $a \mid c$. From the definition of divisibility, we see that

$$\begin{aligned} b &= ar \quad \text{for some} \quad r \in \mathbb{Z}; \\ c &= as \quad \text{for some} \quad s \in \mathbb{Z}. \end{aligned}$$

Therefore, we see

$$\begin{aligned} b + 3c &= ar + 3as \\ &= a(r + 3s), \quad \text{with } r + 3s \in \mathbb{Z}. \end{aligned}$$

Consequently, we conclude $a \mid (b + 3c)$, since $b + 3c = a(r + 3s)$ with $r + 3s \in \mathbb{Z}$. This completes the proof that, for all a, b, $c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid (b + 3c)$. $\square$

Problem 5.2 (5 points)**.** Complete each of the following questions.

(1) Write down the standard factored form of 450.
(2) For $n = 97$ and $d = 13$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.
(3) For $n = -57$ and $d = 23$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.
(4) Compute $\left\lfloor \dfrac{33}{4} \right\rfloor$.
(5) Compute $\left\lceil -\dfrac{43}{6} \right\rceil$.

Solution. The answers are provided below.

(1) $450 = 2^1 \cdot 3^2 \cdot 5^2 = 2 \cdot 3^2 \cdot 5^2$.
(2) $97 = 13(7) + 6$, with $q = 7$ and $r = 6$.
(3) $-57 = 23(-3) + 12$, with $q = -3$ and $r = 12$.
(4) $\left\lfloor \dfrac{33}{4} \right\rfloor = 8$, because $8 \leqslant \dfrac{33}{4} < 8 + 1$.
(5) $\left\lceil -\dfrac{43}{6} \right\rceil = -7$, because $(-7) - 1 < -\dfrac{43}{6} \leqslant -7$.

PROBLEMS SOLUTIONS

Problem 5.1 (5 points)**.** Prove that, for all p, m, $n \in \mathbb{Z}$, if $p \mid m$ and $p \mid n$ then $p \mid (2m-n)$.

Proof. Let p, m, $n \in \mathbb{Z}$; assume $p \mid m$ and $p \mid n$. From the definition of divisibility, we see that

$$\begin{aligned} m &= pr \quad \text{for some} \quad r \in \mathbb{Z}; \\ n &= ps \quad \text{for some} \quad s \in \mathbb{Z}. \end{aligned}$$

Therefore, we see

$$\begin{aligned} 2m - n &= 2pr - ps \\ &= p(2r - s), \quad \text{with } 2r - s \in \mathbb{Z}. \end{aligned}$$

Consequently, we conclude $p \mid (2m-n)$, since $2m - n = p(2r - s)$ with $2r - s \in \mathbb{Z}$. This completes the proof that, for all p, m, $n \in \mathbb{Z}$, if $p \mid m$ and $p \mid n$ then $p \mid (2m - n)$. ☐

Problem 5.2 (5 points)**.** Complete each of the following questions.

(1) Write down the standard factored form of 540.
(2) For $n = -97$ and $d = 13$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.
(3) For $n = 57$ and $d = 23$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.
(4) Compute $\left\lfloor -\dfrac{33}{4} \right\rfloor$.
(5) Compute $\left\lceil \dfrac{43}{6} \right\rceil$.

Solution. The answers are provided below.

(1) $540 = 2^2 \cdot 3^3 \cdot 5^1 = 2^2 \cdot 3^3 \cdot 5$.
(2) $-97 = 13(-8) + 7$, with $q = -8$ and $r = 7$.
(3) $57 = 23(2) + 11$, with $q = 2$ and $r = 11$.
(4) $\left\lfloor -\dfrac{33}{4} \right\rfloor = -9$, because $-9 \leqslant -\dfrac{33}{4} < (-9) + 1$.
(5) $\left\lceil \dfrac{43}{6} \right\rceil = 8$, because $8 - 1 < \dfrac{43}{6} \leqslant 8$.

PROBLEMS SOLUTIONS

Problem 6.1 (5 points)**.** Complete the following by filling in the blanks.

(1) Write down the standard factored form: $5500 = \underline{\quad 2^2 \cdot 5^3 \cdot 11 \quad}$.
(2) For $n = 187$ and $d = 13$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$: $187 = 13(\underline{\ 14\ }) + \underline{\ 5\ }$.
(3) For $n = -179$ and $d = 24$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$: $-179 = 24(\underline{\ -8\ }) + \underline{\ 13\ }$.
(4) $\left\lfloor \dfrac{93}{8} \right\rfloor = \underline{\ 11\ }$.
(5) $\left\lceil -\dfrac{93}{7} \right\rceil = \underline{\ -13\ }$.

Solution. The answers are provided below (as well as above).

(1) $5500 = \underline{\quad 2^2 \cdot 5^3 \cdot 11 \quad}$.
(2) $187 = 13(\underline{\ 14\ }) + \underline{\ 5\ }$, with $q = 14$ and $r = 5$.
(3) $-179 = 24(\underline{\ -8\ }) + \underline{\ 13\ }$, with $q = -8$ and $r = 13$.
(4) $\left\lfloor \dfrac{93}{8} \right\rfloor = \underline{\ 11\ }$, because $11 \leqslant \dfrac{93}{8} < 11 + 1$.
(5) $\left\lceil -\dfrac{93}{7} \right\rceil = \underline{\ -13\ }$, because $(-13) - 1 < -\dfrac{93}{7} \leqslant -13$.

Problem 6.2 (5 points)**.** Complete the following (true or false, or filling in the blanks, where m, n, p are integers whose specific values not not known):

(1) $-44 \equiv 66 \pmod{10}$ .... $\boxed{\textbf{True}}$ **False**
(2) $-44 \equiv -66 \pmod{13}$ .... **True** $\boxed{\textbf{False}}$
(3) Suppose m mod $13 = 7$ and n mod $13 = 8$. Then mn mod $13 = \underline{\ 4\ }$.
(4) Suppose m mod $13 = 7$ and n mod $13 = 8$. Then $n^2 + m$ mod $13 = \underline{\ 6\ }$.
(5) Suppose p mod $13 = 3$. Then p^4 mod $13 = \underline{\ 3\ }$.

Solution. We provide answers (with brief explanations) as follows.

(1) True, because $10 \mid (-44 - 66)$.
(2) False, because $13 \nmid (-44 - (-66))$.
(3) Given m mod $13 = 7$ and n mod $13 = 8$, we have
$$mn \bmod 13 = 7 \cdot 8 \bmod 13 = 56 \bmod 13 = 4.$$
(4) Given m mod $13 = 7$ and n mod $13 = 8$, we have
$$n^2 + m \bmod 13 = 8^2 + 7 \bmod 13 = 71 \bmod 13 = 6.$$
(5) Given p mod $13 = 3$, we have
$$p^4 \bmod 13 = 3^4 \bmod 13 = 81 \bmod 13 = 3.$$

PROBLEMS SOLUTIONS

Problem 6.1 (5 points)**.** Complete the following by filling in the blanks.

(1) Write down the standard factored form: $4400 = \underline{\quad 2^4 \cdot 5^2 \cdot 11 \quad}$.
(2) For $n = -187$ and $d = 13$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$: $-187 = 13(\underline{\ -15\ }) + \underline{\quad 8 \quad}$.
(3) For $n = 179$ and $d = 24$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$: $179 = 24(\underline{\quad 7 \quad}) + \underline{\quad 11 \quad}$.
(4) $\left\lfloor -\dfrac{93}{8} \right\rfloor = \underline{\ -12\ }$.
(5) $\left\lceil \dfrac{93}{7} \right\rceil = \underline{\quad 14 \quad}$.

Solution. The answers are provided below (as well as above).

(1) $4400 = \underline{\quad 2^4 \cdot 5^2 \cdot 11 \quad}$.
(2) $-187 = 13(\underline{\ -15\ }) + \underline{\quad 8 \quad}$, with $q = -15$ and $r = 8$.
(3) $179 = 24(\underline{\quad 7 \quad}) + \underline{\quad 11 \quad}$, with $q = 7$ and $r = 11$.
(4) $\left\lfloor -\dfrac{93}{8} \right\rfloor = \underline{\ -12\ }$, because $-12 \leqslant \dfrac{93}{8} < (-12) + 1$.
(5) $\left\lceil \dfrac{93}{7} \right\rceil = \underline{\quad 14 \quad}$, because $(14) - 1 < -\dfrac{93}{7} \leqslant 14$.

Problem 6.2 (5 points)**.** Complete the following (true or false, or filling in the blanks, where m, n, p are integers whose specific values not not known):

(1) $-44 \equiv -66 \pmod{10}$ .... **True** $\boxed{\textbf{False}}$
(2) $-44 \equiv 66 \pmod{13}$ .... **True** $\boxed{\textbf{False}}$
(3) Suppose $m \bmod 13 = 6$ and $n \bmod 13 = 9$. Then $mn \bmod 13 = \underline{\quad 2 \quad}$.
(4) Suppose $m \bmod 13 = 6$ and $n \bmod 13 = 9$. Then $n^2 + m \bmod 13 = \underline{\quad 9 \quad}$.
(5) Suppose $p \bmod 13 = 2$. Then $p^6 \bmod 13 = \underline{\quad 12 \quad}$.

Solution. We provide answers (with brief explanations) as follows.

(1) False, because $10 \nmid (-44 - (-66))$.
(2) False, because $13 \nmid (-44 - 66)$.
(3) Given $m \bmod 13 = 6$ and $n \bmod 13 = 9$, we have
$$mn \bmod 13 = 6 \cdot 9 \bmod 13 = 54 \bmod 13 = 2.$$
(4) Given $m \bmod 13 = 6$ and $n \bmod 13 = 9$, we have
$$n^2 + m \bmod 13 = 9^2 + 6 \bmod 13 = 87 \bmod 13 = 9.$$
(5) Given $p \bmod 13 = 2$, we have
$$p^6 \bmod 13 = 2^6 \bmod 13 = 64 \bmod 13 = 12.$$

PROBLEMS SOLUTIONS

Materials covered earlier: Midterm I; Quizzes 1, 2, 3.

Proofs (concerning even/odd numbers, divisibility, etc.): Problems 4.1, 5.1.

Standard factored form, quotient-remainder, floor/ceiling, etc.: Problems 5.2, 6.1.

Modular congruence, etc.: Problems 6.2.

Lecture notes and textbooks: All we have covered.

Problem. Prove that, for all m, $n \in \mathbb{Z}$, if $m \bmod 6 = 3$ and $n \bmod 6 = 5$ then mn is odd.

Proof. Let m, $n \in \mathbb{Z}$; assume $m \bmod 6 = 3$ and $n \bmod 6 = 5$. Then

$$m = 6p + 3 \quad \text{and} \quad n = 6q + 5 \quad \text{for some } p,\ q \in \mathbb{Z}.$$

Consequently, we see

$$mn = (6p + 3)(6q + 5) = 36pq + 30p + 18q + 15 = 2(18pq + 15p + 9q + 7) + 1$$

with $18pq + 15p + 9q + 7 \in \mathbb{Z}$. *[That is, mn is 2 times an integer and then plus 1.]* This shows that mn is odd, completing the proof. □

Problem. Prove that, for all d, m, $n \in \mathbb{Z}$, if $d \mid m$ and $d \mid n$ then $d \mid (5m + 8n)$.

Proof. Let d, m, $n \in \mathbb{Z}$; assume $d \mid m$ and $d \mid n$. From definition, we see

$$m = dr \quad \text{and} \quad n = ds \quad \text{for some } r,\ s \in \mathbb{Z}.$$

Therefore, we see

$$5m + 8n = 5(dr) + 8(ds) = 5dr + 8ds = d(5r + 8s)$$

with $5r+8s \in \mathbb{Z}$. *[That is, $5m+8n$ equals d times an integer.]* This shows that $d \mid (5m+8n)$, completing the proof. □

Problem. Prove that, for all d, $m \in \mathbb{Z}$, if $d \mid m$ then $d^2 \mid m^2$.

Proof. Let d, $m \in \mathbb{Z}$ and assume $d \mid m$. From definition, we see

$$m = dr \quad \text{for some } r \in \mathbb{Z}.$$

Therefore, we see

$$m^2 = (dr)^2 = d^2 r^2 \quad \text{with } r^2 \in \mathbb{Z}.$$

[That is, m^2 equals d^2 times an integer.] This shows that $d^2 \mid m^2$, completing the proof. □

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply rote-memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Solutions

have been withdrawn

from the site

PROBLEMS SOLUTIONS

Problem 7.1. Prove by **mathematical induction** that $\sum\limits_{i=4}^{n}(2i+1) = n^2+2n-15$ for all integers $n \geqslant 4$ (that is, $(2(4)+1)+\cdots+(2n+1) = n^2+2n-15$ for all integers $n \geqslant 4$).

Proof. When $n=4$, the left hand side is $\sum_{i=4}^{4}(2i+1) = 2(4)+1 = 9$ while the right hand side is $4^2+2(4)-15 = 9$. As both sides of the equation agree, this verifies $\sum_{i=4}^{4}(2i+1) = 4^2+2(4)-15$; that is, the claim holds when $n=4$. *[This is the basic step.]*

[Now is the inductive step.] Assume that the claim $\sum\limits_{i=4}^{n}(2i+1) = n^2+2n-15$ holds for $n=k$ for some integer $k \geqslant 4$; that is, assume

$$(*) \qquad \sum_{i=4}^{k}(2i+1) = k^2+2k-15,$$

where $k \geqslant 4$ is an integer. *[The above $(*)$ is the induction hypothesis.] [Next, we show that the claim holds for $n=k+1$, i.e., $\sum\limits_{i=4}^{k+1}(2i+1) = (k+1)^2+2(k+1)-15$.]* Then we have

$$\begin{aligned}
\sum_{i=4}^{k+1}(2i+1) &= \sum_{i=4}^{k}(2i+1) + [2(k+1)+1] \\
&= [k^2+2k-15] + [2(k+1)+1] \quad \text{(by induction hypothesis } (*)) \\
&= k^2+2k-15+2(k+1)+1 \\
&= (k^2+2k+1)+2(k+1)-15 \\
&= (k+1)^2+2(k+1)-15.
\end{aligned}$$

In short, $\sum\limits_{i=4}^{k+1}(2i+1) = (k+1)^2+2(k+1)-15$, which verifies the claim $\sum\limits_{i=4}^{n}(2i+1) = n^2+2n-15$ when $n=k+1$. *[Overall, we have proved that if the claim is true for an integer $n=k$, where $k \geqslant 4$, then the claim is true for $n=k+1$, which completes the inductive step.]*

Therefore, by the principle of mathematical induction, we conclude that the equation $\sum_{i=4}^{n}(2i+1) = n^2+2n-15$ holds for all integers $n \geqslant 4$. $\square$

PROBLEMS SOLUTIONS

Problem 7.1. Prove by **mathematical induction** that $\sum\limits_{i=3}^{n}(2i+1) = n^2+2n-8$ for all integers $n \geqslant 3$ (that is, $(2(3)+1)+\cdots+(2n+1) = n^2+2n-8$ for all integers $n \geqslant 3$).

Proof. When $n=3$, the left hand side is $\sum_{i=3}^{3}(2i+1) = 2(3)+1 = 7$ while the right hand side is $3^2+2(3)-8 = 7$. As both sides of the equation agree, this verifies $\sum_{i=3}^{3}(2i+1) = 3^2+2(3)-8$; that is, the claim holds when $n=3$. *[This is the basic step.]*

[Now is the inductive step.] Assume that the claim $\sum\limits_{i=3}^{n}(2i+1) = n^2+2n-8$ holds for $n=k$ for some integer $k \geqslant 3$; that is, assume

$$(*) \qquad \sum_{i=3}^{k}(2i+1) = k^2+2k-8,$$

where $k \geqslant 3$ is an integer. *[The above $(*)$ is the induction hypothesis.] [Next, we show that the claim holds for $n=k+1$, i.e., $\sum\limits_{i=3}^{k+1}(2i+1) = (k+1)^2+2(k+1)-8$.]* Then we have

$$\begin{aligned}
\sum_{i=3}^{k+1}(2i+1) &= \sum_{i=3}^{k}(2i+1) + [2(k+1)+1] \\
&= [k^2+2k-8]+[2(k+1)+1] \quad \text{(by induction hypothesis $(*)$)} \\
&= k^2+2k-8+2(k+1)+1 \\
&= (k^2+2k+1)+2(k+1)-8 \\
&= (k+1)^2+2(k+1)-8.
\end{aligned}$$

In short, $\sum\limits_{i=3}^{k+1}(2i+1) = (k+1)^2+2(k+1)-8$, which verifies the claim $\sum\limits_{i=3}^{n}(2i+1) = n^2+2n-8$ when $n=k+1$. *[Overall, we have proved that if the claim is true for an integer $n=k$, where $k \geqslant 3$, then the claim is true for $n=k+1$, which completes the inductive step.]*

Therefore, by the principle of mathematical induction, we conclude that the equation $\sum_{i=3}^{n}(2i+1) = n^2+2n-8$ holds for all integers $n \geqslant 3$. $\square$

PROBLEMS SOLUTIONS

Problem 8.1 (5 points)**.** Let $A = (-5,\ 1]$, $B = [-3,\ 4)$ and $C = [-2,\ 6]$, all being intervals of $\mathbb{R}$. Compute the following:

(1) $(A \cup B) - C$.
(2) $A \cup (B - C)$.
(3) $A - (B \cap C)$.
(4) $(A - B) \cap C$.
(5) $(A \cap B) \cup C$.

Solution. We provide answers (with some intermediate steps) as follows:

(1) $(A \cup B) - C = (-5,\ 4) - [-2,\ 6] = (-5,\ -2)$.
(2) $A \cup (B - C) = (-5,\ 1] \cup [-3,\ -2) = (-5,\ 1]$.
(3) $A - (B \cap C) = (-5,\ 1] - [-2,\ 4) = (-5,\ -2)$.
(4) $(A - B) \cap C = (-5,\ -3) \cap [-2,\ 6] = \varnothing$. *[Note that $\varnothing$ denotes the empty set.]*
(5) $(A \cap B) \cup C = [-3,\ 1] \cup [-2,\ 6] = [-3,\ 6]$.

Problem 8.2 (5 points)**.** Consider function $f\colon \mathbb{R} \to \mathbb{R}$ that is defined as follows:

$$f(x) = x^2 + 9 \qquad \text{for all } x \in \mathbb{R}.$$

(1) Find the image of 5 under f.
(2) Find the set of *all* pre-images of 7 under f. If the answer is the empty set, state so.
(3) Find the set of *all* pre-images of 13 under f. If the answer is the empty set, state so.
(4) Determine whether f is a one-to-one (i.e., injective) function.
(5) Determine whether f is an onto (i.e., surjective) function.

Solution. We provide answers (with some brief explanations) as follows:

(1) The image of 5 under f is $f(5) = 5^2 + 9 = 34$.
(2) The set of all pre-images of 7 under f is $\varnothing$, the empty set. *[Solving $f(x) = 7$, i.e., $x^2 + 9 = 7$, we obtain $x^2 = -2$, which has no solution in $\mathbb{R}$.]*
(3) The set of all pre-images of 13 under f is $\{2,\ -2\}$. *[By solving $f(x) = 13$, i.e., $x^2 + 9 = 13$, we get $x^2 = 4$, which yields $x = \pm 2$.]*
(4) No, f is not a one-to-one (i.e., injective) function. *[Indeed, there exist $2 \neq -2 \in \mathbb{R}$ such that $f(2) = f(-2)$.]*
(5) No, f is not an onto (i.e., surjective) function. *[Indeed, for all $x \in \mathbb{R}$, we see $f(x) = x^2 + 9 \geqslant 9$. Thus, for example, there exist no $x \in \mathbb{R}$ such that $f(x) = 8.9$. Alternatively, from (2) above, we see that there is no $x \in \mathbb{R}$ such that $f(x) = 7$.]*

PROBLEMS SOLUTIONS

Problem 8.1 (5 points)**.** Let $A = (-4,\ 3]$, $B = [-2,\ 5)$ and $C = [-1,\ 8]$, all being intervals of $\mathbb{R}$. Compute the following:

(1) $A \cup (B - C)$.
(2) $(A \cup B) - C$.
(3) $(A - B) \cap C$.
(4) $A - (B \cap C)$.
(5) $A \cap (B \cup C)$.

Solution. We provide answers (with some intermediate steps) as follows:

(1) $A \cup (B - C) = (-4,\ 3] \cup [-2,\ -1) = (-4,\ 3]$.
(2) $(A \cup B) - C = (-4,\ 5) - [-1,\ 8] = (-4,\ -1)$.
(3) $(A - B) \cap C = (-4,\ -2) \cap [-1,\ 8] = \varnothing$. *[Note that* $\varnothing$ *denotes the empty set.]*
(4) $A - (B \cap C) = (-4,\ 3] - [-1,\ 5) = (-4,\ -1)$.
(5) $A \cap (B \cup C) = (-4,\ 3] \cap [-2,\ 8] = [-2,\ 3]$.

Problem 8.2 (5 points)**.** Consider function $f \colon \mathbb{R} \to \mathbb{R}$ that is defined as follows:

$$f(x) = x^2 + 4 \qquad \text{for all } x \in \mathbb{R}.$$

(1) Find the image of 6 under f.
(2) Find the set of *all* pre-images of 3 under f. If the answer is the empty set, state so.
(3) Find the set of *all* pre-images of 13 under f. If the answer is the empty set, state so.
(4) Determine whether f is a one-to-one (i.e., injective) function.
(5) Determine whether f is an onto (i.e., surjective) function.

Solution. We provide answers (with some brief explanations) as follows:

(1) The image of 2 under f is $f(6) = 6^2 + 4 = 40$.
(2) The set of all pre-images of 3 under f is $\varnothing$, the empty set. *[Solving* $f(x) = 3$*, i.e.,* $x^2 + 4 = 3$*, we obtain* $x^2 = -1$*, which has no solution in* $\mathbb{R}$*.]*
(3) The set of all pre-images of 13 under f is $\{3,\ -3\}$. *[By solving* $f(x) = 13$*, i.e.,* $x^2 + 4 = 13$*, we get* $x^2 = 9$*, which yields* $x = \pm 3$*.]*
(4) No, f is not a one-to-one (i.e., injective) function. *[Indeed, there exist* $2 \neq -2 \in \mathbb{R}$ *such that* $f(2) = f(-2)$*.]*
(5) No, f is not an onto (i.e., surjective) function. *[Indeed, for all* $x \in \mathbb{R}$*, we see* $f(x) = x^2 + 4 \geqslant 4$*. Thus, for example, there exist no* $x \in \mathbb{R}$ *such that* $f(x) = 3.9$*. Alternatively, from (2) above, we see that there is no* $x \in \mathbb{R}$ *such that* $f(x) = 3$*.]*

PROBLEMS SOLUTIONS

Problem 9.1 (5 points)**.** Let $S = \{100, 101, \ldots, 999\}$, the set of all 3-digit integers, from 100 to 999 inclusive. Complete the following:

(1) Compute $N(S)$, the total number of objects in the set S.
(2) How many 3-digit integers are there from 100 to 799 inclusive?
(3) How many 3-digit integers are there from 100 to 799 inclusive without repeated digits?
(4) How many 3-digit integers are there from 100 to 799 inclusive with repeated digits?
(5) What is the probability that a randomly number chosen from S is from 100 to 799 inclusive without repeated digits?

Solution. We provide answers (with some intermediate steps) as follows:

(1) $N(S) = 9 \cdot 10 \cdot 10 = 900$.
(2) There are $7 \cdot 10 \cdot 10 = 700$ many 3-digit integers from 100 to 799 inclusive.
(3) There are $7 \cdot 9 \cdot 8 = 504$ many 3-digit integers from 100 to 799 inclusive without repeated digits.
(4) By the difference rule, there are $700 - 504 = 196$ many 3-digit integers from 100 to 799 inclusive with repeated digits.
(5) The probability is $\dfrac{504}{900} = \dfrac{7 \cdot 9 \cdot 8}{9 \cdot 10 \cdot 10} = \dfrac{56}{100} = 56\%$.

Problem 9.2 (5 points)**.** A certain personal identification number (PIN) is required to be a sequence of any three (3) symbols chosen from the first six (6) uppercase letters A, B, C, D, E, F. Complete the following:

(1) How many different PINs are possible if repetition of symbols is allowed?
(2) How many different PINs are possible if repetition of symbols is not allowed?
(3) How many different PINs are possible that contain at least one repeated symbol?
(4) How many different PINs are possible that do not contain the symbol C if repetition of symbols is not allowed?
(5) How many different PINs are possible that contain the symbol C if repetition of symbols is not allowed?

Solution. We provide answers as follows:

(1) There are $6 \cdot 6 \cdot 6 = 6^3 = 216$ many PINs if repetition of symbols is allowed.
(2) There are $P(6, 3) = 6 \cdot 5 \cdot 4 = 120$ many PINs if repetition of symbols is not allowed.
(3) By the difference rule, there are $216 - 120 = 96$ many PINs that contain at least one repeated symbol.
(4) There are $P(5, 3) = 5 \cdot 4 \cdot 3 = 60$ many PINs that do not contain the symbol C if repetition of symbols is not allowed.
(5) By the difference rule, there are $P(6, 3) - P(5, 3) = 120 - 60 = 60$ many PINs that contain the symbol C if repetition of symbols is not allowed.

PROBLEMS SOLUTIONS

Problem 9.1 (5 points)**.** Let $S = \{100, 101, \ldots, 999\}$, the set of all 3-digit integers, from 100 to 999 inclusive. Complete the following:

(1) Compute $N(S)$, the total number of objects in the set S.
(2) How many 3-digit integers are there from 300 to 899 inclusive?
(3) How many 3-digit integers are there from 300 to 899 inclusive without repeated digits?
(4) How many 3-digit integers are there from 300 to 899 inclusive with repeated digits?
(5) What is the probability that a randomly number chosen from S is from 300 to 899 inclusive without repeated digits?

Solution. We provide answers (with some intermediate steps) as follows:

(1) $N(S) = 9 \cdot 10 \cdot 10 = 900$.
(2) There are $6 \cdot 10 \cdot 10 = 600$ many 3-digit integers from 300 to 899 inclusive.
(3) There are $6 \cdot 9 \cdot 8 = 432$ many 3-digit integers from 300 to 899 inclusive without repeated digits.
(4) By the difference rule, there are $600 - 432 = 168$ many 3-digit integers from 300 to 899 inclusive with repeated digits.
(5) The probability is $\dfrac{432}{900} = \dfrac{6 \cdot 9 \cdot 8}{9 \cdot 10 \cdot 10} = \dfrac{48}{100} = 48\%$.

Problem 9.2 (5 points)**.** A certain personal identification number (PIN) is required to be a sequence of any three (3) symbols chosen from the first seven (7) uppercase letters A, B, C, D, E, F, G. Complete the following:

(1) How many different PINs are possible if repetition of symbols is allowed?
(2) How many different PINs are possible if repetition of symbols is not allowed?
(3) How many different PINs are possible that contain at least one repeated symbol?
(4) How many different PINs are possible that do not contain the symbol E if repetition of symbols is not allowed?
(5) How many different PINs are possible that contain the symbol E if repetition of symbols is not allowed?

Solution. We provide answers as follows:

(1) There are $7 \cdot 7 \cdot 7 = 7^3 = 343$ many PINs if repetition of symbols is allowed.
(2) There are $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$ many PINs if repetition of symbols is not allowed.
(3) By the difference rule, there are $343 - 210 = 133$ many PINs that contain at least one repeated symbol.
(4) There are $P(6, 3) = 6 \cdot 5 \cdot 4 = 120$ many PINs that do not contain the symbol E if repetition of symbols is not allowed.
(5) By the difference rule, there are $P(7, 3) - P(6, 3) = 210 - 120 = 90$ many PINs that contain the symbol E if repetition of symbols is not allowed.

PROBLEMS SOLUTIONS