$\diamond \diamond \diamond \diamond$ **MATH 2420: DISCRETE MATHEMATICS** $\diamond \diamond \diamond \diamond$

QUIZZES AND EXAMS

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Contents

Note. Quizzes are administered in class (face-2-face), on Mondays or Wednesdays. In the (very rare) occasion when a quiz is take-home, the quiz must be submitted by the next lecture day before the class starts.

When solving the problems, make sure that your arguments are rigorous, accurate, and complete. Present your step-by-step work in your solutions.

There are two (2) PDF files for the quizzes and exams, one with the problems, and one with solutions. Links are available below.

PROBLEMS SOLUTIONS

Problem 1.1. Let P and Q be statement variables. Consider statements $\sim(P \to Q)$ and $P \wedge \sim Q$. *[Note that $P \wedge \sim Q$ is short for $P \wedge (\sim Q)$, while $P \to \sim Q$ is short for $P \to (\sim Q)$.]*

(1) [3 points] Complete the following truth table

P	Q	$P \to Q$	$\sim(P \to Q)$	$\sim Q$	$P \wedge \sim Q$	$P \to \sim Q$
T	T					
T	F					
F	T					
F	F					

(2) [1 point] The statement $\sim(P \to Q)$ is equivalent to $P \wedge \sim Q$ .... **True** **False**

(3) [1 point] The statement $\sim(P \to Q)$ is equivalent to $P \to \sim Q$ .... **True** **False**

(4) [1 point] The statement $P \wedge \sim Q$ is a contradiction .... **True** **False**

(5) [1 point] The statement $P \to \sim Q$ is a tautology .... **True** **False**

Problem 1.2. Determine whether the following statements are true or false. *[Pay close attention to 'if', 'only if", and 'if and only if'.]*

(1) [1 point] If turtles can fly then $1 + 2 = 5$. .... **True** **False**

(2) [1 point] Humans can walk only if turtles can fly .... **True** **False**

(3) [1 point] Turtles can fly if and only if eagles can fly .... **True** **False**

PROBLEMS SOLUTIONS

Problem 2.1. Determine whether the following statements are valid or invalid.

(1) [1 point] The following argument is .... **Valid** **Invalid**

If Alice has a laptop, then Alice knows logic
Alice does not know logic
$\therefore$ Alice does not have a laptop

(2) [1 point] The following argument is .... **Valid** **Invalid**

If Alice has a laptop, then Alice knows logic
Alice knows logic
$\therefore$ Alice has a laptop

(3) [1 point] The following argument is .... **Valid** **Invalid**

If Alice has a laptop, then Alice knows logic
Alice does not have a laptop
$\therefore$ Alice does not know logic

(4) [1 point] The following argument is .... **Valid** **Invalid**

If Alice has a laptop, then Alice knows logic
Alice has a laptop
$\therefore$ Alice knows logic

Problem 2.2. For each of statement, determine true or false:

(1) [1 point] $\forall x \in \mathbb{R}$, if $x < 3$ then $x^2 < 9$ .... **True** **False**

(2) [1 point] $\forall x \in \mathbb{R}$, if $x \geqslant 3$ then $x^2 \geqslant 9$ .... **True** **False**

(3) [1 point] $\exists x \in \mathbb{R}$ such that $x^2 = 7$ .... **True** **False**

(4) [1 point] $\exists x \in \mathbb{Z}$ such that $x^2 = 7$ .... **True** **False**

(5) [1 point] $\forall x \in \mathbb{R}$, if $x \geqslant -3$ then $x^2 \geqslant 9$ .... **True** **False**

(6) [1 point] $\forall x \in \mathbb{Z}$, if $x < -3$ then $x^2 \geqslant 16$ .... **True** **False**

PROBLEMS SOLUTIONS

Problem 3.1. For each of the statements, determine true or false:

(1) [1 point] $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{Z}$ such that $x + y = 5$ .... **True** **False**

(2) [1 point] $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{R}$ such that $x + y = 4$ .... **True** **False**

(3) [1 point] $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x + y = 7$ .... **True** **False**

(4) [1 point] $\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $x + y = 6$ .... **True** **False**

(5) [1 point] $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, if $x \neq y$ then $x^2 \neq y^2$ .... **True** **False**

(6) [1 point] $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, if $x = -y$ then $x^2 = y^2$ .... **True** **False**

Problem 3.2. For each statement (regardless of its truth value), re-write its negation. *[You are not supposed to add $\sim$ or the phrase "it is not the case that" before the given statements to form their negations. Instead, try to paraphrase their negations.]*

(1) [1 point] There exists a person in Atlanta whose height is $\leqslant 8$ feet.
(2) [1 point] All people in Atlanta have height $\leqslant 8$ feet.
(3) [1 point] $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x + y = 3$.
(4) [1 point] $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, if $x < y$ then $x^2 < y^2$.

PROBLEMS SOLUTIONS

Truth table, conjunction, disjunction, negation, tautology, contradiction, (bi)conditional, etc.: Problems 1.1, 1.2.

Valid arguments, invalid arguments: Problem 2.1.

Statements with quantifiers ($\forall$, $\exists$), their negations: Problems 2.2, 3.1, 3.2.

Lecture notes and textbooks: All we have covered in class.

Problem. Determine the truth values of the following statements:

(1) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x - y^2 = 13$ .... **True** **False**

(2) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y^2 > 135$ .... **True** **False**

Problem. Let p, q and r be statement variables. Determine valid or invalid:

(1) $\begin{cases} p \to (q \vee r) \\ p \wedge (\sim q) \\ \therefore r \end{cases}$ ..... **Valid** **Invalid** (2) $\begin{cases} p \to q \\ p \vee r \\ \therefore q \end{cases}$ .... **Valid** **Invalid**

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply rote-memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

Problems

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PROBLEMS SOLUTIONS

Problem 4.1. Prove that, for all m, $n \in \mathbb{Z}$, if both m and n are odd then $m + n$ is even. (That is, prove that the sum of any two odd integers is even.)

Hint. Start with "Let m, $n \in \mathbb{Z}$; assume that both m and n are odd." From the assumption that both m and n are odd, what can we say about m and n? Study $m+n$, and try to show that $m + n$ is even. Present your proof carefully.

PROBLEMS SOLUTIONS

Problem 5.1 (5 points)**.** Prove that, for all a, b, $c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid (b + 3c)$.

Hint. Start the proof with "Let a, b, $c \in \mathbb{Z}$. Assume $a \mid b$ and $a \mid c$." Based on the assumption/definition, what can we say about b and c? Study $b + 3c$ and (try to) conclude that $a \mid (b + 3c)$.

Problem 5.2 (5 points)**.** Complete each of the following questions.

(1) Write down the standard factored form of 450.
(2) For $n = 97$ and $d = 13$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.
(3) For $n = -57$ and $d = 23$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.
(4) Compute $\left\lfloor \dfrac{33}{4} \right\rfloor$.
(5) Compute $\left\lceil -\dfrac{43}{6} \right\rceil$.

PROBLEMS SOLUTIONS

Problem 5.1 (5 points). Prove that, for all p, m, $n \in \mathbb{Z}$, if $p \mid m$ and $p \mid n$ then $p \mid (2m-n)$.

Hint. Start the proof with "Let p, m, $n \in \mathbb{Z}$. Assume $p \mid m$ and $p \mid n$." Based on the assumption/definition, what can we say about m and n? Study $2m-n$ and (try to) conclude that $p \mid (2m-n)$.

Problem 5.2 (5 points). Complete each of the following questions.

(1) Write down the standard factored form of 540.

(2) For $n = -97$ and $d = 13$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.

(3) For $n = 57$ and $d = 23$, write down the equation $n = dq + r$ such that q, $r \in \mathbb{Z}$ and $0 \leqslant r < d$.

(4) Compute $\left\lfloor -\frac{33}{4} \right\rfloor$.

(5) Compute $\left\lceil \frac{43}{6} \right\rceil$.

PROBLEMS SOLUTIONS

Problem 6.1 (5 points)**.** Complete the following by filling in the blanks.

(1) Write down the standard factored form: $5500 =$ ________________.

(2) For $n = 187$ and $d = 13$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$: $187 = 13(\_\_\_\_\_\_) + \_\_\_\_\_\_$.

(3) For $n = -179$ and $d = 24$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$: $-179 = 24(\_\_\_\_\_\_) + \_\_\_\_\_\_$.

(4) $\left\lfloor \dfrac{93}{8} \right\rfloor = \_\_\_\_\_\_$.

(5) $\left\lceil -\dfrac{93}{7} \right\rceil = \_\_\_\_\_\_$.

Problem 6.2 (5 points)**.** Complete the following (true or false, or filling in the blanks, where m, n, p are integers whose specific values not not known):

(1) $-44 \equiv 66 \pmod{10}$ .... **True** **False**

(2) $-44 \equiv -66 \pmod{13}$ .... **True** **False**

(3) Suppose $m \bmod 13 = 7$ and $n \bmod 13 = 8$. Then $mn \bmod 13 =$ ______.

(4) Suppose $m \bmod 13 = 7$ and $n \bmod 13 = 8$. Then $n^2 + m \bmod 13 =$ ______.

(5) Suppose $p \bmod 13 = 3$. Then $p^4 \bmod 13 =$ ______.

PROBLEMS SOLUTIONS

Problem 6.1 (5 points)**.** Complete the following by filling in the blanks.

(1) Write down the standard factored form: $4400 =$ __________________.

(2) For $n = -187$ and $d = 13$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$:
$-187 = 13(\_\_\_\_\_\_) + \_\_\_\_\_\_$.

(3) For $n = 179$ and $d = 24$, write $n = dq + r$ such that $q, r \in \mathbb{Z}$ and $0 \leqslant r < d$:
$179 = 24(\_\_\_\_\_\_) + \_\_\_\_\_\_$.

(4) $\left\lfloor -\frac{93}{8} \right\rfloor = \_\_\_\_\_\_$.

(5) $\left\lceil \frac{93}{7} \right\rceil = \_\_\_\_\_\_$.

Problem 6.2 (5 points)**.** Complete the following (true or false, or filling in the blanks, where m, n, p are integers whose specific values not not known):

(1) $-44 \equiv -66 \pmod{10}$ .... **True** **False**

(2) $-44 \equiv 66 \pmod{13}$ .... **True** **False**

(3) Suppose $m \bmod 13 = 6$ and $n \bmod 13 = 9$. Then $mn \bmod 13 = \_\_\_\_\_\_$.

(4) Suppose $m \bmod 13 = 6$ and $n \bmod 13 = 9$. Then $n^2 + m \bmod 13 = \_\_\_\_\_\_$.

(5) Suppose $p \bmod 13 = 2$. Then $p^6 \bmod 13 = \_\_\_\_\_\_$.

PROBLEMS SOLUTIONS

Materials covered earlier: Midterm I; Quizzes 1, 2, 3.

Proofs (concerning even/odd numbers, divisibility, etc.): Problems 4.1, 5.1.

Standard factored form, quotient-remainder, floor/ceiling, etc.: Problems 5.2, 6.1.

Modular congruence, etc.: Problems 6.2.

Lecture notes and textbooks: All we have covered.

Problem. Prove that, for all m, $n \in \mathbb{Z}$, if $m \bmod 6 = 3$ and $n \bmod 6 = 5$ then mn is odd.

Problem. Prove that, for all d, m, $n \in \mathbb{Z}$, if $d \mid m$ and $d \mid n$ then $d \mid (5m + 8n)$.

Problem. Prove that, for all d, $m \in \mathbb{Z}$, if $d \mid m$ then $d^2 \mid m^2$.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply rote-memorizing the solutions of the problems may not help you as much.

You are strongly encouraged to practice more problems (than the ones listed above) on your own.

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PROBLEMS SOLUTIONS

Problem 7.1. Prove by **mathematical induction** that $\sum_{i=4}^{n}(2i+1) = n^2 + 2n - 15$ for all integers $n \geqslant 4$ (that is, $(2(4)+1) + \cdots + (2n+1) = n^2 + 2n - 15$ for all integers $n \geqslant 4$).

Hint. You need to complete both the **basis step** $n = 4$ and the **inductive step**. Show your step-by-step work.

PROBLEMS SOLUTIONS

Problem 7.1. Prove by **mathematical induction** that $\sum_{i=3}^{n}(2i+1) = n^2 + 2n - 8$ for all integers $n \geqslant 3$ (that is, $(2(3)+1) + \cdots + (2n+1) = n^2 + 2n - 8$ for all integers $n \geqslant 3$).

Hint. You need to complete both the **basis step** $n = 3$ and the **inductive step**. Show your step-by-step work.

PROBLEMS SOLUTIONS

Problem 8.1 (5 points)**.** Let $A = (-5, 1]$, $B = [-3, 4)$ and $C = [-2, 6]$, all being intervals of $\mathbb{R}$. Compute the following:

(1) $(A \cup B) - C$.
(2) $A \cup (B - C)$.
(3) $A - (B \cap C)$.
(4) $(A - B) \cap C$.
(5) $(A \cap B) \cup C$.

Hint. Be careful with the computations. It suffices to provide answers, i.e., no need to justify. Label your answers clearly. Use $\varnothing$ to denote the empty set, if needed.

Problem 8.2 (5 points)**.** Consider function $f\colon \mathbb{R} \to \mathbb{R}$ that is defined as follows:

$$f(x) = x^2 + 9 \qquad \text{for all } x \in \mathbb{R}.$$

(1) Find the image of 5 under f.
(2) Find the set of *all* pre-images of 7 under f. If the answer is the empty set, state so.
(3) Find the set of *all* pre-images of 13 under f. If the answer is the empty set, state so.
(4) Determine whether f is a one-to-one (i.e., injective) function.
(5) Determine whether f is an onto (i.e., surjective) function.

Hint. It suffices to provide answers, i.e., no need to justify. In (2) and (3), your answers should be sets. Label your answers clearly. Use $\varnothing$ to denote the empty set, if needed.

PROBLEMS SOLUTIONS

Problem 8.1 (5 points)**.** Let $A = (-4, 3]$, $B = [-2, 5)$ and $C = [-1, 8]$, all being intervals of $\mathbb{R}$. Compute the following:

(1) $A \cup (B - C)$.
(2) $(A \cup B) - C$.
(3) $(A - B) \cap C$.
(4) $A - (B \cap C)$.
(5) $A \cap (B \cup C)$.

Hint. Be careful with the computations. It suffices to provide answers, i.e., no need to justify. Label your answers clearly. Use $\varnothing$ to denote the empty set, if needed.

Problem 8.2 (5 points)**.** Consider function $f : \mathbb{R} \to \mathbb{R}$ that is defined as follows:

$$f(x) = x^2 + 4 \qquad \text{for all } x \in \mathbb{R}.$$

(1) Find the image of 6 under f.
(2) Find the set of *all* pre-images of 3 under f. If the answer is the empty set, state so.
(3) Find the set of *all* pre-images of 13 under f. If the answer is the empty set, state so.
(4) Determine whether f is a one-to-one (i.e., injective) function.
(5) Determine whether f is an onto (i.e., surjective) function.

Hint. It suffices to provide answers, i.e., no need to justify. In (2) and (3), your answers should be sets. Label your answers clearly. Use $\varnothing$ to denote the empty set, if needed.

PROBLEMS SOLUTIONS

Problem 9.1 (5 points)**.** Let $S = \{100, 101, \ldots, 999\}$, the set of all 3-digit integers, from 100 to 999 inclusive. Complete the following:

(1) Compute $N(S)$, the total number of objects in the set S.
(2) How many 3-digit integers are there from 100 to 799 inclusive?
(3) How many 3-digit integers are there from 100 to 799 inclusive without repeated digits?
(4) How many 3-digit integers are there from 100 to 799 inclusive with repeated digits?
(5) What is the probability that a randomly number chosen from S is from 100 to 799 inclusive without repeated digits?

Hint. Use multiplication rule, the difference rule, etc..

Problem 9.2 (5 points)**.** A certain personal identification number (PIN) is required to be a sequence of any three (3) symbols chosen from the first six (6) uppercase letters A, B, C, D, E, F. Complete the following:

(1) How many different PINs are possible if repetition of symbols is allowed?
(2) How many different PINs are possible if repetition of symbols is not allowed?
(3) How many different PINs are possible that contain at least one repeated symbol?
(4) How many different PINs are possible that do not contain the symbol C if repetition of symbols is not allowed?
(5) How many different PINs are possible that contain the symbol C if repetition of symbols is not allowed?

Hint. The multiplication rule, the difference rule, the formula of r-permutations $P(n, r)$, etc..

PROBLEMS SOLUTIONS

Problem 9.1 (5 points)**.** Let $S = \{100, 101, \ldots, 999\}$, the set of all 3-digit integers, from 100 to 999 inclusive. Complete the following:

(1) Compute $N(S)$, the total number of objects in the set S.
(2) How many 3-digit integers are there from 300 to 899 inclusive?
(3) How many 3-digit integers are there from 300 to 899 inclusive without repeated digits?
(4) How many 3-digit integers are there from 300 to 899 inclusive with repeated digits?
(5) What is the probability that a randomly number chosen from S is from 300 to 899 inclusive without repeated digits?

Hint. Use multiplication rule, the difference rule, etc..

Problem 9.2 (5 points)**.** A certain personal identification number (PIN) is required to be a sequence of any three (3) symbols chosen from the first seven (7) uppercase letters A, B, C, D, E, F, G. Complete the following:

(1) How many different PINs are possible if repetition of symbols is allowed?
(2) How many different PINs are possible if repetition of symbols is not allowed?
(3) How many different PINs are possible that contain at least one repeated symbol?
(4) How many different PINs are possible that do not contain the symbol E if repetition of symbols is not allowed?
(5) How many different PINs are possible that contain the symbol E if repetition of symbols is not allowed?

Hint. The multiplication rule, the difference rule, the formula of r-permutations $P(n, r)$, etc..

PROBLEMS SOLUTIONS

Materials covered earlier: Midterm I; Midterm II; Quizzes 1, 2, 3, 4, 5, 6, 7, 8, 9.

Mathematical induction: Problem 7.1.

Set theory: Problem 8.1.

Functions: Problem 8.2.

Probabilities: Problem 9.1.

Lecture notes and textbooks: All we have covered.

Problem. Prove by **mathematical induction** that $\sum_{i=3}^{n}(2i-1) = n^2 - 4$ for all integers $n \geqslant 3$ (that is, $(2(3)-1) + \cdots + (2n-1) = n^2 - 4$ for all integers $n \geqslant 3$).

Hint. You need to complete both the **basis step (for $n = 3$)** and the **inductive step**.

Problem. Re-attempt Problems 7.1, 8.1, 8.2, 9.1 and 9.2.

Note: The above list is not intended to be complete. The problems in the actual test may vary in difficulty as well as in content. Going over, understanding, and digesting the problems listed above will definitely help. However, simply rote-memorizing the solutions of the problems may not help you as much.

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