Problem 1. Let R be a Noetherian ring of characteristic p, I an ideal, and M an R-module.

(1) For any given $e \in \mathbb{N}$, show that $\mathrm{H}^i_I({}^eM) \cong {}^e(\mathrm{H}^i_I(M))$ for every i.

(2) Suppose that ${}^{e_0}M$ is finitely generated over R for some $e_0 \geq 1$. Show that $R/\operatorname{Ann}_R(M)$ is an F-finite ring. Consequently, ${}^eM$ is finitely generated over R for every e.

Proof. (1). Given any ring homomorphism $\phi : S \to T$ of Noetherian rings, any T-module M (which is naturally an S-module), and any ideal I of S, we always have $\mathrm{H}^i_I(M) \cong \mathrm{H}^i_{\phi(I)T}(M)$ as S-modules (and as T-modules as well) for all i. To prove (1), apply the above with $S = R$, $T = R$, and ϕ being the Frobenius ring homomorphism $F^e : S \to T$, which gives $\mathrm{H}^i_I({}^eM) \cong {}^e(\mathrm{H}^i_{I^{[q]}}(M))$ for every i. Now notice that $\mathrm{H}^i_{I^{[q]}}(M)) \cong \mathrm{H}^i_I(M))$ for every i.

(2). Since $e_0 \geq 1$, we have that both $M = {}^0M$ and ${}^1M$ are finitely generated over R. Say M is generated by $x_1, \ldots, x_n$. Then there is an injective R-linear map $R/\operatorname{Ann}_R(M) \to M^n$ sending the class of 1 to $(x_1, \ldots, x_n) \in M^n$. This induces an injective R-linear map ${}^1(R/\operatorname{Ann}_R(M)) \to {}^1(M^n) \cong ({}^1M)^n$, which forces ${}^1(R/\operatorname{Ann}_R(M))$ to be finitely generated over R, i.e. $R/\operatorname{Ann}_R(M)$ is an F-finite ring. Consequently, ${}^e(R/\operatorname{Ann}_R(M))$ is finitely generated over R for every e. Also notice that there is a surjective R-linear map $(R/\operatorname{Ann}_R(M))^n \to M$, which induces a surjective R-linear map $({}^e(R/\operatorname{Ann}_R(M)))^n \to {}^eM$ for every e. Thus ${}^eM$ is finitely generated over R for every e. ☐

Problem 2. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p and $P \in \operatorname{Spec}(R)$ be any prime ideal of R. Suppose R is F-finite and say $[k : k^p] = p^a$.

(1) Prove that $\dim(\widehat{R}/Q) = \dim(R/P)$ for every $Q \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/P\widehat{R}) = \min_{\widehat{R}}(\widehat{R}/P\widehat{R})$.

(2) Show that $[(R/P)_P : ((R/P)_P)^p] = p^{a+\dim(R/P)}$. (We have proved this when R is complete.)

Proof. Without loss of generality, we assume R is a domain and $P = 0$. Then, as proved in class, $\widehat{R}$ is reduced so that $\operatorname{Ass}_{\widehat{R}}(\widehat{R}) = \min_{\widehat{R}}(\widehat{R})$ and $\widehat{R}_Q$ is the fraction field of $\widehat{R}/Q$ for every $Q \in \min_{\widehat{R}}(\widehat{R})$. By going-down, $Q \cap R = P = 0$ for every $Q \in \min_{\widehat{R}}(\widehat{R})$. Thus there is a natural isomorphism $\widehat{R}_Q \cong R_P \otimes_{R_P} \widehat{R}_Q$ for every $Q \in \min_{\widehat{R}}(\widehat{R})$. We have shown in class that $\widehat{R} \otimes_R {}^1R \cong {}^1(\widehat{R})$, which is the same as that $R^{1/p} \otimes_R \widehat{R} \cong (\widehat{R})^{1/p}$. Say $[R_P : (R_P)^p] = n$, i.e. $(R_P)^{1/p} = (R^{1/p})_P \cong (R_P)^n$. Then $(\widehat{R}_Q)^{1/p} \cong ((\widehat{R})^{1/p})_Q \cong (R^{1/p} \otimes_R \widehat{R})_Q \cong R^{1/p} \otimes_R \widehat{R}_Q \cong (R^{1/p})_P \otimes_{R_P} \widehat{R}_Q \cong (R_P)^n \otimes_{R_P} \widehat{R}_Q \cong (R_P \otimes_{R_P} \widehat{R}_Q)^n \cong (\widehat{R}_Q)^n$ for any $Q \in \min_{\widehat{R}}(\widehat{R})$. In other words, $[(\widehat{R}_Q)^{1/p} : \widehat{R}_Q] = n = [R_P : (R_P)^p]$ for *all* $Q \in \min_{\widehat{R}}(\widehat{R})$.

We have shown in class that $[(\widehat{R}_Q)^{1/p} : \widehat{R}_Q] = p^{a+\dim(\widehat{R}/Q)}$ for any $Q \in \operatorname{Spec}(\widehat{R})$. Thus $\dim(\widehat{R}/Q)$ is constant for all $Q \in \min_{\widehat{R}}(\widehat{R})$. Since $\dim(\widehat{R}/Q) = \dim(\widehat{R}) = \dim(R/P)$ for some $Q \in \min_{\widehat{R}}(\widehat{R})$, we conclude that (1) $\dim(\widehat{R}/Q) = \dim(R/P)$ for all $Q \in \min_{\widehat{R}}(\widehat{R})$ and (2) $[R_P : (R_P)^p] = p^{a+\dim(R)}$. ☐

Problem 3. Let R be a Noetherian ring of prime characteristic p, M a finitely generated R-module such that $\operatorname{Ann}_R(M) \subseteq \sqrt{0}$. For any R-modules $N \subseteq L$ and $x \in L$, prove $x \in N^*_L \iff$ there exists $c \in R^\circ$ such that $\operatorname{Image}(x \otimes_R {}^e(cM) \to L \otimes_R {}^eM) \subseteq \operatorname{Image}(N \otimes_R {}^eM \to L \otimes_R {}^eM)$ for all $e \gg 0$.

Proof. The direction '$\Rightarrow$' is straightforward. To show the implication '$\Leftarrow$', it suffices to prove it for $(N + \sqrt{0}M)/\sqrt{0}M \subseteq M/\sqrt{0}M \ni x + \sqrt{0}M$ over the reduced ring $R/\sqrt{0}M$. In other words, we may assume R is reduced (hence $\operatorname{Ann}_R(M) = 0$) without loss of generality. Let $W = R^\circ$. Then there exists a surjective $(W^{-1}R)$-map $W^{-1}M \to W^{-1}R$, which implies the existence of an R-linear map $\phi : M \to R$ such that $\phi(M) \cap R^\circ \neq \emptyset$. Say $c' \in \phi(M) \cap R^\circ$. Observe that $\phi \in \operatorname{Hom}_R(M, R)$ implies that $\phi \in \operatorname{Hom}_R({}^eM, {}^eR)$ for every e. Then the assumption that $\operatorname{Image}(x \otimes_R {}^e(cM) \to L \otimes_R {}^eM) \subseteq \operatorname{Image}(N \otimes_R {}^eM \to L \otimes_R {}^eM)$ for all $e \gg 0$ would imply that $x \otimes_R (cc') \in \operatorname{Image}(N \otimes_R {}^eR \to L \otimes_R {}^eR) = N^{[q]}_L$ for all $e \gg 0$, which shows that $x \in N^*_L$ as $cc' \in R^\circ$. ☐

Problem 4. Let R be a Noetherian F-finite ring of prime characteristic p, M a finitely generated R-module with FFRT by finitely generated R-modules $M_1, M_2, \ldots, M_r$, and L a finitely generated

R-module. Show that $\cup_{e\in\mathbb{N}} \operatorname{Ass}(L\otimes_R {}^e\!M)$ is a finite set and, moreover, there exists an integer $k\in\mathbb{N}$ such that the following are satisfied.

(1) For every $e\in\mathbb{N}$, there exists a primary decomposition

$$0 = Q_{e1}\cap Q_{e2}\cap\cdots\cap Q_{es_e} \quad \text{of } \ 0 \text{ in } L\otimes_R {}^e\!M,$$

where $\operatorname{Ass}(L\otimes {}^e\!M) = \{P_{ej} \,|\, 1\le j\le s_e\}$ and Q_{ej} are P_{ej}-primary components of $0\subseteq L\otimes_R {}^e\!M$ satisfying $P_{ej}^k(L\otimes_R {}^e\!M)\subseteq Q_{ej}$ for all $1\le j\le s_e$.

(2) We have $J^k(0:_{L\otimes_R {}^e\!M} J^\infty)=0$, i.e., $J^k\,\mathrm{H}^0_J(L\otimes_R {}^e\!M)=0$ for all $J\subseteq R$ and for all $e\in\mathbb{N}$.

(In case $L=R/I$, the above may be stated in terms of $\cup_{e\in\mathbb{N}}\operatorname{Ass}(M/I^{[q]}M)$ and $\mathrm{H}^0_J(M/I^{[q]}M)$.)

Proof. For each $i=1,2,\ldots,r$, choose a primary decomposition of 0 in $L\otimes_R M_i$ (ignore the possible cases of i where $L\otimes_R M_i=0$) as follows

$$0 = Q'_{i1}\cap Q'_{i2}\cap\cdots\cap Q'_{it_i},$$

where Q'_{ij} are P'_{ij}-primary components of $0\subset L\otimes_R M_i$ for $1\le i\le r, 1\le j\le t_i$. Since ${}^e\!M$ is a direct sum of the M_i (implying that $L\otimes_R {}^e\!M$ is a sum of the $L\otimes_R M_i$), we naturally get an induced primary decomposition $0=Q_{e1}\cap Q_{e2}\cap\cdots\cap Q_{es_e}$ of 0 in $L\otimes_R {}^e\!M$ for every e. Therefore $\cup_{e\in\mathbb{N}}\operatorname{Ass}_R(L\otimes_R {}^e\!M)$ is finite as it is contained in $\cup_{i=1}^r \operatorname{Ass}_R(L\otimes_R M_i)$. Choose $k\in\mathbb{N}$ such that ${P'_{ij}}^k(L\otimes_R M_i)\subseteq Q'_{ij}$ for all $i=1,2,\ldots,r$ and all $j=1,2,\ldots,t_i$. Then (1) is evidently true.

To see (2), recall that, for any ideal J, $(0:_{L\otimes_R {}^e\!M} J^\infty) = \cap_{J\not\subseteq P_{ej}} Q_{ej}$. Thus, for any ideal J and any $e\in\mathbb{N}$, we have $J^k(0:_{L\otimes_R {}^e\!M} J^\infty)\subseteq \big(\cap_{J\not\subseteq P_{ej}} Q_{ej}\big)\cap\big(\cap_{J\subseteq P_{ej}} Q_{ej}\big)=0$.

(In case $L=R/I$, the result implies that $\cup_{e\in\mathbb{N}}\operatorname{Ass}(M/I^{[q]}M)$ is finite (cf. Homework Set #3 Problem 2(2)) and, for any ideal J, $J^{(k+\mu(J))q}\,\mathrm{H}^0_J(\frac{M}{I^{[q]}M})\subseteq (J^k)^{[q]}\,\mathrm{H}^0_J(M/I^{[q]}M)=0$, where $\mu(J)$ is the least number of generators of the ideal J.) $\square$

Problem 5. Let $R\to S$ be a homomorphism of Noetherian rings of prime characteristic p and M a finitely generated R-module. (In this problem, we treat ${}^e\!M$ as an R-R-bimodule where $r_1\cdot x\cdot r_2 = r_1^q r_2 x$ for any $r_1,r_2\in R, x\in M$. Also recall that $\#_R({}^e\!M) = \ell^r_R(\operatorname{Image}(k\otimes_R {}^e\!M \xrightarrow{\psi\otimes 1} E_R(k)\otimes_R {}^e\!M))$ where $\psi: k\to E_R(k)$ is any injective R-map and $\ell^r_R(-)$ denotes length as a right R-module.)

(1) For any R-module E, there is an isomorphism $(E\otimes {}^e\!M)\otimes_R S\cong (E\otimes_R S)\otimes_S {}^e(M\otimes_R S)$. Moreover, the isomorphism is natural in the sense that for any R-linear map $E_1\to E_2$, the following diagram commutes:

$$\begin{array}{ccc}
(E_1\otimes {}^e\!M)\otimes_R S & \xrightarrow{\cong} & (E_1\otimes_R S)\otimes_S {}^e(M\otimes_R S)\\
\downarrow & & \downarrow\\
(E_2\otimes {}^e\!M)\otimes_R S & \xrightarrow{\cong} & (E_2\otimes_R S)\otimes_S {}^e(M\otimes_R S)
\end{array}$$

(2) Assume, furthermore, that $(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ is a flat homomorphism of local rings such that $\mathfrak{m}S=\mathfrak{n}$. Show that (a) $E_R(k)\otimes S\cong E_S(l)$ and (b) $\#_R({}^e\!M)=\#_S({}^e(M\otimes_R S))$ for all e.

Proof. (1). For every $e\in\mathbb{N}$, we have a series of natural isomorphisms

$$(E\otimes_R {}^e\!M)\otimes_R S\cong E\otimes_R {}^e(M\otimes_R S)\cong E\otimes_R(S\otimes_S {}^e(M\otimes_R S))\cong (E\otimes_R S)\otimes_S {}^e(M\otimes_R S).$$

And it is routine to verify that the diagram commutes.

(2). Since every element of $E_R(k)$ is killed by a power of $\mathfrak{m}$, every element of $E_R(k)\otimes S$ is killed by a power of $\mathfrak{m}S=\mathfrak{n}$. Thus $E_R(k)\otimes S$ is an essential extension of its socle $(0:_{E_R(k)\otimes S}\mathfrak{n})$ while

$$(0:_{E_R(k)\otimes S}\mathfrak{n}) = (0:_{E_R(k)\otimes S}\mathfrak{m}S)\cong(0:_{E_R(k)}\mathfrak{m})\otimes_S S\cong (R/\mathfrak{m})\otimes_S S\cong S/\mathfrak{m}S = S/\mathfrak{n} = l.$$

This implies that $E_R(k)\otimes S$ is isomorphic to an S-submodule of $E_S(l)$. To prove (a), it will then suffice to show $\operatorname{Ann}_S(E_R(k)\otimes_R S)=0$. For any $i\in\mathbb{N}$, let $E_i=(0:_{E_R(k)}\mathfrak{m}^i)$, which is a finitely generated R-module with $\operatorname{Ann}_R(E_i)=\mathfrak{m}^i$. Then $\operatorname{Ann}_S(E_R(k)\otimes_R S)\subseteq \operatorname{Ann}_S(E_i\otimes_R S) = \operatorname{Ann}_R(E_i)S=\mathfrak{m}^iS=\mathfrak{n}^i$ for all $i\in\mathbb{N}$, which forces $\operatorname{Ann}_S(E_R(k)\otimes_R S)=0$. Finally, (b) follows from (a) and (1) immediately: Simply apply the commutative diagram in (1) with $E_1=k$, $E_2=E_R(k)$, and $\psi: E_1=k\to E_R(k)=E_2$ being the injective R-linear map as in the definition of $\#({}^e\!M)$. $\square$