Math 615 (Winter 2005) **Homework Set #4** **Due: 04/18/05 (Mon.)**

Problem 1. Let R be a Noetherian ring of characteristic p, I an ideal, and M an R-module.

(1) For any given $e \in \mathbb{N}$, show that $\mathrm{H}^i_I({}^{e}M) \cong {}^{e}(\mathrm{H}^i_I(M))$ for every i.

(2) Suppose that ${}^{e_0}M$ is finitely generated over R for some $e_0 \geq 1$. Show that $R/\operatorname{Ann}_R(M)$ is an F-finite ring. Consequently, ${}^{e}M$ is finitely generated over R for every e.

Problem 2. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p and $P \in \operatorname{Spec}(R)$ be any prime ideal of R. Suppose R is F-finite and say $[k : k^p] = p^a$.

(1) Prove that $\dim(\widehat{R}/Q) = \dim(R/P)$ for every $Q \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/P\widehat{R}) = \min_{\widehat{R}}(\widehat{R}/P\widehat{R})$.

(2) Show that $[(R/P)_P : ((R/P)_P)^p] = p^{a+\dim(R/P)}$. (We have proved this when R is complete.)

Problem 3. Let R be a Noetherian ring of prime characteristic p, M a finitely generated R-module such that $\operatorname{Ann}_R(M) \subseteq \sqrt{0}$. For any R-modules $N \subseteq L$ and $x \in L$, prove $x \in N^*_L \iff$ there exists $c \in R^\circ$ such that $\operatorname{Image}(x \otimes_R {}^{e}(cM) \to L \otimes_R {}^{e}M) \subseteq \operatorname{Image}(N \otimes_R {}^{e}M \to L \otimes_R {}^{e}M)$ for all $e \gg 0$.

Problem 4. Let R be a Noetherian F-finite ring of prime characteristic p, M a finitely generated R-module with FFRT by finitely generated R-modules $M_1, M_2, \ldots, M_r$, and L a finitely generated R-module. Show that $\cup_{e\in\mathbb{N}} \operatorname{Ass}(L \otimes_R {}^{e}M)$ is a finite set and, moreover, there exists an integer $k \in \mathbb{N}$ such that the following are satisfied.

(1) For every $e \in \mathbb{N}$, there exists a primary decomposition

$$0 = Q_{e1} \cap Q_{e2} \cap \cdots \cap Q_{es_e} \quad \text{of } \ 0 \text{ in } L \otimes_R {}^{e}M,$$

where $\operatorname{Ass}(L \otimes {}^{e}M) = \{P_{ej} \mid 1 \leq j \leq s_e\}$ and Q_{ej} are P_{ej}-primary components of $0 \subseteq L \otimes_R {}^{e}M$ satisfying $P^k_{ej}(L \otimes_R {}^{e}M) \subseteq Q_{ej}$ for all $1 \leq j \leq s_e$.

(2) We have $J^k(0 :_{L\otimes_R {}^{e}M} J^\infty) = 0$, i.e., $J^k\, \mathrm{H}^0_J(L \otimes_R {}^{e}M) = 0$ for all $J \subseteq R$ and for all $e \in \mathbb{N}$.

(In case $L = R/I$, the above may be stated in terms of $\cup_{e\in\mathbb{N}} \operatorname{Ass}(M/I^{[q]}M)$ and $\mathrm{H}^0_J(M/I^{[q]}M)$.)

Problem 5. Let $R \to S$ be a homomorphism of Noetherian rings of prime characteristic p and M a finitely generated R-module. (In this problem, we treat ${}^{e}M$ as an R-R-bimodule where $r_1 \cdot x \cdot r_2 = r_1^q r_2 x$ for any $r_1, r_2 \in R, x \in M$. Also recall that $\#_R({}^{e}M) = \ell^r_R(\operatorname{Image}(k \otimes_R {}^{e}M \xrightarrow{\psi\otimes 1} E_R(k) \otimes_R {}^{e}M))$ where $\psi : k \to E_R(k)$ is any injective R-map and $\ell^r_R(-)$ denotes length as a right R-module.)

(1) For any R-module E, there is an isomorphism $(E \otimes {}^{e}M) \otimes_R S \cong (E \otimes_R S) \otimes_S {}^{e}(M \otimes_R S)$. Moreover, the isomorphism is natural in the sense that for any R-linear map $E_1 \to E_2$, the following diagram commutes:

$$\begin{array}{ccc} (E_1 \otimes {}^{e}M) \otimes_R S & \xrightarrow{\cong} & (E_1 \otimes_R S) \otimes_S {}^{e}(M \otimes_R S) \\ \downarrow & & \downarrow \\ (E_2 \otimes {}^{e}M) \otimes_R S & \xrightarrow{\cong} & (E_2 \otimes_R S) \otimes_S {}^{e}(M \otimes_R S) \end{array}$$

(2) Assume, furthermore, that $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is a flat homomorphism of local rings such that $\mathfrak{m}S = \mathfrak{n}$. Show that (a) $E_R(k) \otimes S \cong E_S(l)$ and (b) $\#_R({}^{e}M) = \#_S({}^{e}(M \otimes_R S))$ for all e.