Problem 1. Let R be a Noetherian ring of prime characteristic p. Show that R has a weak test element if and only if $R/\sqrt{0}$ has a weak test element. (Here a weak test element is, by definition, a q-weak test element for some q.)

Proof. If $c \in R^\circ$ is a q_1-weak test element for R, then it is straightforward to verify that $c + \sqrt{0}$ is a q_1-weak test element for $R/\sqrt{0}$. Conversely, suppose $R/\sqrt{0}$ has a q_2-weak test element, say $d + \sqrt{0} \in (R/\sqrt{0})^\circ$ so that $d \in R^\circ$. Say $\sqrt{0}^{[q_3]} = 0$. Then direct checking shows that d^{q_3}, which is in $\in R^\circ$, is a $(q_2 q_3)$-weak test element for R. $\square$

Problem 2. Let R be a Noetherian ring of prime characteristic p and M an R-module. Recall that ${}^eM$ is the derived R-module structure on M via the Frobenius homomorphism $F^e : R \to R$.

(1) If ${}^{e_0}M$ is a faithful R-module for some $e_0 > 0$, then R is reduced and ${}^eM$ (including $M = {}^0M$) are faithful for all $e \in \mathbb{N}$.
(2) Show that $\operatorname{Ass}_R(M) = \operatorname{Ass}_R({}^eM)$ for every $e \in \mathbb{N}$.

Proof. (1). Denote $q_0 = p^{e_0}$, which is $\geq p$. Suppose R is not reduced. Then there exists $0 \neq x \in \sqrt{0}$ such that $x^{q_0} = 0$. Then we see that $x \in \operatorname{Ann}_R({}^{e_0}M)$, a contradiction. Now that R is reduced, the claim that ${}^eM$ is faithful for all e follows immediately from the easy assertion that, quite generally, $\operatorname{Ann}_R({}^{e_1}M) \subseteq \operatorname{Ann}_R({}^{e_2}M) \subseteq \sqrt{\operatorname{Ann}_R({}^{e_1}M)}$ for every $e_1 \leq e_2$ and any R-module M.

(2). Firstly, we observe an easy claim that $\operatorname{Ann}_R(x \in M) \subseteq \operatorname{Ann}_R(x \in {}^eM) \subseteq \sqrt{\operatorname{Ann}_R(x \in M)}$ for any $x \in M$ and any $e \in \mathbb{N}$. Then for any $P \in \operatorname{Ass}_R(M)$, there is $y \in M$ such that $\operatorname{Ann}_R(y \in M) = P$. Hence $\operatorname{Ann}_R(y \in {}^eM) = P$ and therefore $P \in \operatorname{Ass}_R({}^eM)$. Conversely, suppose $P \in \operatorname{Ass}_R({}^eM)$, i.e. there is z such that $\operatorname{Ann}_R(z \in {}^eM) = P$. Let Rz be the R-submodule of M generated by z. Then $P \in \min(R/\operatorname{Ann}_R(z \in M)) = \min(Rz) \subseteq \operatorname{Ass}_R(Rz) \subseteq \operatorname{Ass}_R(M)$. $\square$

Problem 3. Let $(R, \mathfrak{m}, k)$ be a Noetherian equidimensional catenary local ring of prime characteristic p with $\dim(R) = d$. Suppose $q^d < \ell_R(R/\mathfrak{m}^{[q]}) < q^d + q$ for some $q = p^e \geq p$. Prove $\operatorname{Sing}(R) = \{\mathfrak{m}\}$, where $\operatorname{Sing}(R) = \{P \in \operatorname{Spec}(R) \mid R_P \text{ is not regular}\}$ is the singular locus of R.

Proof. The assumption of $(R, \mathfrak{m}, k)$ being equidimensional catenary guarantees that $\dim(R/P) + \dim(R_P) = \dim(R)$ for every $P \in \operatorname{Spec}(R)$. And R is not regular as $q^d < \ell_R(R/\mathfrak{m}^{[q]})$.

If $\dim(R) = 0$, then there is nothing to prove. So we assume $\dim(R) \geq 1$ and it suffices to show R_P is regular for any prime ideal P such that $\dim(R/P) = 1$ (and hence $\dim(R_P) = d - 1$). For any such P, $\ell_{R_P}(R_P/P_P^{[q]}) \leq \frac{1}{q}\ell_R(R/\mathfrak{m}^{[q]}) < q^{d-1} + 1 = q^{\dim(R_P)} + 1$, which implies R_P is regular. $\square$

Problem 4. Let R be a ring (not necessarily of characteristic p). Given R-modules M, N and $f \in \operatorname{Hom}_R(M, N)$, we say f is pure if the induced map $f \otimes_R 1_L : M \otimes_R L \to N \otimes_R L$ is injective for every R-module L. (Denote by m-Spec(R) the set consisting of all maximal ideals of R.)

(1) If $f \in \operatorname{Hom}_R(M, N)$ is pure, then f is injective. (Therefore, $f \in \operatorname{Hom}_R(M, N)$ is pure if and only if f is injective and the inclusion map $f(M) \subseteq N$ is pure.)
(2) $f \in \operatorname{Hom}_R(M, N)$ is pure if and only if $f_P : M_P \to N_P$ is pure for every $P \in \operatorname{Spec}(R)$ if and only if $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is pure for every $\mathfrak{m} \in$ m-Spec(R).
(3) Show (A) $f \in \operatorname{Hom}_R(M, N)$ is pure if and only if $f \otimes_R 1_L : M \otimes_R L \to N \otimes_R L$ is injective for every finitely generated R-module L; and (B) If R is Noetherian and M is finitely generated, then $f \in \operatorname{Hom}_R(M, N)$ is pure if and only if $f \otimes_R 1_L : M \otimes_R L \to N \otimes_R L$ is injective for every finitely generated R-module L such that $\operatorname{Ass}_R(L) = \{\mathfrak{m}\}$ for some $\mathfrak{m} \in$ m-Spec(R).
(4) Suppose R is Noetherian and M, N are finitely generated R-modules. Then $f \in \operatorname{Hom}_R(M, N)$ is pure if and only if $\widehat{f_\mathfrak{m}} : \widehat{M_\mathfrak{m}} \to \widehat{N_\mathfrak{m}}$ is pure for every $\mathfrak{m} \in$ m-Spec(R) if and only if $f : M \to N$ splits (meaning there exists $g \in \operatorname{Hom}_R(N, M)$ such that $g \circ f = 1_M$.)
(5) Suppose R is Noetherian and F is a free R-module. Then $f \in \operatorname{Hom}_R(F, N)$ is pure if and only if the induced map $f \otimes_R 1_E : F \otimes_R E \to N \otimes_R E$ is injective, where $E = \oplus_{\mathfrak{m} \in \text{m-Spec}(R)} E_R(R/\mathfrak{m})$.

Proof. (1). This follows from the injectivity of $f \otimes_R 1_R : M \otimes_R R \to N \otimes_R R$.

(2). This is standard.

(3). In both (A) and (B), we only need to show 'if'. Suppose $f : M \to N$ is not pure. Then there exists an R-module L such that $f \otimes_R 1_L$ has a non-zero kernel. Then, by property of tensor product, there exists a (sufficiently large) finitely generated R-submodule $L' \subseteq L$ such that $f \otimes_R 1_{L'}$ is not injective, which proves (A). If, moreover, R is Noetherian and M is finitely generated, then $0 \neq \ker(f \otimes_R 1_{L'}) \subseteq M \otimes_R L'$ are all finitely generated R-modules. Choose $\mathfrak{m} \in \text{m-Spec}(R)$ such that $0 \neq \big(\ker(f \otimes_R 1_{L'})\big)_{\mathfrak{m}}$. Then, by Krull intersection theorem, Artin-Rees Lemma etc., $\ker(f \otimes_R 1_{L'/\mathfrak{m}^n L'}) \neq 0$ for some integer $n \gg 0$, which proves case (B) as $\mathrm{Ass}_R(L'/\mathfrak{m}^n L') = \{\mathfrak{m}\}$.

(4). Without loss of generality, we assume $(R, \mathfrak{m}, k)$ is local. Denote $-^\vee = \mathrm{Hom}_{\widehat{R}}(-, E_{\widehat{R}}(k))$. Then f is pure $\Longrightarrow f \otimes_R L$ is injective for all R-module L such that $\ell_R(L) < \infty \Longrightarrow \widehat{f} \otimes_{\widehat{R}} L$ is injective for all $\widehat{R}$-module L such that $\ell_{\widehat{R}}(L) < \infty \Longrightarrow \widehat{f}$ is pure $\Longrightarrow \widehat{f} \otimes_{\widehat{R}} \widehat{M}^\vee : \widehat{M} \otimes_{\widehat{R}} \widehat{M}^\vee \to \widehat{N} \otimes_{\widehat{R}} \widehat{M}^\vee$ is injective $\Longrightarrow \big(\widehat{f} \otimes_{\widehat{R}} \widehat{M}^\vee\big)^\vee : \big(\widehat{N} \otimes_{\widehat{R}} \widehat{M}^\vee\big)^\vee \to \big(\widehat{M} \otimes_{\widehat{R}} \widehat{M}^\vee\big)^\vee$ is surjective $\Longleftrightarrow \mathrm{Hom}_{\widehat{R}}(\widehat{f}, \widehat{M}) : \mathrm{Hom}_{\widehat{R}}(\widehat{N}, \widehat{M}) \to \mathrm{Hom}_{\widehat{R}}(\widehat{M}, \widehat{M})$ is surjective $\Longrightarrow \widehat{f}$ splits $\Longrightarrow f$ splits $\Longrightarrow f$ is pure.

(5). Without loss of generality, we assume $(R, \mathfrak{m}, k)$ is local. We only need to show 'if'. Suppose $f \in \mathrm{Hom}_R(F, N)$ is not pure. Then, as F is free (not necessarily of finite rank), an argument similar to the one in part (3) above shows $\ker(f \otimes_R 1_L) \neq 0$ for some L with $\ell(L) < \infty$. Then there exists an integer $n > 0$ such that L is embedded into E^n where $E = E_R(k)$. Then, as F is free, we have $\ker(f \otimes_R 1_{E^n}) \neq 0 \Longrightarrow \ker(f \otimes_R 1_E) \neq 0$, a contradiction. □

Problem 5. Given a local Noetherian ring $(R, \mathfrak{m}, k)$ of prime characteristic p (not necessarily F-finite), one could define R to be strongly F-regular if, for any $c \in R^\circ$, there exists an integer $e \geq 1$ such that the R-linear map $R \to {}^eR$ sending 1 to c is pure. In general, one could define R is strongly F-regular if $R_{\mathfrak{m}}$ is strongly F-regular for every $\mathfrak{m} \in \text{m-Spec}(R)$. (By Problem 4, we see that the above definition agrees with the one given in class when R is F-finite.)

1. If there exists a pure R-linear map $R \to {}^eR$ sending 1 to c with $e \geq 1$, then R is reduced and, for every $e' \geq e$, the R-linear map $R \to {}^{e'}R$ sending 1 to c is pure. (Thus the above definition of strong F-regularity forces R to be reduced.)
2. Show that $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if $0^*_{E_R(k)} = 0$.

Proof. (1). The given pure map shows ${}^eR$ is faithful, implying R is reduced by Problem 2(1). So we are free to identify ${}^eR$ with $R^{1/q}$ as R-modules for any $q = p^e$. Thus the given pure map may be considered as $f : R \to R^{1/q}$ sending 1 to $c^{1/q}$. But $f = g \circ i : R \overset{i}{\subseteq} R^{1/p} \xrightarrow{g} R^{1/q}$ in which g is the $R^{1/p}$-linear map sending 1 to $c^{1/q}$. Thus the purity of f forces the purity of inclusion map i (which is easy to check). Also the purity of f amounts to the purity of the $R^{1/p}$-linear map $f' : R^{1/p} \to R^{1/qp}$ sending 1 to $c^{1/qp}$, which readily implies the purity of f' as an R-linear map. Therefore the R-linear map $f' \circ i : R \to R^{1/qp}$ is pure (which is easy to check) and it sends 1 to $c^{1/qp}$. In other words, the R-linear map $R \to {}^{e+1}R$ sending 1 to c is pure. This in enough to prove (1).

(2). First of all, for any $c \in R$ and $e \in \mathbb{N}$, let us denote by $f_{c,e} : R \to {}^eR$ the R-linear map sending 1 to $c \in {}^eR$.

To show 'only if', suppose R is strongly F-regular. For any $x \in 0^*_{E_R(k)}$, by the definition of tight closure, there exists $c \in R^\circ$ such that $0 = c \otimes_R x \in {}^eR \otimes_R E_R(k)$ for all $e \gg 0$. Thus $1 \otimes_R x \in R \otimes_R E_R(k)$ is in $\ker(f_{c,q} \otimes_R 1_{R_R(k)})$ for all $e \gg 0$. But, by part (1) above, we know that $f_{c,q} : R \to {}^eR$ are pure for all $e \gg 0$, which forces $0 = 1 \otimes_R x \in R \otimes_R E_R(k)$, implying $x = 0$. So $0^*_{E_R(k)} = 0$.

Finally, let us prove 'if'. Choose $0 \neq w \in (0 :_{E_R(k)} \mathfrak{m})$ so that w generates the socle of $E_R(k)$. The assumption $0^*_{E_R(k)} = 0$ implies that $w \notin 0^*_{E_R(k)}$. Thus, for any $c \in R^\circ$, there exists an integer $e \geq 1$ such that $0 \neq c \otimes_R w \in {}^eR \otimes_R E_R(k)$, which means that $1 \otimes_R w \in R \otimes_R E_R(k)$ is not in $\ker(f_{c,q} \otimes_R 1_{E_R(k)})$, which implies that $\ker(f_{c,q} \otimes_R 1_{E_R(k)}) = 0$ as every non-zero R-submodule of $R \otimes_R E_R(k)$ contains $1 \otimes_R w$. Thus $f_{c,q} \otimes_R 1_{E_R(k)} : R \otimes_R 1_{E_R(k)} \to {}^eR \otimes_R 1_{E_R(k)}$ is injective and, therefore, $f_{c,q} : R \to {}^eR$ is pure by Problem 4(5). Hence R is strongly F-regular. □