
Problem 1. Let R be a Noetherian ring of prime characteristic p. Show that R has a weak test element if and only if $R/\sqrt{0}$ has a weak test element. (Here a weak test element is, by definition, a q-weak test element for some q.)

Problem 2. Let R be a Noetherian ring of prime characteristic p and M an R-module. Recall that ${}^eM$ is the derived R-module structure on M via the Frobenius homomorphism $F^e : R \to R$.

(1) If ${}^{e_0}M$ is a faithful R-module for some $e_0 > 0$, then R is reduced and ${}^eM$ (including $M = {}^0M$) are faithful for all $e \in \mathbb{N}$.

(2) Show that $\mathrm{Ass}_R(M) = \mathrm{Ass}_R({}^eM)$ for every $e \in \mathbb{N}$.

Problem 3. Let $(R, \mathfrak{m}, k)$ be a Noetherian equidimensional catenary local ring of prime characteristic p with $\dim(R) = d$. Suppose $q^d < \ell_R(R/\mathfrak{m}^{[q]}) < q^d + q$ for some $q = p^e \geq p$. Prove $\mathrm{Sing}(R) = \{\mathfrak{m}\}$, where $\mathrm{Sing}(R) = \{P \in \mathrm{Spec}(R) \mid R_P \text{ is not regular}\}$ is the singular locus of R.

Problem 4. Let R be a ring (not necessarily of characteristic p). Given R-modules M, N and $f \in \mathrm{Hom}_R(M, N)$, we say f is pure if the induced map $f \otimes_R 1_L : M \otimes_R L \to N \otimes_R L$ is injective for every R-module L. (Denote by m-Spec(R) the set consisting of all maximal ideals of R.)

(1) If $f \in \mathrm{Hom}_R(M, N)$ is pure, then f is injective. (Therefore, $f \in \mathrm{Hom}_R(M, N)$ is pure if and only if f is injective and the inclusion map $f(M) \subseteq N$ is pure.)

(2) $f \in \mathrm{Hom}_R(M, N)$ is pure if and only if $f_P : M_P \to N_P$ is pure for every $P \in \mathrm{Spec}(R)$ if and only if $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is pure for every $\mathfrak{m} \in$ m-Spec(R).

(3) Show (A) $f \in \mathrm{Hom}_R(M, N)$ is pure if and only if $f \otimes_R 1_L : M \otimes_R L \to N \otimes_R L$ is injective for every finitely generated R-module L; and (B) If R is Noetherian and M is finitely generated, then $f \in \mathrm{Hom}_R(M, N)$ is pure if and only if $f \otimes_R 1_L : M \otimes_R L \to N \otimes_R L$ is injective for every finitely generated R-module L such that $\mathrm{Ass}_R(L) = \{\mathfrak{m}\}$ for some $\mathfrak{m} \in$ m-Spec(R).

(4) Suppose R is Noetherian and M, N are finitely generated R-modules. Then $f \in \mathrm{Hom}_R(M, N)$ is pure if and only if $\widehat{f_\mathfrak{m}} : \widehat{M_\mathfrak{m}} \to \widehat{N_\mathfrak{m}}$ is pure for every $\mathfrak{m} \in$ m-Spec(R) if and only if $f : M \to N$ splits (meaning there exists $g \in \mathrm{Hom}_R(N, M)$ such that $g \circ f = 1_M$.)

(5) Suppose R is Noetherian and F is a free R-module. Then $f \in \mathrm{Hom}_R(F, N)$ is pure if and only if the induced map $f \otimes_R 1_E : F \otimes_R E \to N \otimes_R E$ is injective, where $E = \oplus_{\mathfrak{m} \in \text{m-Spec}(R)} E_R(R/\mathfrak{m})$.

Problem 5. Given a local Noetherian ring $(R, \mathfrak{m}, k)$ of prime characteristic p (not necessarily F-finite), one could define R to be strongly F-regular if, for any $c \in R^\circ$, there exists an integer $e \geq 1$ such that the R-linear map $R \to {}^eR$ sending 1 to c is pure. In general, one could define R is strongly F-regular if $R_\mathfrak{m}$ is strongly F-regular for every $\mathfrak{m} \in$ m-Spec(R). (By Problem 4, we see that the above definition agrees with the one given in class when R is F-finite.)

(1) If there exists a pure R-linear map $R \to {}^eR$ sending 1 to c with $e \geq 1$, then R is reduced and, for every $e' \geq e$, the R-linear map $R \to {}^{e'}R$ sending 1 to c is pure. (Thus the above definition of strong F-regularity forces R to be reduced.)

(2) Show that $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if $0^*_{E_R(k)} = 0$.