**Problem 1.** Let $(R, \mathfrak{m})$ be a homomorphic image of a Noetherian Cohen-Macaulay local ring $(S, \mathfrak{n})$. Assume that $R$ is equidimensional with $\dim(R) = d$. For any system of parameters $x_1, \dots, x_d$ of $R$, show that $((x_1, \dots, x_i)^*_R :_R x_{i+1}) = (x_1, \dots, x_i)^*_R$ for every $i < d$. (The case $i = 0$ may be interpreted as $(0^* :_R x_1) = 0^*$, the proof of which should be well incorporated into the general case.)

*Proof.* Say $R = S/Q$ with $\text{height}(Q) = h$. As shown in class, there exist $\underline{y} = y_1, \dots, y_d \in S$, $\underline{z} = z_1, \dots, z_h \in Q$, and $c \in S$ such that (a) the images of $y_i$ are $x_i$ for $i = 1, \dots, d$; (b) $y_1, \dots, y_d, z_1, \dots, z_h$ form part of a system of parameters of $S$; (c) the image of $c$ is $\bar{c} \in R^\circ$; and (d) there exists $q_0 \in \mathbb{N}$ such that $cQ^{[q_0]} \subseteq (\underline{z})S$.

For any $i < d$ and any $x \in ((x_1, \dots, x_i)^*_R :_R x_{i+1})$, we have $xx_{i+1} \in (x_1, \dots, x_i)^*_R$, meaning that there exists $\bar{b} \in R^\circ$ and $q_1$ such that $\bar{b}(xx_{i+1})^q \in (x_1, \dots, x_i)^{[q]}$ for all $q \geq q_1$. Say $y, b \in S$ maps to $x, \bar{b} \in R$ respectively. Then $b(yy_{i+1})^q \in (y_1, \dots, y_i)^{[q]} + Q$ for all $q \geq q_1$. Thus $b^{q_0}(yy_{i+1})^{q_0 q} \in (y_1, \dots, y_i)^{[q_0 q]} + Q^{[q_0]}$ for all $q \geq q_1$. Hence $cb^{q_0}y^{q_0 q}y_{i+1}^{q_0 q} \in (y_1, \dots, y_i)^{[q_0 q]} + (\underline{z})$, which implies $cb^{q_0}y^{q_0 q} \in (y_1, \dots, y_i)^{[q_0 q]} + (\underline{z})$ for all $q \geq q_1$ (as $S$ is Cohen-Macaulay). Applying the homomorphism $S \to R$, we get $\bar{c}\bar{b}^{q_0}x^{q_0 q} \in (x_1, \dots, x_i)^{[q_0 q]}$ for all $q \geq q_1$. In other words, we have $\bar{c}\bar{b}^{q_0}x^q \in (x_1, \dots, x_i)^{[q]}$ for all $q \geq q_0 q_1$. As $\bar{c}\bar{b}^{q_0} \in R^\circ$, we conclude that $x \in (x_1, \dots, x_i)^*_R$. Hence $((x_1, \dots, x_i)^*_R :_R x_{i+1}) \subseteq (x_1, \dots, x_i)^*_R$. Consequently, $((x_1, \dots, x_i)^*_R :_R x_{i+1}) = (x_1, \dots, x_i)^*_R$ as the other containment, $((x_1, \dots, x_i)^*_R :_R x_{i+1}) \supseteq (x_1, \dots, x_i)^*_R$, is obvious. □

**Problem 2.** Let $R \subseteq S$ be an extension of Noetherian domains of prime characteristic $p$ such that $R$ is complete local and $S$ is weakly $F$-regular. Moreover, assume that $(IS) \cap R = I$ for all ideals $I$ of $R$ (e.g. $R$ is a direct summand of $S$ as $R$-modules). Show $R$ is Cohen-Macaulay.

*Proof.* Choose a system of parameters $x_1, \dots, x_d$ of $R$ (with $d = \dim(R)$). Observe that $R$ has 'colon-capture' property: $((x_1, \dots, x_i) :_R x_{i+1}) \subseteq (x_1, \dots, x_i)^*_R$ for every $i < d$. Hence it suffices to show that $(x_1, \dots, x_i)^*_R = (x_1, \dots, x_i)R$ for every $i < d$. We are actually to prove $I^* = I$ for every ideal $I$ of $R$. Indeed, since $R^\circ \subseteq S^\circ$ and $S$ is weakly $F$-regular, we have $I^*_R S \subseteq (IS)^*_S = IS$ and therefore $I^*_R = (I^*_R S) \cap R = (IS) \cap R = I$ for every ideal $I$ of $R$. □

**Problem 3.** Let $R \subseteq S$ be any integral extension of commutative rings of prime characteristic $p$ in which $R$ is Noetherian.

(1) Assume that $S$ is module-finite over $R$. Then $(IS)^*_S \cap R \subseteq I^*_R$ for any ideal $I$ of $R$.
(2) Show that $(IS) \cap R \subseteq I^*_R$ for any ideal $I$ of $R$.

*Proof.* Say $\min(R) = \{P_1, \dots, P_r\}$. By lying over, there exists $Q_i \in \text{Spec}(S)$ such that $Q_i \cap R = P_i$ for every $1 \leq i \leq r$. By incomparability, each $Q_i$ is a minimal prime of $S$. For every $1 \leq i \leq r$, there is an induced embedding $R/P_i \to S/Q_i$. For this reason, we simply treat $R_i := R/P_i$ as a subring of $S_i := S/Q_i$ for every $1 \leq i \leq r$. Remember that we have proved the claims of this problem in case of extensions of domains.

(1). For any $x \in (IS)^* \cap R$, we have $x + Q_i \in (IS_i)^*_{S_i}$ over $S_i$, which implies $x + P_i \in (IS_i)^*_{S_i} \cap R_i = (IR_i)^*_{R_i}$ over $R_i$ for every $1 \leq i \leq r$. This shall force $x \in I^*_R$ over $R$.

(2). For any $x \in (IS) \cap R$, we can write $x = x_1 s_1 + \cdots + x_n s_n$ for some $x_i \in I$, $s_i \in S$. Then we may apply part (1) to $R \subseteq R[s_1, \dots, s_n]$ and conclude $x \in I^*_R$. Alternatively, we observe that $x \in (IS) \cap R$ implies that $x + Q_i \in IS_i$, which implies $x + P_i \in (IS_i) \cap R_i \subseteq (IR_i)^*_{R_i}$ over $R_i$ for every $1 \leq i \leq r$. This shall force $x \in I^*_R$ over $R$. □

**Problem 4.** Let $R$ be a Noetherian ring of prime characteristic $p$ and $N \subsetneq M$ be finitely generated $R$-modules. Let $\Lambda = \{L \mid N \subseteq L \subseteq M \text{ and } \sqrt{\text{Ann}_R(M/L)} \text{ is a maximal ideal}\}$.

(1) Show that $N = \cap_{L \in \Lambda} L$. (This claim does not depend on characteristic $p$.)
(2) Suppose that $J^* = J$ for all ideals $J$ of $R$ such that $\sqrt{J}$ are maximal ideals. Show that $R$ is weakly $F$-regular, i.e. every ideal is tightly closed.

(3) Show that $R$ is weakly $F$-regular if and only if $R_\mathfrak{m}$ is weakly $F$-regular for all maximal ideals $\mathfrak{m}$ of $R$.

*Proof.* (1). We assume $N = 0$ without loss of generality. For any $0 \neq x \in M$, choose a maximal ideal $\mathfrak{m}$ of $R$ such that $(0 :_R x) \subseteq \mathfrak{m}$. Then $x/1$ is nonzero in $M_\mathfrak{m}$. But we have $\cap_{n>0}\mathfrak{m}^n M_\mathfrak{m} = 0$, which implies that $x/1 \notin \mathfrak{m}^n M_\mathfrak{m}$ and, hence, $x \notin \mathfrak{m}^n M$ for some $0 < n \in \mathbb{N}$. Let $L = \mathfrak{m}^n M$ and, clearly, $\sqrt{\mathrm{Ann}_R(M/L)} = \mathfrak{m}$. (The argument actually implies that $N = \cap_\mathfrak{m} \cap_{n\in\mathbb{N}} (N + \mathfrak{m}^n M)$, in which $\mathfrak{m}$ runs through all maximal ideals of $R$.)

(2). For any ideal $I$ of $R$, we have $I = \cap_{J\in\Lambda} J$ with $\Lambda = \{J \mid I \subseteq J \text{ and } \sqrt{J} \text{ is maximal}\}$ by part (1). Therefore $I^* \subseteq \cap_{J\in\Lambda} J^* = \cap_{J\in\Lambda} J = I$. So $I^* = I$ for every ideal $I$ of $R$.

(3). If $R$ is weakly $F$-regular, then $I^* = I$ for all ideals $I$ of $R$. In particular, $J^* = J$ for all ideals $J$ primary to any given maximal ideal $\mathfrak{m}$. For any $\mathfrak{m}R_\mathfrak{m}$-primary ideal $H$ of $R_\mathfrak{m}$, $H = JR_\mathfrak{m}$ for some $\mathfrak{m}$-primary ideal $J$ of $R$. Therefore $H^*_{R_\mathfrak{m}} = J^*_R R_\mathfrak{m} = JR_\mathfrak{m} = H$, implying $R_\mathfrak{m}$ is weakly $F$-regular by part (2). Conversely, suppose $R_\mathfrak{m}$ is weakly $F$-regular for all maximal ideals $\mathfrak{m}$ of $R$. For any ideal $J$ that is primary to a maximal ideal, say $\mathfrak{m}$, we have $J^*_R R_\mathfrak{m} = (JR_\mathfrak{m})^*_{R_\mathfrak{m}} = JR_\mathfrak{m}$. This implies that $J^*_R = J$. Hence $R$ is weakly $F$-regular by part (2). □

**Problem 5.** Let $R$ be a commutative ring (not necessarily Noetherian or with characteristic $p$). Given an ideal $I \subseteq R$ and $x \in R$, prove that $x$ is in the integral closure of $I$ if and only if $x + P$ is in the integral closure of $(I + P)/P \subseteq R/P$ for all $P$ in $\min(R)$, the set of minimal primes of $R$.

*Proof.* Let $W = \{x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \mid n \in \mathbb{Z}, a_i \in I^i\}$. Then we see that $W$ is a multiplicatively closed subset of $R$.

Suppose that $x$ is not in the integral closure of $I$ on the contrary. Then $0 \notin W$, meaning $W^{-1}R$ is not a zero ring and hence $\mathrm{Spec}(W^{-1}R) \neq \emptyset$. Then, by Zorn lemma, there is a minimal prime ideal $Q$ of $W^{-1}R$, in which $Q$ has to be of the form $W^{-1}P$ for some $P \in \min(R)$. Clearly $P \cap W = \emptyset$. But this contradicts to the assumption that $x + P$ is in the integral closure of $(I + P)/P \subseteq R/P$. □

**Problem 6.** Let $R$ be a Noetherian ring of prime characteristic $p$ and $N \subseteq M$ $R$-modules such that $(N_P)^*_{M_P} = (N^*_M)_P$ for every $P \in \mathrm{Ass}(M/N^*_M)$. Show that $(W^{-1}N)^*_{W^{-1}M} = W^{-1}(N^*_M)$ for any multiplicatively closed set $W \subset R$.

*Proof.* Let $W$ be any multiplicatively closed subset of $R$. Since $W^{-1}(N^*_M) \subseteq (W^{-1}N)^*_{W^{-1}M}$ and, hence, $(W^{-1}(N^*_M))^*_{W^{-1}M} = (W^{-1}N)^*_{W^{-1}M}$, we may mod out $N^*_M$. In other words, we may assume $N = 0 = 0^*_M$ without loss of generality. Now we need to show that $0 = 0^*_{W^{-1}M}$.

Say $\mathrm{Ass}_R(M) = \{P_1, \ldots, P_t\}$ and $\mathrm{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}P_1, \ldots, W^{-1}P_s\}$ for some $s \leq t$. Clearly, we have $\mathrm{Ass}_{W^{-1}R}(0^*_{W^{-1}M}) \subseteq \mathrm{Ass}_{W^{-1}R}(W^{-1}M)$.

Suppose, on the contrary, that $0 \subsetneq 0^*_{W^{-1}M}$. Then $\mathrm{Ass}_{W^{-1}R}(0^*_{W^{-1}M}) \neq \emptyset$. Say $\mathrm{Ass}_{W^{-1}R}(0^*_{W^{-1}M}) = \{W^{-1}P_1, \ldots, W^{-1}P_r\}$ for some $1 \leq r \leq s$. Let $P = P_1$. Then $M_P$ can be viewed as a further localization of $W^{-1}M$ at $W^{-1}P$. Thus we have $(0^*_{W^{-1}M})_{W^{-1}P} \subseteq 0^*_{M_P}$.

By our choice, $W^{-1}P \in \mathrm{Ass}_{W^{-1}R}(0^*_{W^{-1}M})$, which implies that $0 \neq (0^*_{W^{-1}M})_{W^{-1}P} \subseteq 0^*_{M_P}$. On the other hand, the assumption, together with the fact that $P \in \mathrm{Ass}_R(M)$ (which is $\mathrm{Ass}_R(M/N^*_M)$ since we are assuming $N^*_M = 0$), forces $0^*_{M_P} = 0$. We have reached a contradiction. □