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**Problem 1.** Let $(R, \mathfrak{m})$ be a homomorphic image of a Noetherian Cohen-Macaulay local ring $(S, \mathfrak{n})$. Assume that $R$ is equidimensional with $\dim(R) = d$. For any system of parameters $x_1, \ldots, x_d$ of $R$, show that $((x_1, \ldots, x_i)^*_R :_R x_{i+1}) = (x_1, \ldots, x_i)^*_R$ for every $i < d$. (The case $i = 0$ may be interpreted as $(0^* :_R x_1) = 0^*$, the proof of which should be well incorporated into the general case.)

**Problem 2.** Let $R \subseteq S$ be an extension of Noetherian domains of prime characteristic $p$ such that $R$ is complete local and $S$ is weakly $F$-regular. Moreover, assume that $(IS) \cap R = I$ for all ideals $I$ of $R$ (e.g. $R$ is a direct summand of $S$ as $R$-modules). Show $R$ is Cohen-Macaulay.

**Problem 3.** Let $R \subseteq S$ be any integral extension of commutative rings of prime characteristic $p$ in which $R$ is Noetherian.

(1) Assume that $S$ is module-finite over $R$. Then $(IS)^*_S \cap R \subseteq I^*_R$ for any ideal $I$ of $R$.
(2) Show that $(IS) \cap R \subseteq I^*_R$ for any ideal $I$ of $R$.

**Problem 4.** Let $R$ be a Noetherian ring of prime characteristic $p$ and $N \subsetneq M$ be finitely generated $R$-modules. Let $\Lambda = \{L \,|\, N \subseteq L \subseteq M \text{ and } \sqrt{\mathrm{Ann}_R(M/L)} \text{ is a maximal ideal}\}$.

(1) Show that $N = \cap_{L \in \Lambda} L$. (This claim does not depend on characteristic $p$.)
(2) Suppose that $J^* = J$ for all ideals $J$ of $R$ such that $\sqrt{J}$ are maximal ideals. Show that $R$ is weakly $F$-regular, i.e. every ideal is tightly closed.
(3) Show that $R$ is weakly $F$-regular if and only if $R_\mathfrak{m}$ is weakly $F$-regular for all maximal ideals $\mathfrak{m}$ of $R$.

**Problem 5.** Let $R$ be a commutative ring (not necessarily Noetherian or with characteristic $p$). Given an ideal $I \subseteq R$ and $x \in R$, prove that $x$ is in the integral closure of $I$ if and only if $x + P$ is in the integral closure of $(I + P)/P \subseteq R/P$ for all $P$ in $\min(R)$, the set of minimal primes of $R$.

**Problem 6.** Let $R$ be a Noetherian ring of prime characteristic $p$ and $N \subseteq M$ $R$-modules such that $(N_P)^*_{M_P} = (N^*_M)_P$ for every $P \in \mathrm{Ass}(M/N^*_M)$. Show that $(W^{-1}N)^*_{W^{-1}M} = W^{-1}(N^*_M)$ for any multiplicatively closed set $W \subset R$.