Problem 1. Let $(R, \mathfrak{m})$ be a 0-dimensional Noetherian (i.e. Artinian) local ring of prime characteristic p and M be a finitely generated R-module. Show that there exists $e_0 \in \mathbb{N}$ such that $F^e(M) \cong F^{e_0}(M)$ for all $e \geq e_0$. (Actually, this is true as long as R is 0-dimensional Noetherian, the proof of which reduces to local case.)

Proof. Write down an representation $R^m \xrightarrow{(a_{ij})} R^n \longrightarrow M \longrightarrow 0$ such that $a_{ij} \in \mathfrak{m}$. Then, for any $q = p^e$, $F^e(M)$ is represented by exact sequence $R^m \xrightarrow{(a_{ij}^q)} R^n \longrightarrow F^e(M) \longrightarrow 0$. By assumption, there exists $q_0 = p^{e_0}$ such that $\mathfrak{m}^{[q]} = 0$ for all $q \geq q_0$. Therefore $F^e(M) \cong R^n$ for all $e \geq e_0$.

(In case R is Artinian (not necessarily local), $R = \prod_{i=1}^s R_i$ is a direct product of Artinian local rings R_i. So the claim remains true although the stabled $F^e(M)$ is not necessarily free over R. Instead, $F^e(M)$ is isomorphic to $\oplus_{i=1}^s R_i^{b_i}$ for all $e \gg 0$.) ☐

Problem 2. Let R be a Noetherian ring of prime characteristic p. It is known that $I^* \subseteq \sqrt{I}$ for every ideal $I \subseteq R$ (by comparing tight closure with integral closure). Here we show this from definition of tight closure.

(1) If $(R, \mathfrak{m})$ is local (with maximal ideal $\mathfrak{m}$), show $\mathfrak{m}^* = \mathfrak{m}$.
(2) Show $P^* = P$ for any prime ideal of R, which is not assumed to be local.
(3) Show $I^* \subseteq \sqrt{I}$ for every ideal I of R.
(4) Compute 0_R^*. Here 0 refers to the zero ideal of R.

Proof. (1). Suppose $\mathfrak{m}^* \supsetneq \mathfrak{m}$. Then $1 \in \mathfrak{m}^*$, meaning that there exists $c \in R^\circ$ such that $c1^q \in \mathfrak{m}^{[q]}$ for all $q \gg 0$. But this forces $c \in \cap_{q \gg 0} \mathfrak{m}^{[q]} = 0$, a contradiction.

(2). By (1), the ideal P_P is tightly closed in R_P for any $P \in \operatorname{Spec}(R)$. This gives the conclusion $P^* = P$ as $(P_R^*)R_P \subseteq (P_P)_{R_P}^*$ for any $P \in \operatorname{Spec}(R)$.

(3). For any ideal I of R, say $\sqrt{I} = \cap_{i=1}^n P_i$ for $P_i \in \operatorname{Spec}(R)$. Then $I^* \subseteq \cap_{i=1}^n P_i^* = \cap_{i=1}^n P_i = \sqrt{I}$.

(4). We always have $\sqrt{0} \subseteq 0^*$. By (3), we also have $0^* \subseteq \sqrt{0}$. Thus $0^* = \sqrt{0}$. ☐

Problem 3. Let k be a field of characteristic 2, $S = k[X, Y]$ be a polynomial ring over k with indeterminates X, Y, and $R = S/(X^3)S$. Let $I = (X^2)R \subset (X)R = J$ be ideals of R. (Then $I \subset J$ are also modules over S under the natural ring homomorphism $S \to R$.)

(1) Let $F_R^e(-)$ be the Frobenius functor over R. Up to isomorphism, how many distinct R-modules are there among $0, I, J, R, F_R(I), F_R(J), I_R^{[2]}, J_R^{[2]}, I_J^{[2]}$? (Everything is considered as an R-module, including I as in the notation $F_R(I)$, for example.) Group isomorphic R-modules together.
(2) Determine I_R^* and I_J^* over R. (Everything is considered as an R-module.)
(3) Determine I_R^* and I_J^* over S. (Everything is considered as an S-module, including R as in I_R^*.)

Proof. Denote the images of X, Y in R by x, y. Therefore $x^3 = 0$ in R.

(1). As R-modules, $I = xJ \cong R/xR$ and $J \cong R/x^2R$. Hence $F_R(I) \cong R/x^2R$, $F_R(J) \cong R/x^4R = R$ and $I_J^{[2]} = x^2 F_R(J) \cong x^2R = I$. We also have $I_R^{[2]} = x^4R = 0$ and $J_R^{[2]} = x^2R = I$. To sum up, there are four (4) distinct R-modules up to isomorphism, namely $0 \cong I_R^{[2]}$, $I \cong J_R^{[2]} \cong I_J^{[2]}$, $J \cong F_R(I)$ and $R \cong F_R(J)$.

(2). First, $I_R^* = \sqrt{0} = J$ because $I \subset \sqrt{0}$. Then notice that $J_R^* = J$ as $J = \sqrt{0}$, which means that $0_{R/J}^* = 0$. But $R/J \cong J/I$. So $0_{J/I}^* = 0$, meaning $I_J^* = I$.

(3). Over regular ring S, $N_M^* = N$ for any S-modules $N \subseteq M$. So $I_R^* = I$ and $I_J^* = I$ over S. ☐

Problem 4. Let R be a Noetherian ring of prime characteristic p. Suppose

$$G_\bullet : \qquad 0 \longrightarrow G_n \xrightarrow{\phi_n} G_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} G_1 \xrightarrow{\phi_1} G_0$$

is a complex with G_i free of finite rank for $0 \leq i \leq n$. Apply Frobenius functor F^e to get $F^e(G_\bullet)$ for every $e \in \mathbb{N}$.

(1) If $G_\bullet$ is exact, show $F^e(G_\bullet)$ remains exact for every $e \in \mathbb{N}$.

(2) If $\ell_R(\mathrm{H}_i(G_\bullet)) < \infty$ for every $1 \leq i \leq n$, show $\ell_R(\mathrm{H}_i(F^e(G_\bullet))) < \infty$ for every $1 \leq i \leq n$ and for every $e \in \mathbb{N}$. (Here $\ell_R(-)$ represents the length of an R-module.)

Do (1) and (2) still hold if, instead, we assume G_i are finitely generated projective R-modules for $0 \leq i \leq n$?

Proof. (1). Since Frobenius function F^e commutes with localization and a complex is exact if and only if every localization of it is exact, we may assume R is local. Then the assumption that $G_\bullet$ is exact implies that $\operatorname{depth}_{I(\phi_i)}(R) \geq i$ and $\operatorname{rank}(G_i) = \operatorname{rank}(\phi_i) + \operatorname{rank}(\phi_{i+1})$ for every $1 \leq i \leq n$. In particular, $I(\phi_i)$ contains a non-zero-divisor for every $1 \leq i \leq n$, which guarantees that $\operatorname{rank}(\phi_i) = \operatorname{rank}(F^e(\phi_i))$ for every i and every e. This also implies that $I(F^e(\phi_i)) = (I(\phi_i))^{[q]}$ and thus $\operatorname{depth}_{I(F^e(\phi_i))}(R) = \operatorname{depth}_{I(\phi_i)}(R)$ for every $q = p^e$. Therefore, the complex $F^e(G_\bullet)$ still satisfies the 'rank and depth' condition. Consequently, $F^e(G_\bullet)$ is exact for every $e \in \mathbb{N}$. (If, for any given i, ϕ_i is represented by a matrix (a_{jk}), then $F^e(\phi_i)$ may be represented by the matrix (a_{jk}^q).)

(2). This follows from (1). Indeed, for any given e, $\ell_R(\mathrm{H}_i(F^e(G_\bullet))) < \infty$ for all $1 \leq i \leq n$ if and only if $F^e(G_\bullet) \otimes_R R_P$ is exact for all prime ideals P that are not maximal.

Finally, (1) and (2) still hold if we assume G_i are finitely generated projective R-modules for $0 \leq i \leq n$. As seen in the above proof, both (1) and (2) reduce to local case. □

Problem 5. Let R be a Noetherian ring of prime characteristic p and $N \subseteq M$ be R-modules. Define $N_M^F = \{x \in M \mid x^q \in N_M^{[q]} \subseteq F^e(M) \text{ for some } q = p^e\}$. ($N_M^F$ is called the *Frobenius closure* of N in M.)

(1) Show that $N_M^F \subseteq N_M^*$.

(2) Show that Frobenius closure commutes with localization, i.e. $(W^{-1}N)^F_{W^{-1}M} = W^{-1}(N_M^F)$ for any multiplicatively closed set $W \subset R$.

Proof. (It is routine to check that N_M^F is an R-submodule of M.)

(1). For any $x \in N_M^F$, we have $x^{q_0} \in N_M^{[q_0]} \subseteq F^{e_0}(M)$ for some $q_0 = p^{e_0}$. This actually implies that $x^q \in N_M^{[q]} \subseteq F^e(M)$ for all $q \geq q_0$, showing $x \in N_M^*$ (with $c = 1 \in R^\circ$).

(2). Let $W \subset R$ be a multiplicatively closed subset of R. If $x \in N_M^F$, then $x^q \in N_M^{[q]} \subseteq F^e(M)$ for some $q = p^e$. This gives $(x/1)^q \in (W^{-1}N)^{[q]}_{W^{-1}M} \subseteq F^e(W^{-1}M)$. (Here we use the fact that Frobenius functor commutes with localization.) Hence $x/1 \in (W^{-1}N)^F_{W^{-1}M}$, showing $(W^{-1}N)^F_{W^{-1}M} \supseteq W^{-1}(N_M^F)$. On the other hand, for any $x/w \in (W^{-1}N)^F_{W^{-1}M}$ with $x \in M$ and $w \in W$, there exists $q = p^e$ such that $(x/w)^q \in (W^{-1}N)^{[q]}_{W^{-1}M} \subseteq F^e(W^{-1}M)$. (We may simply assume $w = 1$ as Frobenius closure is a submodule.) Using the fact that Frobenius functor commutes (naturally) with localization, we may write $(x/w)^q \in W^{-1}(N_M^{[q]}) \subseteq W^{-1}(F^e(M))$ for the same $q = p^e$. Therefore there exists $w' \in W$ such that $w'x^q \in N^{[q]} \subseteq F^e(M)$ and hence $(w'x)^q \in N^{[q]} \subseteq F^e(M)$, which implies $w'x \in N_M^F$. This shows that $(W^{-1}N)^F_{W^{-1}M} \subseteq W^{-1}(N_M^F)$. Finally, $(W^{-1}N)^F_{W^{-1}M} = W^{-1}(N_M^F)$. □