Problem 1. Let $(R, \mathfrak{m})$ be a 0-dimensional Noetherian (i.e. Artinian) local ring of prime characteristic p and M be a finitely generated R-module. Show that there exists $e_0 \in \mathbb{N}$ such that $F^e(M) \cong F^{e_0}(M)$ for all $e \geq e_0$. (Actually, this is true as long as R is 0-dimensional Noetherian, the proof of which reduces to local case.)

Problem 2. Let R be a Noetherian ring of prime characteristic p. It is known that $I^* \subseteq \sqrt{I}$ for every ideal $I \subseteq R$ (by comparing tight closure with integral closure). Here we show this from definition of tight closure.

(1) If $(R, \mathfrak{m})$ is local (with maximal ideal $\mathfrak{m}$), show $\mathfrak{m}^* = \mathfrak{m}$.
(2) Show $P^* = P$ for any prime ideal of R, which is not assumed to be local.
(3) Show $I^* \subseteq \sqrt{I}$ for every ideal I of R.
(4) Compute 0^*_R. Here 0 refers to the zero ideal of R.

Problem 3. Let k be a field of characteristic 2, $S = k[X, Y]$ be a polynomial ring over k with indeterminates X, Y, and $R = S/(X^3)S$. Let $I = (X^2)R \subset (X)R = J$ be ideals of R. (Then $I \subset J$ are also modules over S under the natural ring homomorphism $S \to R$.)

(1) Let $F^e_R(-)$ be the Frobenius functor over R. Up to isomorphism, how many distinct R-modules are there among $0, I, J, R, F_R(I), F_R(J), I^{[2]}_R, J^{[2]}_R, I^{[2]}_J$? (Everything is considered as an R-module, including I as in the notation $F_R(I)$, for example.) Group isomorphic R-modules together.
(2) Determine I^*_R and I^*_J over R. (Everything is considered as an R-module.)
(3) Determine I^*_R and I^*_J over S. (Everything is considered as an S-module, including R as in I^*_R.)

Problem 4. Let R be a Noetherian ring of prime characteristic p. Suppose

$$G_\bullet : \qquad 0 \longrightarrow G_n \xrightarrow{\phi_n} G_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} G_1 \xrightarrow{\phi_1} G_0$$

is a complex with G_i free of finite rank for $0 \leq i \leq n$. Apply Frobenius functor F^e to get $F^e(G_\bullet)$ for every $e \in \mathbb{N}$.

(1) If $G_\bullet$ is exact, show $F^e(G_\bullet)$ remains exact for every $e \in \mathbb{N}$.
(2) If $\ell_R(\mathrm{H}_i(G_\bullet)) < \infty$ for every $1 \leq i \leq n$, show $\ell_R(\mathrm{H}_i(F^e(G_\bullet))) < \infty$ for every $1 \leq i \leq n$ and for every $e \in \mathbb{N}$. (Here $\ell_R(-)$ represents the length of an R-module.)

Do (1) and (2) still hold if, instead, we assume G_i are finitely generated projective R-modules for $0 \leq i \leq n$?

Problem 5. Let R be a Noetherian ring of prime characteristic p and $N \subseteq M$ be R-modules. Define $N^F_M = \{x \in M \,|\, x^q \in N^{[q]}_M \subseteq F^e(M) \text{ for some } q = p^e\}$. ($N^F_M$ is called the *Frobenius closure* of N in M.)

(1) Show that $N^F_M \subseteq N^*_M$.
(2) Show that Frobenius closure commutes with localization, i.e. $(W^{-1}N)^F_{W^{-1}M} = W^{-1}(N^F_M)$ for any multiplicatively closed set $W \subset R$.

(Assume R is Noetherian and M is a finitely generated R-module. Note that, for every ideal I of R, $\sqrt{I}$ is defined to be $\{r \in R \,|\, r^n \in I \text{ for some } n \in \mathbb{N}\}$. We may use the fact that $\mathrm{depth}_I(M) = \mathrm{depth}_{\sqrt{I}}(M)$ without proof. We may also use the fact without proof that $F^e_{W^{-1}R}(W^{-1}M) \cong W^{-1}F^e_R(M)$ for any multiplicatively closed set $W \subset R$ although the proof of it is, actually, very short/easy. Finally, a polynomial ring over a field is regular.)