Modified Bayesian approach for the reconstruction of dynamical systems from time series

D. N. Mukhin,* A. M. Feigin, E. M. Loskutov,† and Ya. I. Molkov
Institute of Applied Physics, Russian Academy of Sciences, Russia

(Received 5 May 2005; revised manuscript received 17 January 2006; published 16 March 2006)

Some recent papers were concerned with applicability of the Bayesian (*statistical*) approach to reconstruction of *dynamic* systems (DS) from experimental data. A significant merit of the approach is its universality. But, being correct in terms of meeting conditions of the underlying theorem, the Bayesian approach to reconstruction of DS is hard to realize in the most interesting case of noisy *chaotic* time series (TS). In this work we consider a modification of the Bayesian approach that can be used for reconstruction of DS from noisy TS. We demonstrate efficiency of the modified approach for solution of two types of problems: (1) finding values of parameters of a known DS by noisy TS; (2) classification of modes of behavior of such a DS by short TS with pronounced noise.

DOI: 10.1103/PhysRevE.73.036211 PACS number(s): 05.45.Tp, 02.70.Rr, 02.30.Zz

I. STATISTICAL APPROACH TO RECONSTRUCTION OF DYNAMIC SYSTEMS

Reconstruction of DS from TS generated by this system is usually understood as seeking its evolution operator. When a DS is known, it is necessary to find values of parameters that determined evolution of the system during TS generation. Such a formulation of the problem arises, for instance, when chaotic regimes of DS behavior are used for solution of the problem of coded transmission of information (see, e.g., Ref. [1]). In situations typical for most natural systems (atmospheric-oceanic, tectonic, biological), the DS that generated the observed TS is *unknown*. In this case, reconstruction of DS means construction on the basis of the information contained in the TS of a parametrized *model* of unknown evolution operator. Apparently, such a model cannot be ideal as, generally, there does not exist such a set of parameter values for any class of models that would make the model absolutely adequate to the modeled DS. Inevitable discrepancy between them is called "defect of the model." In this sense, the two above formulations of the problem of DS retrieval are sometimes classified as the "perfect model class scenario" and the "imperfect model class scenario" [2].

Assume that we have at our disposal the vector TS $\mathbf{x}$, which is formed by M observed quantities $\mathbf{x}=\{\mathbf{x}^{(m)}\}_{m=1}^{M}$ and connected with DS state $\mathbf{u}=\{\mathbf{u}^{(k)}\}_{k=1}^{d}$ via an observer h, $\mathbf{x}=h(\mathbf{u})$. Here d is DS dimensionality, $d\geqslant M\geqslant 1$. Under the perfect model class scenario d value is naturally known to us. When DS is unknown this quantity may be estimated by minimal embedding dimension of the attractor responsible for its observed evolution. Methods of obtaining this information and quite substantial restrictions were discussed by numerous authors and summarized in different review papers and books (see, e.g., Refs. [3,4]). In this work we restrict our consideration to the first situation. The second situation (the "imperfect model class scenario") will be considered elsewhere.

In what will follow we will use the following formulation of the Bayes theorem [5]. Suppose that the system under experiment possesses a set of properties (parameters) $\boldsymbol{\mu}$ that cannot be measured directly and let values of $\mathbf{x}$ be recorded in experiment. Then, *posterior* conditional probability density of *unobservable* parameters (frequently referred to as likelihood) $p(\boldsymbol{\mu}|\mathbf{x})$ is proportional to the product of their *prior* probability density $p(\boldsymbol{\mu})$ and *prior* conditional probability density of the obtained experimental results, $p(\mathbf{x}|\boldsymbol{\mu})$:

$$p(\boldsymbol{\mu}|\mathbf{x}) = C \times p(\boldsymbol{\mu}) \times p(\mathbf{x}|\boldsymbol{\mu}). \tag{1}$$

It will be clear from what will follow that conditional probability density $p(\mathbf{x}|\boldsymbol{\mu})$ depends wholly on the way TS becomes noisy and on probability densities of all noises present in the TS. Factor $p(\boldsymbol{\mu})$ takes into account *a priori* information about the system. If such information is not available, probability density $p(\boldsymbol{\mu})$ must be chosen to be constant, with the width allowing for all possible values of parameters $\boldsymbol{\mu}$. Constant C in (1) is determined by the normalization condition: $C=[\int p(\boldsymbol{\mu})p(\mathbf{x}|\boldsymbol{\mu})d\boldsymbol{\mu}]^{-1}$.

The presence in experimental data of noise component [6] justifies application of the *probability* Bayesian approach to construction of models of *dynamic* systems. Consider as an example a DS with discrete time and assume for definiteness that measurement error ("noise") $\boldsymbol{\xi}$ is additive:

$$\boldsymbol{\xi}_t = \mathbf{x}_t - h(\mathbf{u}_t), \quad \mathbf{u}_{t+1} = \mathbf{f}(\mathbf{u}_t, \boldsymbol{\mu}). \tag{2}$$

Here, the subscript numbers discrete time counts, vector $\mathbf{u}_t=(u_t^{(k)})_{k=1}^{d}$ specifies now "true" (*latent*) state of the DS at the time instant t ($t=0,\ldots,T-1$) in d-dimensional phase space (embedding space), the discrete time map $f(\mathbf{u}_t,\boldsymbol{\mu})$ describes evolution operator of the DS, and $\boldsymbol{\mu}=\{\mu_m\}_{m=1}^{M}$ is the vector of parameters.

As "true" states of the DS are unknown, the probability densities entering (1) depend not only on parameters $\boldsymbol{\mu}$, but also on latent variables $\mathbf{u}=\{\mathbf{u}_t\}_{t=0}^{T-1}: p(\boldsymbol{\mu}|\mathbf{x})\Rightarrow p(\mathbf{u},\boldsymbol{\mu}|\mathbf{x})$; $p(\boldsymbol{\mu})\Rightarrow p(\mathbf{u},\boldsymbol{\mu}), p(\mathbf{x}|\boldsymbol{\mu})\Rightarrow p(\mathbf{x}|\mathbf{u},\boldsymbol{\mu})$, with *prior* conditional probability density $p(\mathbf{x}|\mathbf{u},\boldsymbol{\mu})$ determined wholly by the properties of random quantities $\boldsymbol{\xi}$. If they are mutually independent and their probability densities are described by the

*Electronic address: mukhin@appl.sci-nnov.ru
†Electronic address: loskutov@appl.sci-nnov.ru

