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BISTABLE PHASE SYNCHRONIZATION AND CHAOS IN A SYSTEM OF COUPLED VAN DER POL–DUFFING OSCILLATORS

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Analysis of numerical solutions for a system of two van der Pol–Duffing oscillators with nonlinear coupling showed that there exist chaotic switchings (occurring at irregular time intervals) between two oscillatory regimes differing by nearly time-constant phase shifts between the coupled subsystems. The analysis includes the investigation of bifurcations of the periodic motions corresponding to synchronization of two subsystems, finding stability regions of synchronization regimes and scenarios of the transitions to chaos.

1. Introduction

Low-dimensional models are frequently used for description of qualitative changes in the behavior of even relatively complex systems near critical values of control parameters. It is quite natural, taking into consideration the center manifold theorem and the ideas about typical or "generic" systems, which are the foundations of the synergetic approach to modeling collective effects in large systems [Haken, 1987, 1988]. It is essential that not only simple but also rather complicated regimes in large systems and transitions between them may often be described adequately by low-dimensional models. This makes details of their dynamics attractive for both theoreticians who traditionally regard low-dimensional models as a testing ground for rigorous mathematical methods and researchers interested in applied problems. Application-oriented aspects play a significant role in choosing directions of investigation in those cases when even low-dimensional models prove to be too complicated for comprehensive analysis. In particular, they stimulated investigation of the regimes for which the phase difference between interacting subsystems is a meaningful variable (order parameter). In recent years, particular attention to such research has been associated with the investigation of dynamical mechanisms of coordination of rhythmic processes and movements in living organisms, for instance, in psychophysiological experiments with humans [Haken *et al.*, 1985; Kelso *et al.*, 1986; Schöner & Kelso, 1988; Sternad *et al.*, 1992; Buchanan *et al.*, 1995; Fuchs *et al.*, 1996]. In connection with the latter, the Haken–Kelso–Bunz (HKB) model was proposed and further developed more than a decade ago. This model consists of two oscillators with nonlinear coupling that imitates combined action of antagonistic forces, e.g. excitatory and inhibitory synaptic couplings in neural networks. The use of such a combinatory coupling makes possible dynamical modeling of coordination of the movements of human fingers relative to periodic external stimulus [Kelso *et al.*, 1990], mutual

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coordination of the movements of different limbs (hand and leg) [Kelso & Jeka, 1992] and coordination of different groups of muscles of the human leg during locomotor-like movements evoked by nonresonant vibration of muscles [Molkov *et al.*, 1998]. This coupling proved to be most effective in that it produces a rich variety of dynamic regimes when the interacting oscillators possess strong intrinsic nonlinearity and their oscillation frequencies depend on amplitude [Molkov *et al.*, 1998]. In particular, chaotic transitions between sufficiently time-extended portions of oscillations with almost constant phase differences between the coupled oscillators are observed in a rather wide range of coupling parameters. It was found [Molkov *et al.*, 1998] that these transitions resemble spontaneous switchings between "forward" and "backward" stepping movements observed in experiments on human locomotion evoked by continuous muscle vibration [Gurfinkel *et al.*, 1998].

Primary attention in this research is focused on the regime of such switchings, and on the bifurcations leading to it. Investigation of the role of antagonistic couplings may also be useful for other applications, e.g. for modeling the phenomena of dynamic coding, storing and recognition of information [Kozlov *et al.*, 1997].

2. Basic Equations

The dynamic system considered here consists of two nonlinear coupled identical van der Pol–Duffing oscillators:

$$\begin{aligned} \dot{x}_1 &= y_1 + F(x_1, x_2)\,, \\ \dot{y}_1 &= \lambda(1 - x_1^2)y_1 - \varepsilon x_1^3 - x_1\,, \\ \dot{x}_2 &= y_2 + F(x_2, x_1)\,, \\ \dot{y}_2 &= \lambda(1 - x_2^2)y_2 - \varepsilon x_2^3 - x_2\,. \end{aligned} \tag{1}$$

The coupling function $F(x_1, x_2)$, like in the model proposed in [Molkov *et al.*, 1998], is chosen to be a cubic one:

$$F(x_1, x_2) = \gamma(x_1 - x_2)[\alpha - (x_1 - x_2)^2] \tag{2}$$

and approximates general tendencies in the action of synaptic couplings that are the most pronounced in the following limiting cases. For $\alpha \gg 1$ (hereinafter we take γ, ε, $\lambda > 0$), the linear term in (2) guarantees the onset of out-of-phase π-oscillations x_1 and x_2 (each oscillator is out of phase with the other one by half a period). For $\alpha < 0$ or $0 < \alpha \ll 1$, in-phase or nearly in-phase periodic oscillations are established, respectively, which reproduces the action of excitatory couplings (see e.g. [Abarbanel *et al.*, 1996]). For finite values of $\alpha > 0$, cubic nonlinearity of coupling provides competition of these two opposite tendencies, which leads to multistable synchronization and intermitting regimes with almost constant phase shift at rather long time intervals.

The tendencies mentioned above are apparent in a quasiharmonic approximation at γ, ε, $\lambda \sim \mu \ll 1$, when the variables x, y may be represented in the form

$$\begin{aligned} x_j(t) &= a_j(t)e^{i\varphi_j(t)+it} + \text{c.c.}\,, \\ y_j(t) &= i(a_j(t)e^{i\varphi_j(t)+it} - \text{c.c.})\,, \end{aligned} \tag{3}$$

where a_j and φ_j, $(j = 1, 2)$ are the amplitudes and phases of oscillations of coupled oscillators slowly varying in time which, in the first approximation with respect to the small parameter μ, satisfy the reduced equations

$$\begin{aligned} \dot{a}_1 &= a_1(1 - a_1^2) - \Gamma \cdot (a_1 - a_2 \cos\varphi)\,, \\ \dot{a}_2 &= a_2(1 - a_2^2) - \Gamma \cdot (a_2 - a_1 \cos\varphi)\,, \\ \dot{\varphi} &= C(a_2^2 - a_1^2) - \Gamma \cdot \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) \sin\varphi\,. \end{aligned} \tag{4}$$

Here, $\varphi \equiv \varphi_1 - \varphi_2$, $\Gamma \equiv \Gamma(a_1, a_2, \varphi) = \gamma'(a_1^2 + a_2^2 - 2a_1a_2\cos\varphi - \alpha')$, $a_1a_2 \neq 0$, and new time $t' = t\lambda/2$ and parameters $C = -3\varepsilon/\lambda$, $\alpha' = \alpha/3$, $\gamma' = 3\gamma/\lambda$ are introduced. A trivial equilibrium point $a_j = 0$ is unstable in (4), and nontrivial states $a_j = a_j^*$, $\varphi = \varphi^* = \varphi_1^* - \varphi_2^*$ further denoted as $S(a_1^*, a_2^*, \varphi^*)$ correspond to periodic solutions, i.e. to synchronized oscillations of two subsystems with constant phase difference $\varphi = \varphi^*$. For $\alpha' < 0$, Eq. (4) has a stable equilibrium point $S_0 = S(1, 1, 0)$ and a saddle $S_\pi = S(a^\pi, a^\pi, \pi)$ which correspond to in-phase and out-of-phase oscillations in (1). When the sign of α' is changed, the equilibrium point S_0 becomes a saddle point and gives birth to two nonsymmetric ($\varphi^* = \pm\psi$, $\psi \neq 0, \pi$) stable equilibrium states $S_{+\psi}(1, 1, \psi)$ and $S_{-\psi}(1, 1, -\psi)$. For $\alpha^* = 4$, the equilibrium points $S_{+\psi}$ and $S_{-\psi}$ merge with the equilibrium point S_π that becomes stable at $\alpha' > 4$ (see Fig. 1). Thus, at small supercritical values of $\alpha' > 0$, the nonlinearity of coupling after loss of stability of the regime of in-phase synchronization gives rise to nonsymmetric regimes with constant phase shift ($\varphi \neq 0, \pi$) between subsystems, even in

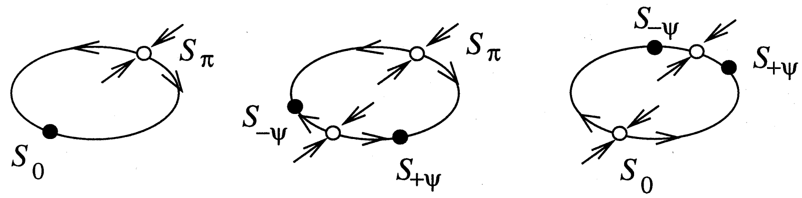


Fig. 1. Sequence of bifurcations in (4) with increasing parameter α' (from left to right).

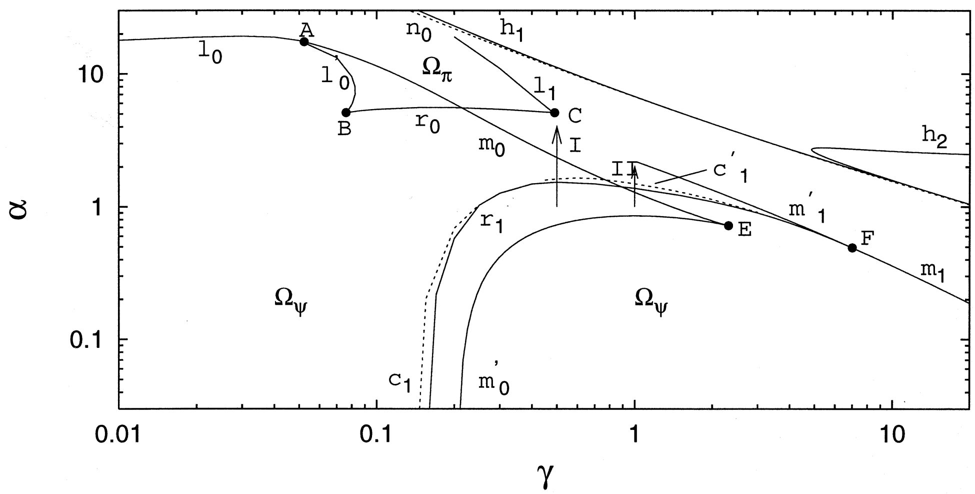


Fig. 2. Bifurcation diagram in the plane of parameters γ, α.

the case of identical oscillators without frequency mismatch.

3. Basic Bifurcations

Nonlinear interaction of the van der Pol–Duffing oscillators (1) is studied for two control parameters, γ and α. The two-parametric bifurcation diagram obtained for (1) at $\lambda = 0.5$, $\varepsilon = 1$ is shown in Fig. 2

For small values of γ, the basic system (1) agrees well with the averaged equations (4). Namely, as α is increased, the periodic orbit Γ_0 corresponding to identical synchronization of the subsystems loses stability at $\alpha = 0$ and becomes a saddle one. For $\alpha > 0$, it gives birth to two stable periodic orbits $\Gamma_{+\psi}$ and $\Gamma_{-\psi}$ differing, mainly, by the phase shift between the time series $x_1(t)$ and $x_2(t)$ (see Fig. 3). As α grows at small values of γ, the periodic orbits $\Gamma_{+\psi}$ and $\Gamma_{-\psi}$ disappear, merging with the saddle periodic orbit Γ_π corresponding to out-of-phase synchronization. Γ_π becomes stable and its multiplier intersects the unit circle across $+1$. The curve l_0 corresponds to this bifurcation in the two-parametric diagram shown in Fig. 2.

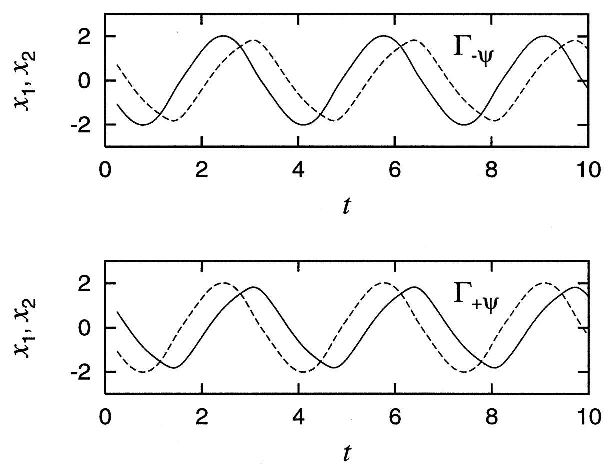


Fig. 3. Time series corresponding to periodic orbits $\Gamma_{+\psi}$ and $\Gamma_{-\psi}$ ($x_1(t)$ are plotted as solid lines, and $x_2(t)$ as dashed lines).

The stable periodic orbit Γ_π exists for (γ, α) from domain Ω_π. Ω_π is bounded by bifurcation curves l_0, l_0', l_1 (one multiplier equals $+1$, pitchfork bifurcation) and r_0 (a pair of complex-conjugate multipliers $e^{\pm i\sigma}$).

The existence domain of the stable periodic orbits $\Gamma_{+\psi}$, $\Gamma_{-\psi}$ (or similar periodic orbits $\Gamma'_{+\psi}$, $\Gamma'_{-\psi}$ which differ from $\Gamma_{+\psi}$ and $\Gamma_{-\psi}$ basically by amplitudes) is marked by Ω_ψ. Ω_ψ is above the line $\alpha > 0$ and below the bifurcation curves l_0 (described above), m_0, m_1 (saddle-node bifurcation of periodic orbits where $\Gamma_{+\psi}$, $\Gamma_{-\psi}$ disappear merging with saddle periodic orbits), and r_1 (a pair of complex-conjugate multipliers $e^{\pm i\sigma}$ for $\Gamma'_{+\psi}$, $\Gamma'_{-\psi}$).

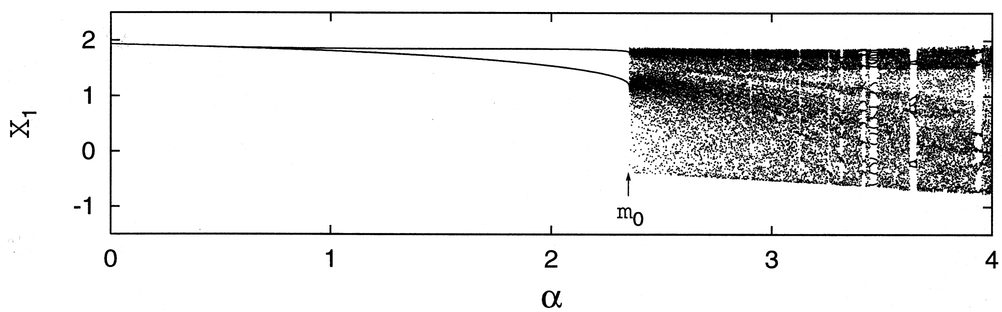
Besides the unstable equilibrium point (0, 0, 0, 0), there are two pairs of symmetrical equilibrium points above the dashed line n_0. Two of them become stable in the domain bounded by the curves h_1, h_2 (Andronov–Hopf bifurcations).

Complex behavior of the system (1) is observed in a wide range of parameter values beyond domains Ω_π, Ω_ψ and below h_1.

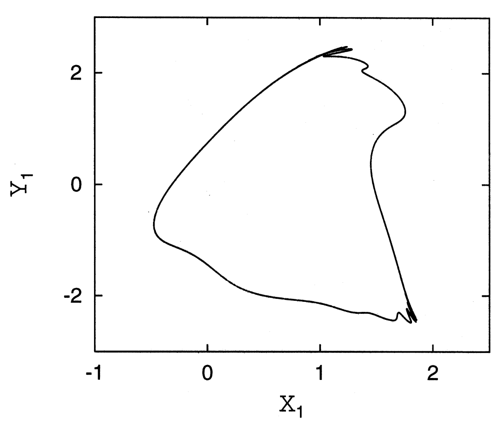
4. Transitions to Chaos

The key phenomena studied in Eqs. (1) are the irregular switchings between two quasisynchronized modes described in the Introduction. They appear in different scenarios of the transitions to chaos in (1) when the periodic orbits $\Gamma_{+\psi}$, $\Gamma_{-\psi}$ or $\Gamma'_{+\psi}$, $\Gamma'_{-\psi}$ lose their stability or disappear. If, immediately after the bifurcation point, the state of the system is not in the attraction basin of another stable trajectory, then one of the scenarios is observed numerically, e.g. in the transition across the boundary m_0 in the region between curves r_0 and r_1 (Scenario I) and across the boundary r_1 in the region between curve m_0 and point F (Scenario II). Based on numerical studies, we come to the following conclusions concerning these two scenarios.

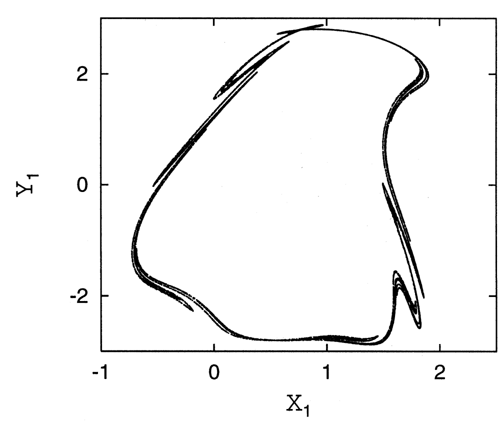
Scenario I. The one-parametric bifurcation diagram and Poincaré maps in Fig. 4 illustrate rearrangement of attracting sets in (1) for $\gamma = 0.5$ and quasistatic variation of parameter α. X_1 and Y_1 are the values of coordinates x_1, y_1 taken at the intersection of the trajectory with the hyperplane $\Pi : y_1 + y_2 = 0$. The solid curves in the diagram correspond to stable periodic orbits, and the diffusive point distribution to complex oscillations. One can see from Fig. 4(a) that complex oscillations are established immediately after intersection



(a)



(b)

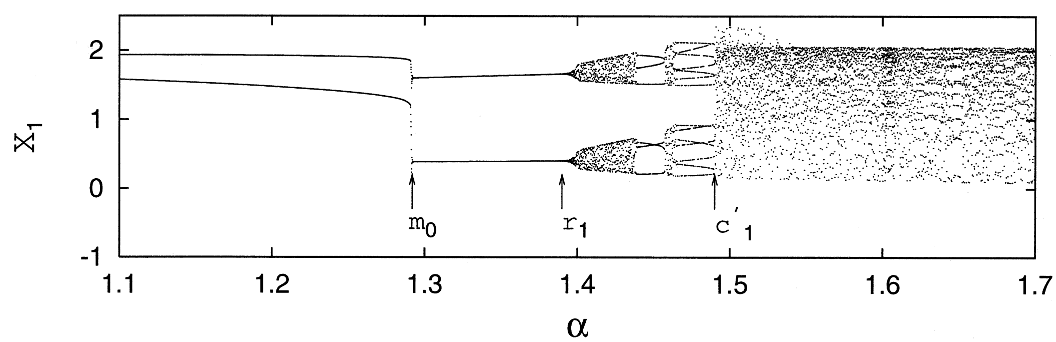


(c)

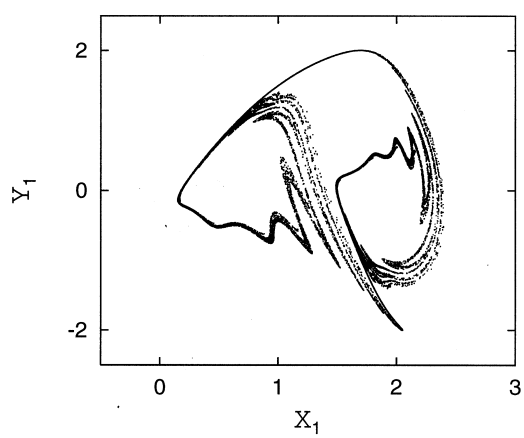
Fig. 4. Transition to chaos via intermittency I (Scenario I): (a) one-parametric bifurcation diagram for $\gamma = 0.5$ and slowly increasing α; the arrow indicates the bifurcation value of parameter α corresponding to the boundary m_0 in Fig. 2; two diagrams corresponding to the orbits $\Gamma_{+\psi}$ and $\Gamma_{-\psi}$ are collated to the left of m_0; (b) trajectories of Poincaré map for $\alpha = 2.8$; (c) $\alpha = 3.8$.

of the boundary m_0 and exist in a wide range of values of α. The structure of the attracting set in the neighborhood of the boundary m_0, the cross-section of which by the hyperplane Π is shown in Fig. 4(b), suggests that a chaotic attractor is born at the saddle-node bifurcation of the periodic orbits $\Gamma_{+\psi}$ and $\Gamma_{-\psi}$ (multipliers +1) on the surface of a nonsmooth torus [Afraimovich *et al.*, 1991]. This scenario corresponds to the transition to chaos via intermittency I [Bergé *et al.*, 1988].

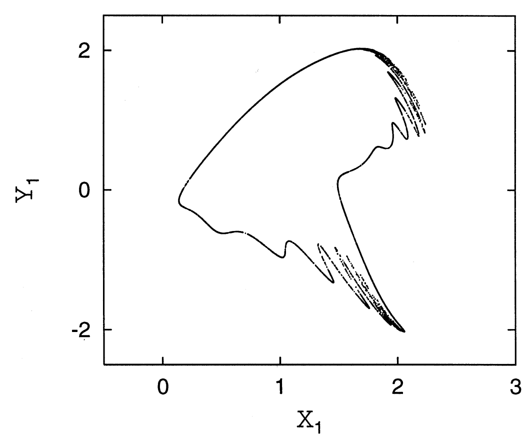
Scenario II. The periodic orbits $\Gamma'_{+\psi}$, $\Gamma'_{-\psi}$ give birth to stable tori when the boundary r_1 is intersected between curve m_0 and point F, as the parameter α is changed in the direction indicated by arrow II in Fig. 2. The tori exist in the near vicinity of the bifurcation boundary r_1. With the increase of α, the tori break down at the boundary c'_1 as a result of formation of resonance periodic orbits which undergo a complex of related bifurcations until an "attracting homoclinic structure" is formed [Neimark & Landa, 1987]. A one-parametric bifurcation diagram and typical trajectories of Poincaré maps are given in Fig. 5 for $\gamma = 1.0$. One can see from the figure that a chaotic attractor similar to the one described in Scenario I is formed with the increase of α. The chaotic attractors appearing in (1) by Scenarios I and II are symmetric relative to the commutation of variables $x_1 \leftrightarrow x_2$, $y_1 \leftrightarrow y_2$ due to the symmetry of the system itself. Symmetry breaking of a chaotic attractor in a system of two van der Pol oscillators with nonidentical nonlinear couplings was considered in [Pastor-Díaz & López-Fraguas, 1995]. In the present paper, not a chaotic attractor but a periodic orbit undergoes the symmetry-breaking bifurcation in the system (1) and chaos multistability is not observed for the parameter values considered.



(a)

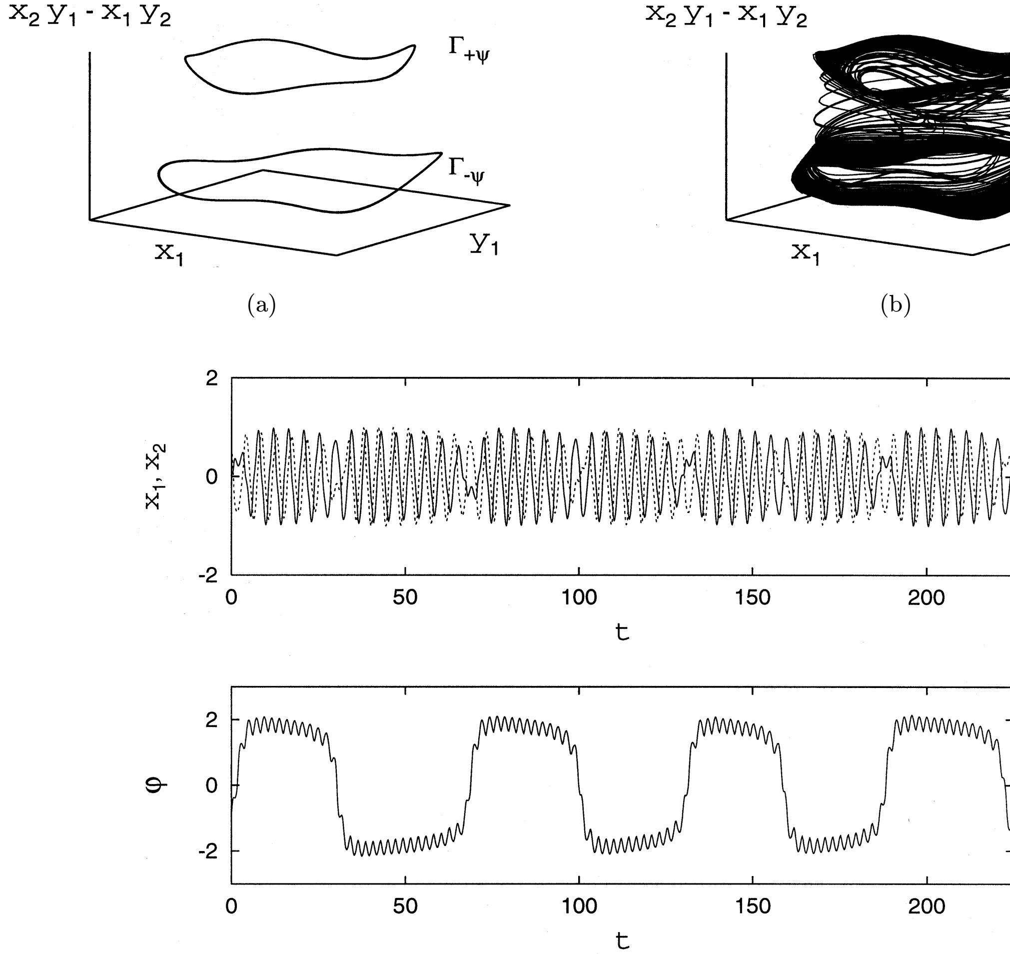


(b)



(c)

Fig. 5. Transition to chaos via breaking of tori and formation of attracting homoclinic structure (Scenario II): (a) one-parametric bifurcation diagram for $\gamma = 1.0$ and slowly increasing α; the arrows indicate the values of parameter α corresponding to the bifurcation boundaries shown in Fig. 2; diagrams corresponding to the orbits $\Gamma_{+\psi}$, $\Gamma_{-\psi}$ and $\Gamma'_{+\psi}$, $\Gamma'_{-\psi}$ are collated to the left of c'_1; (b) trajectories of Poincaré map for $\alpha = 1.42$; (c) $\alpha = 1.525$.



(c)

Fig. 6. Regime of chaotic switching between oscillations with phase advance and phase lag: (a) stable periodic orbits $\Gamma_{+\psi}, \Gamma_{-\psi}$ at subcritical values of parameters $\gamma = 0.5, \alpha = 2.3$; (b) chaotic attractor at $\gamma = 0.5$, $\alpha = 2.5$; (c) chaotic time series $x_1(t)$, $x_2(t)$ (solid and dashed lines, respectively) and the phase difference between them.

5. Chaotic Oscillations with Two Metastable States

Analysis of Scenarios I and II of the transitions to chaos shows that chaotic motions may be established in the system and persist in a wide range of coupling parameters. The chaotic attractor that determines these oscillations is formed near the bifurcation boundaries corresponding to the birth of saddle-node periodic orbits or to the formation of an attracting homoclinic structure. A complex limiting set may be born in either case. The trajectories on this chaotic attractor stay relatively long in the neighborhood of the periodic orbits $\Gamma'_{+\psi}$, $\Gamma'_{-\psi}$ or $\Gamma_{+\psi}$, $\Gamma_{-\psi}$ that have lost stability (or disappeared) and that correspond to synchronous oscillations of partial subsystems with phase shifts of different signs [see Figs. 6(a) and 6(b)]. In the present paper, the phase difference $\varphi(t) = \varphi_2(t) - \varphi_1(t)$ of oscillations $x_1(t)$, $x_2(t)$ was calculated from the definition of instantaneous (Hilbert) phase [Panter, 1965], which is quite correct in the case of oscillations close to the harmonic ones [Pikovsky *et al.*, 1997].

The most pronounced "switchings" between two distinct nearly periodic metastable states are observed in Scenario I as shown in Fig. 6. In Scenario II, irregular switching between two *chaotic* states is observed but none of them can any longer be associated with the regime of subsystem synchronization.

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References

Abarbanel, H. D. I., Rabinovich, M. I., Selverston, A., Bazhenov, M. V., Huerta, R., Sushchik, M. M. & Rubchinsky, L. L. [1996] "Synchronization in neural networks," *Physics–Uspekhi* **39**, 337–362.

Afraimovich, V. S., Arnold, V. I., Ilyashenko, Yu. S. & Shilnikov, L. P. [1991] *Theory of Bifurcations in Dynamic Systems* (Springer-Verlag, NY).

Bergé, P., Pomeau, Y. & Vidal, C. [1988] *L'Ordre dans le Chaos* (Hermann, Paris).

Buchanan, J. J., Kelso, J. A. S. & Fuchs, A. [1995] "Coordination dynamics of trajectory formation," *Biol. Cybern.* **74**(1), 41–54.

Fuchs, A., Jirsa, V. K., Haken, H. & Kelso, J. A. S. [1996] "Extending the HKB model of coordinated movement to oscillators with different eigen frequencies," *Biol. Cybern.* **74**(1), 21–30.

Gurfinkel, V. S., Levik, Yu. S., Kazennikov, O. V. & Selionov, V. A. [1998] "Locomotor-like movements evoked by leg muscle vibration in humans," *European J. Neurosci.* **10**, 1608–1612.

Haken, H. [1987] *Advanced Synergetics*, 2nd edition (Springer-Verlag, Berlin).

Haken, H. [1988] *Information and Self-Organization* (Springer-Verlag, Berlin).

Haken, H., Kelso, J. A. S. & Bunz, H. [1985] "A theoretical model of phase transitions in human hand movements," *Biol. Cybern.* **51**, 347–356.

Kelso, J. A. S., Delcolle, J. D. & Schöner, G. [1990] "Action-perception as a pattern formation process," in *Attention and Performance XIII*, ed. Jeannerod, M. (Erlbaum, Hillsdale, NJ), pp. 136–169.

Kelso, J. A. S. & Jeka, J. J. [1992] "Symmetry breaking dynamics of human multilimb coordination," *J. Exp. Psychol. Hum. Percept. Perform.* **18**, 645–668.

Kelso, J. A. S., Scholz, J. P. & Schöner, G. [1986] "Nonequilibrium phase transitions in coordinated biological motions: Critical fluctuations," *Phys. Lett.* **118**, 279–284.

Kozlov, A. K., Huerta, R., Rabinovich, M. I., Abarbanel, H. D. I. & Bazhenov, M. V. [1997] "Neuronal ensembles with balanced interconnection as receptors of information," *Physics-Doklady* **42**(12), 664–669.

Molkov, Ya. I., Sushchik, M. M., Kuznetsov, A. S., Kozlov, A. K. & Zakharov, D. G. [1998] "Dynamical model for locomotor-like movements of humans," *Proc. NOLTA'98, Le Regent*, Crans-Montana, Switzerland, September 14–17, 1998, Vol. 3, pp. 1325–1328.

Neimark, Yu. I. & Landa, P. S. [1987] *Stochastic and Chaotic Oscillations* (Nauka, Moscow), (in Russian).

Panter, P. [1965] *Modulation, Noise, and Spectral Analysis* (McGraw-Hill, NY).

Pastor-Díaz, I. & López-Fraguas, A. [1995] "Dynamics of two van der Pol oscillators," *Phys. Rev.* **E52**(2), 1480–1489.

Pikovsky, A. S., Rosenblum, M. G., Osipov, G. V. & Kurths, J. "Phase synchronization of chaotic oscillators by external driving," *Physica* **D104**, 219–238.

Schöner, G. & Kelso, J. A. S. [1988] "Dynamic pattern generation in behavior and neural systems," *Science* **239**, 1513–1520.

Sternad, D., Turvey, M. T. & Schmidt, R. C. [1992] "Average phase difference theory and 1:1 phase entrainment in interlimb coordination," *Biol. Cybern.* **67**, 223–231.