An L^∞ Bound for the Neumann Problem of the Poisson Equations

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Abstract

In this paper we establish an L^∞-bound for the Neumann problem of the Poisson equations. We first develop some estimates for the bounds of solutions in several spaces using Poincarés inequality, Trace theorem and Sobolev's embedding theorem, and then prove our main theorem utilizing the De Giorgi-Nash estimates.

Keywords. L^∞-bound, Poincaré's inequality, Sobolev's theorem, De Giorgi-Nash estimates.

1 Introduction

Suppose Ω is a bounded and connected subset of $\mathbb{R}^n$. Furthermore, Ω satisfies consistent cone condition, and its boundary $\partial\Omega$ with external normal vector $\mathbf{n}$ satisfies local Lipschitiz condition. A-priori maximum estimate for the Neumann problem of the Poisson equations

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega \\ \dfrac{\partial u(x)}{\partial \mathbf{n}} = g(x) & \text{on } \partial\Omega \end{cases} \tag{1}$$

has not been established yet. Our purpose in this paper is to provide an L^∞-bound for the solutions of the Neumann problem.

In this paper we assume that $f(x)$ and $g(x)$ in (1) meet

$$\int_\Omega f(x)dx + \int_{\partial\Omega} g(x)dS(x) = 0. \tag{2}$$

This is a reasonable assumption, since if there is a solution u to (1), then we should have

$$\int_\Omega f(x)dx = -\int_\Omega \Delta u(x)dx = -\int_{\partial\Omega} \frac{\partial u(x)}{\partial \mathbf{n}} dS(x) = -\int_{\partial\Omega} g dS(x). \tag{3}$$

Furthermore, we define two constants F and G as follows,

$$F := \|f\|_{L^\infty(\Omega)} \quad \text{and} \quad G := \|g\|_{L^\infty(\partial\Omega)} \tag{4}$$

and let $u_\Omega := \frac{1}{|\Omega|}\int_\Omega u dx$ be the average of u in Ω. We want to show that $u - u_\Omega$ has a similar a-priori maximum estimate as the third problem of the Poisson equation:

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Theorem 1.1. *There exists a constant C depending only on Ω and its dimension n, such that*

$$\|u - u_\Omega\|_{L^\infty(\Omega)} \le C(F+G) \tag{5}$$

for all $u \in H^2(\Omega) \cap H^1(\bar{\Omega})$ that solves (1).

We first quote several important theorems and corollaries that will be used in this paper.

Theorem 1.2 (Poincaré's inequality)**.** *Let Ω be a bounded, connected subset of $\mathbb{R}^n$ with $\partial\Omega$ satisfying local Lipschitz condition. Assume $1 \le p < \infty$. Then there exists a constatnt C depending only on n, p and Ω such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C\|Du\|_{L^p(\Omega)} \tag{6}$$

for all $u \in W^{1,p}(\Omega)$.

Proof. See, e.g. [2, 6, 5].

Definition 1.3. *For constant p, define p^* by*

$$p^* = \begin{cases} \dfrac{np}{n-p} & \text{if } p < n, \\ q & \text{if } p = n, p \le q < \infty, \\ \infty & \text{if } p > n. \end{cases} \tag{7}$$

Theorem 1.4 (Sobolev's Embedding Theorem)**.** *Let Ω be a bounded, connected subset of $\mathbb{R}^n$ and satisfies consistent cone condition. Assume that $1 \le p \le n$, then there exists a constant C depending only on n, p and Ω (if $p = n$ there is also an q), such that*

$$\|u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \tag{8}$$

for all $u \in W^{1,p}(\Omega)$.

Proof. See [1].

Theorem 1.5 (Trace Theorem)**.** *Let Ω be a bounded set of $\mathbb{R}^n$. Then there exists a linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\Omega) \tag{9}$$

and a constant C depending only on n, p and Ω, such that

$$\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \tag{10}$$

for all $u \in W^{1,p}(\Omega)$.

Proof. See [3, 4].

Corollary 1.6. *Let Ω be a bounded set of $\mathbb{R}^n$. Then there exists a constant C such that*

$$\|u\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}. \tag{11}$$

for all $u \in W^{1,p}(\Omega)$.

Proof. Let $\tilde{u} \in W^{1,p}(\Omega)$ satisfy (1) $\tilde{u} = u$ in $\Omega \setminus \partial\Omega$, and (2) $T\tilde{u} = u$ on $\partial\Omega$. So there is

$$\|u\|_{L^p(\partial\Omega)} = \|T\tilde{u}\|_{L^p(\partial\Omega)} \leq C\|\tilde{u}\|_{W^{1,p}(\Omega)}. \tag{12}$$

Note that $\partial\Omega$ has measure 0 in $\mathbb{R}^n$, we obtain

$$\|u\|_{W^{1,p}(\Omega)} = \|\tilde{u}\|_{W^{1,p}(\Omega)}, \tag{13}$$

and the conclusion follows.

Proposition 1.7. *Let Ω be a bounded, connected subset of $\mathbb{R}^n$ and satisfy consistent cone condition, with $\partial\Omega$ satisfying local Lipschitz condition. Assume that Neumman problem*

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega \\ \dfrac{\partial u(x)}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

and

$$\begin{cases} -\Delta u(x) = 0 & \text{in } \Omega \\ \dfrac{\partial u(x)}{\partial \mathbf{n}} = g(x) & \text{on } \partial\Omega \end{cases} \tag{15}$$

both have a-priori maximum estimate, that is, there exist two constants C_1 and C_2 depending only on n and Ω, such that

$$\|v - v_\Omega\|_{L^\infty(\Omega) \leq C_1 F} \tag{16}$$

and

$$\|w - w_\Omega\|_{L^\infty(\Omega) \leq C_2 G} \tag{17}$$

for all $v \in H^2(\Omega) \cap H^1(\bar{\Omega})$ that solves (14) *and $w \in H^2(\Omega) \cap H^1(\bar{\Omega})$ that solves* (15)*.*

Proof. Assume that u, v solve (1) and (14), respectively. In addition, $u_\Omega = 0$, $v_\Omega = 0$. Since (1) is linear, we know that $u - v$ solves (15), and $(u - v)_\Omega = 0$. So we obtain

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\leq \|u - v\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \\ &\leq C_2 G + C_1 F \\ &\leq C(F + G). \end{aligned} \tag{18}$$

Remark. In the following part of this paper, we will prove Theorem 1.1 in $g(x) = 0$ and $f(x) = 0$ cases separately, and combine the results with proposition 1.7 to prove theorem 1.1.

For notation simplicity, we assume that $u_\Omega = 0$ and let C be a constant depending on n and Ω unless otherwise noted. Note that the exact value may vary in different situation.

2 Case that $g(x) = 0$

In this section we prove Theorem 1.1 in the case that $g(x) = 0$. Also, we consider the $n > 2$ and $n = 2$ cases separately.

2.1 The dimension $n > 2$

Let $p \in \mathbb{R}$ and $p \geq 2$, from $-\Delta u = f$, we know

$$-\int_\Omega |u|^{p-2} u \Delta u dx = \int_\Omega |u|^{p-2} u f dx. \tag{19}$$

In addition, there is

$$\begin{aligned} -\int_\Omega |u|^{p-2} u \Delta u dx &= \frac{4(p-1)}{p^2} \int_\Omega |Du^{p/2}|^2 dx - \int_{\partial\Omega} |u|^{p-2} u \frac{\partial u}{\partial \mathbf{n}} dS(x) \\ &= \frac{4(p-1)}{p^2} \int_\Omega |Du^{p/2}|^2 dx. \end{aligned} \tag{20}$$

From the two equations above we can obtain the following inequality

$$\int_\Omega |Du^{p/2}|^2 dx \leq \frac{p^2 F}{4(p-1)} \int_\Omega |u|^{p-1} dx \tag{21}$$

Using Hölder's inequality, we can obtain

$$\int_\Omega |Du^{p/2}|^2 dx \leq \frac{p^2 |\Omega|^{1/p} F}{4(p-1)} \left(\int_\Omega |u|^p dx \right)^{(p-1)/p}. \tag{22}$$

From Theorem 1.4 we know there is

$$\begin{aligned} \left(\int_\Omega |Du^{p/2}|^{2^*} dx \right)^{2/2^*} &\leq C \left(\int_\Omega |u|^p dx + \int_\Omega |Du^{p/2}|^2 dx \right) \\ &\leq C \left(\int_\Omega |u|^p dx + \frac{p^2 |\Omega|^{1/p} F}{4(p-1)} \left(\int_\Omega |u|^p dx \right)^{(p-1)/p} \right). \end{aligned} \tag{23}$$

Let $\tilde{p} = np/(n-2)$, then there is

$$\left(\int_\Omega |u^{\tilde{p}}|^2 dx \right)^{1/\tilde{p}} \leq C^{1/p} \left(\left(\int_\Omega |u|^p dx \right)^{1/p} + \frac{p^2 |\Omega|^{1/p} F}{4(p-1)} \right)^{1/p} \left(\int_\Omega |u|^p dx \right)^{(p-1)/p^2} \tag{24}$$

Define a sequence $\{p_k\}_{k=0}^\infty$ as follows,

$$p_k := 2 \left(\frac{n}{n-2} \right)^k, \quad k = 0, 1, 2, \cdots. \tag{25}$$

Then we know that

$$S(n) := \sum_{k=0}^\infty p_k = \sum_{k=0}^\infty 2 \left(\frac{n}{n-2} \right)^k = \frac{n}{4}. \tag{26}$$

For notation simplicity, we use $\|u\|_{p_k}$ to denote $\|u\|_{L^{p_k}(\Omega)}$. We can obtain the estimate based on (24) as follows,

$$\|u\|_{p_{k+1}} \leq C^{1/p_k} (\|u\|_{p_k} + \alpha p_k F)^{1/p_k} \|u\|_{p_k}^{(p_k-1)/p_k}, \tag{27}$$

where $\alpha = (1/2) \cdot \max\{1, |\Omega|\}$. First, since there is $-\Delta u = f$, we can obtain that

$$-\int_\Omega u\Delta u dx = \int_\Omega |Du|^2 dx \le F \int_\Omega |u| dx. \tag{28}$$

From Theorem 1.2, we know

$$\int_\Omega |u|^2 dx \le C \int_\Omega |Du|^2 dx. \tag{29}$$

Using Hölder's inequality, we get

$$\|u\|_{p_0} \le CF. \tag{30}$$

From (27) we can obtain that

$$\|u\|_{p_1} \le CF \quad \text{and} \quad \|u\|_{p_2} \le CF. \tag{31}$$

Suppose that there is an estimate as follows,

$$\|u\|_{p_k} \le \alpha\beta C^{\sum_{j=0}^{k-1} \frac{1}{p_j}} (k-1) p_k F, \tag{32}$$

such that it is true for $k = 2$, and $\beta \ge 1$ is a constant depending only on n and Ω. Thus, for all $k \in N$ (32) is true: if (32) is true for some k, then we can show that

$$\begin{aligned} \|u\|_{p_{k+1}} &\le C^{1/p_k} \left(\alpha\beta C^{\sum_{j=0}^{k-1} \frac{1}{p_j}} (k-1) p_k F + \alpha p_k F \right) \\ &\le \alpha\beta C^{\sum_{j=0}^{k} \frac{1}{p_j}} k p_{k+1} F \end{aligned} \tag{33}$$

So by induction, (32) is true for all k. Note that here we assumed that $C > 1$. Otherwise the estimate is much simpler and the results can follow directly. Plugging (32) into (27), we obtain that

$$\begin{aligned} \|u\|_{p_{k+1}} &\le C^{1/p_k} \left(\alpha\beta C^{\sum_{j=0}^{k-1} \frac{1}{p_j}} (k-1) p_k F + \alpha p_k F \right)^{1/p_k} \|u\|_{p_k}^{(p_k-1)/p_k} \\ &\le (\alpha\beta C S(n) k p_k F)^{1/p_k} \|u\|_{p_k}^{(p_k-1)/p_k} \\ &\le (Ckp_k F)^{1/p_k} \|u\|_{p_k}^{(p_k-1)/p_k} \end{aligned} \tag{34}$$

Let $k = 1, 2, \cdots$,, we can obtain the relation between $\|u\|_{p_1}$ and $\|u\|_{p_k}$ as follows,

$$\|u\|_{p_k} \le A_k F^{\mu_k} \|u\|_{p_1}^{\lambda_k}, \forall k \ge 1. \tag{35}$$

Here

$$\begin{aligned} A_k &= C^{\lambda_k} \left\{ \left\{ [p_1^{\frac{1}{p_1}\frac{p_2-1}{p_2}} \times (2p_2)^{\frac{1}{p_2}}]^{\frac{p_3-1}{p_3}} \times \cdots \right\}^{\frac{p_{k-1}-1}{p_{k-1}}} \times [(k-1)p_{k_1}]^{\frac{1}{p_{k-1}}} \right\}, \\ \lambda_k &= \frac{p_1-1}{p_1}\frac{p_2-1}{p_2} \cdots \frac{p_{k-1}-1}{p_{k-1}}, \\ \mu_k &= \left\{ \left(\frac{1}{p_1}\frac{p_2-1}{p_2} + \frac{1}{p_2} \right) \frac{p_3-1}{p_3} + \cdots \right\} \frac{p_{k-1}-1}{p_{k-1}} + \frac{1}{p_{k-1}} \\ &= \left(\frac{1}{p_1}\frac{p_2-1}{p_2} \cdots \frac{p_{k-1}-1}{p_{k-1}} \right) + \cdots + \frac{1}{p_{k-1}} \\ &= 1 - \lambda_k. \end{aligned} \tag{36}$$

Note that $0 \leq \lambda_k < 1$ and λ_k is nonincreasing as $k \to \infty$, we know the limit

$$\lambda := \lim_{k\to\infty} \lambda_k \tag{37}$$

exists and depends only on n and Ω. Thus

$$\mu := \lim_{k\to\infty} \mu_k \tag{38}$$

also exists. In terms of A, we know

$$A_k \leq C^{\lambda_k} \prod_{j=1}^{k-1} (jp_j)^{1/p_j}, \tag{39}$$

and it is easy to prove that

$$\lim_{k\to\infty} \sum_{j=1}^{k-1} \frac{\log(jp_j)}{p_j} \tag{40}$$

exists. Thus we obtain that

$$A := \lim_{k\to\infty} A_k \leq C^{\lambda} \exp\left(\lim_{k\to\infty} \sum_{j=1}^{k-1} \frac{\log(jp_j)}{p_j}\right) \tag{41}$$

exists and depends only on n and Ω. Take the results back to (35) and let $k \to \infty$, we can get

$$\|u\|_{p_\infty} \leq AF^{\mu} \|u\|_{p_1}^{\lambda}. \tag{42}$$

Since $\|u\|_{p_1} \leq CF$, we now can conclude that

$$\|u\|_{L^\infty(\Omega)} \leq CF. \tag{43}$$

2.2 The dimension $n = 2$

In this case we can also obtain (21). With 2^* in Theorem 1.4 being set to 4, we can get a similar inequality as (24) where $\tilde{p} = 2p$. So let p_k we still can get

$$\|u\|_{L^\infty(\Omega)} \leq CF \tag{44}$$

3 Case that $f(x) = 0$

In this case we prove Theorem 1.1 in the case that $g(x) = 0$. Again, we consider two cases $n > 2$ and $n = 2$ separately.

3.1 The dimension $n > 2$

Since there is $-\Delta u = 0$, we have

$$-\int_\Omega |u|^{p-2} u \Delta u dx = \frac{4(p-1)}{p^2} \int_\Omega |Du^{p/2}|^2 dx - \int_{\partial\Omega} |u|^{p-2} ugdS(x) = 0 \tag{45}$$

Thus there is

$$\int_\Omega |Du^{p/2}|^2 dx \le \frac{p^2 G}{4(p-1)} \int_{\partial\Omega} |u|^{p-1} u dS(x) \tag{46}$$

From corollary 1.6, we know that

$$\int_{\partial\Omega} |u|^{p-1} u dS(x) \le \left(\int_\Omega |u|^{p-1} dx + (p-1) \int_\Omega |u|^{p-2} |Du| dx \right). \tag{47}$$

Using Hölder's inequality, we have

$$\begin{aligned} \int_\Omega |u|^{p-2}|Du| dx &\le \left(\int_\Omega |u|^{p-2} dx \right)^{1/2} \left(\int_\Omega |u|^{p-2} |Du|^2 dx \right)^{1/2} \\ &\le \sqrt{\frac{G}{p-1}} \left(\int_\Omega |u|^{p-2} dx \right)^{1/2} \left(\int_{\partial\Omega} |u|^{p-1} dS(x) \right)^{1/2} \end{aligned} \tag{48}$$

Plugging (47) into (48), there is

$$\begin{aligned} &\int_{\partial\Omega} |u|^{p-1} dS(x) \\ &\le C \left(\int_\Omega |u|^{p-1} dx + \sqrt{G(p-1)} \left(\int_\Omega |u|^{p-2} dx \right)^{1/2} \left(\int_{\partial\Omega} |u|^{p-1} dS(x) \right)^{1/2} \right). \end{aligned} \tag{49}$$

Solving this inequality, we can obtain

$$\int_{\partial\Omega} |u|^{p-1} dS(x) \le 4C \int_\Omega |u|^{p-1} dx + CG^2(p-1) \int_\Omega |u|^{p-2} dx \tag{50}$$

Form Theorem 1.4, we can get

$$\begin{aligned} \left(\int_\Omega |u|^{\tilde{p}} dx \right)^{1/\tilde{p}} &\le C^{1/p} \left(\int_\Omega |u|^p dx + \int_\Omega |Du^{p/2}|^2 dx \right)^{1/p} \\ &\le C^{1/p} \left(\int_\Omega |u|^p dx + \frac{p^2 G}{4(p-1)} \int_{\partial\Omega} |u|^{p-1} dx \right)^{1/p} \end{aligned} \tag{51}$$

here $\tilde{p} = np/(n-2)$, and from (49) we can obtain

$$\begin{aligned} &\left(\int_\Omega |u|^{\tilde{p}} dx \right)^{1/\tilde{p}} \\ &\le C^{1/p} \left(\int_\Omega |u|^p dx + CG \frac{p^2}{p-1} \int_\Omega |u|^{p-1} dx + \frac{C^2 p^2 G^2}{4} \int_\Omega |u|^{p-2} dx \right)^{1/p}. \end{aligned} \tag{52}$$

Using Hölder's inequality, we have

$$\begin{aligned} \int_\Omega |u|^{p-1} dx &\le |\Omega|^{1/p} \left(\int_\Omega |u|^p dx \right)^{(p-1)/p} \\ \int_\Omega |u|^{p-2} dx &\le |\Omega|^{2/p} \left(\int_\Omega |u|^p dx \right)^{(p-2)/p} \end{aligned} \tag{53}$$

Plugging them into (49), we obtain that

$$
\begin{aligned}
\|u\|_{\tilde{p}} &\leq C^{1/p}\left(\|u\|_p^2 + CG\frac{p^2}{p-1}|\Omega|^{1/p}\|u\|_p + \frac{C^2p^2G^2|\Omega|^{2/p}}{4}\right)^{1/p}\|u\|_p^{(p-2)/p} \\
&\leq C^{1/p}\left(\|u\|_p + CGp|\Omega|^{1/p}\right)^{2/p}\|u\|_p^{(p-2)/p}
\end{aligned} \tag{54}
$$

We still let

$$
\alpha = \max\{1, |\Omega|\} \quad \text{and} \quad p_k = 2\left(\frac{n}{n-2}\right)^k, \quad k = 0, 1, 2, \cdots . \tag{55}
$$

Then we can get

$$
\|u\|_{p_{k+1}} \leq C^{1/p}\left(\|u\|_{p_k} + \alpha CGp_k\right)^{2/p_k}\|u\|_{p_k}^{(p_k-2)/p_k}, \quad k = 0, 1, 2, \cdots . \tag{56}
$$

Since there is $-\Delta u = 0$, we can obtain

$$
-\int_\Omega u\Delta u dx = \int_\Omega |Du|^2 dx - \int_{\partial\Omega} ugdS(x) = 0. \tag{57}
$$

Therefore, we have

$$
\int_\Omega |Du|^2 dx \leq G\int_{\partial\Omega}|u|dS(x). \tag{58}
$$

By Theorem 1.5, we know that

$$
\int_{\partial\Omega}|u|dS(x) \leq C\left(\int_\Omega |u|dx + \int_\Omega |Du|dx\right). \tag{59}
$$

Theorem 1.2 indicates that

$$
\begin{aligned}
\int_\Omega |u|dx &\leq C\int_\Omega |Du|dx \\
\int_\Omega |u|^2 dx &\leq C\int_\Omega |Du|^2 dx
\end{aligned} \tag{60}
$$

Thus we get

$$
\int_\Omega |u|^2 dx \leq CG. \tag{61}
$$

Using the same method as in section 2.1, form (56) we can prove that

$$
\|u\|_{L^\infty(\Omega)} \leq CG. \tag{62}
$$

3.2 The dimension $n = 2$

Similar to Section 2.2 we can readily show that

$$
\|u\|_{L^\infty(\Omega)} \leq C`G. \tag{63}
$$

in the case that $n = 2$.

4 Conclusion

Combine the results in Sections 2 and 3, and apply Proposition 1.7, we can prove Theorem 1.1.

References

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] Yazhe Chen, Lancheng Wu, *Elliptic Equation of Order 2*. Science Press, 1991.

[3] Lawrence C. Evans, *Partial Differential Equations*, Berkeley Mathematics Lecture Notes, University of California, Berkeley, 1994.

[4] David Gilbarg, Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 1997.

[5] Lishang Jiang, Yazhe Chen, *Method of Mathematical Physics*. Higher Education Press, 1986.

[6] Shulin Zhou, *Partial Differential Equations*. Peking University Press, 2005.