Learnable Descent Algorithm for Nonsmooth Nonconvex Image Reconstruction

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Abstract

We propose a general learning based framework for solving nonsmooth and nonconvex image reconstruction problems. We model the regularization function as the composition of the $l_{2,1}$ norm and a smooth but nonconvex feature mapping parametrized as a deep convolutional neural network. We develop a provably convergent descent-type algorithm to solve the nonsmooth nonconvex minimization problem by leveraging the Nesterov’s smoothing technique and the idea of residual learning, and learn the network parameters such that the outputs of the algorithm match the references in training data. Our method is versatile as one can employ various modern network structures into the regularization, and the resulting network inherits the guaranteed convergence of the algorithm. We also show that the proposed network is parameter-efficient and its performance compares favorably to the state-of-the-art methods in a variety of image reconstruction problems in practice.

1 Introduction

In the past several decades, variational methods and optimization techniques have been extensively studied for solving image reconstruction problems. In particular, a number of regularizers, including total variation (TV), $l_p$ norm ($p \in [0, 1]$), low rank, group Lasso, and nonlocal TV, have been proposed to improve the classical Tikhonov-type regularizers. Advanced optimization techniques were also developed to solve these nonsmooth and/or nonconvex reconstruction models for better computational efficiency, often by leveraging the special structures of the regularizers. However, the image reconstruction quality heavily depends on these hand-crafted regularizers, which are still overly simple and incapable to capture the complex structural features of images. Moreover, the slow convergence and subtle parameter tuning of the optimization algorithms have hindered their applications in real-world image reconstruction problems.

Recent years have witnessed the tremendous success of deep learning in a large variety of real-world application fields [16, 28, 35, 58]. At the heart of deep learning are the deep neural networks (DNNs) which have provable representation power and the substantial amount of data available nowadays for training these DNNs. Deep learning was mostly used as a data-driven approach since the DNNs can be trained with little or no knowledge about the underlying functions to be approximated. However, there are several major issues of such standard deep learning approaches: (i) Generic DNNs may fail to approximate the desired functions if the training data is scarce; (ii) The training of these DNNs are prone to overfitting, noises, and outliers; and (iii) The trained DNNs are mostly “blackboxes” without rigorous mathematical justification and can be very difficult to interpret.

1.1 Background of learnable optimization algorithms

To mitigate the aforementioned issues of DNNs, a class of learnable optimization algorithms (LOAs) has been proposed recently. The main idea of LOA is to map a known iterative optimization algorithm to a DNN. The DNN is restricted to have a small number of blocks, where each block (also called a phase) mimics one iteration of the algorithm but with certain components replaced by network layers, and the network

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parameter \( \theta \) of the DNN is learned such that the outputs of the DNN fit the desired solutions given in the training data.

Consider the standard setting of supervised learning with a set of \( N \) pairs of training data, \( \{(b^{(s)}, \hat{x}^{(s)}) : s \in [N]\} \), such that \( b^{(s)} \) is the input data of the DNN, for instance, a noisy and/or corrupted image or compressed or encoded data, and \( \hat{x}^{(s)} \) is the corresponding ground truth high quality image that the output of the DNN is expected to match. We study a general framework of LOA which can be described by a bi-level optimization problem, where the upper-level minimization is described by the loss function \( \mathcal{L} \), and the lower-level minimization forms the constraint, as follows:

\[
\min_{\theta} \frac{1}{N} \sum_{s=1}^{N} \mathcal{L}(x^{(s)}_{\theta}; x^{(s)}), \quad \text{s.t.} \quad x^{(s)}_{\theta} = \arg \min_{x \in \mathcal{X}} \{ \phi(x; b^{(s)}, \theta) := f(x; b^{(s)}, \theta) + r(x; \theta) \} \tag{1}
\]

where \( f \) is the data fidelity term to ensure that the reconstructed image \( x \) is faithful to the given data \( b \), and \( r \) is the regularization that may incorporate proper prior information of \( x \). The regularization \( r(\cdot; \theta) \) (and possibly \( f \) also) is realized as a DNN with parameter \( \theta \) to be learned. The loss function \( \mathcal{L} \) is to measure the (averaged) difference between \( x_\theta \), which is the minimizer of \( \phi(\cdot; b, \theta) \), and the given ground truth \( \hat{x} \) for every \( s \). The optimal parameter \( \theta \) of \( r \) is then obtained by solving the bi-level optimization \( (1) \). Typically, the actual minimizer \( x_\theta \) is replaced by the output of an LOA-based DNN (different from the DNN \( r \) which mimics an iterative optimization scheme for solving the lower-level minimization in the constraint of \( (1) \). LOA is also widely known as the "unrolling method," as the iteration scheme of the optimization algorithm is "unrolled" into multiple blocks of the LOA-based DNN. However, despite of their promising performance in practice, existing LOA-based DNNs only superficially resemble the steps of optimization algorithms, and hence they do not really yield a convergent algorithm or correspond to solving any interpretable variational model as the one in the constraint of \( (1) \). As a result, they lack theoretical justifications and convergence guarantees.

### 1.2 Our goal and approach

Our goal in this work is to develop a general LOA framework for solving nonsmooth and nonconvex image reconstruction problems. The proposed LOA has the following properties: Versatility—our method is flexible and allows users to plug in various kinds of deep neural networks for learning the objective function; Convergence—we can ensure convergence of our network as its architecture follows exactly the proposed algorithm; and Performance—our method can adaptively learn the regularization function from the training data, such that it is competitive and can even outperform the state-of-the-art methods in terms of both reconstruction accuracy and efficiency in practice.

To this end, we consider to learn the lower-level minimization in \( (1) \) for image reconstruction, such that its solution is close to the ground truth high quality image in the training data. Specifically, we use a composited structure of the regularization term as the \( l_{2,1} \) norm of a learnable feature mapping \( g \) realized by a deep neural network. Both \( f \) and \( g \) are smooth but (possibly) nonconvex, and the overall objective function is nonsmooth and nonconvex. We propose a descent-type algorithm to solve this nonsmooth nonconvex problem as follows: (i) We tackle the nonsmoothness by employing Nesterov’s smoothing technique [38] with automatic diminishing smoothing effect; (ii) We propose two successive residual-type updates (first on \( f \) and then on \( r \)), a key idea proven very effective in deep network training [26], and compute the convex combination of the two updates for the next iteration; and (iii) We employ an iterate selection policy based on objective function value to ensure convergence. Moreover, we prove that a subsequence generated by the proposed LOA has accumulation points and all of them are Clarke stationary points of the nonsmooth nonconvex problem.

### 1.3 Notations and organization

We denote \( [n] := \{1, \ldots, n\} \) for \( n \in \mathbb{N} \). We use regular lower-case letters to denote scalars and scalar-valued functions, and boldfaced lower-case letters for vectors and vector-valued functions. Unless otherwise noted, all vectors are column vectors. The inner product of two vectors \( x \) and \( y \) is denoted by \( \langle x, y \rangle \), and \( \|x\| = \|x\|_2 \) stands for the \( l_2 \) norm of \( x \) and \( \|A\| \) the induced \( l_2 \) norm of the matrix \( A \), and \( A^\top \) is the transpose of \( A \). For any set \( S \subset \mathbb{R}^n \), we denote \( \text{dist}(y, S) := \inf \{\|y - x\| : x \in S\} \), and \( S + y := \{x + y : x \in S\} \).
The remainder of this paper is organized as follows. In Section 2, we review the recent literature on LOA and general nonsmooth nonconvex optimization methods. In Section 3, we present our LOA based on a descent type algorithm to solve the nonsmooth nonconvex image reconstruction problem with comprehensive convergence analysis. In Section 4, we conduct a number of numerical experiments on natural and medical image dataset to show the promising performance of our proposed method. We provide several concluding remarks in Section 5.

2 Related Work

2.1 Learnable optimization algorithms

Learnable optimization algorithm (LOA) is a class of methods developed in recent years to imitate the iterations in optimization algorithms as blocks in a deep neural network with certain components replaced by learnable layers. Existing LOAs can be approximately categorized into two groups.

The first group of LOAs appeared in the literature are motivated by the similarity between the iterative scheme of a traditional optimization algorithm (e.g., proximal gradient algorithm) and a feed forward neural network. Provided instances of training data, such as ground truth solutions, an LOA replaces certain components of the optimization algorithm with parameters to be learned from the data. The pioneer work [24] in this group of LOAs is based on the well-known iterative shrinkage thresholding algorithm (ISTA) for solving the LASSO problem. In [24], a learned ISTA network, called LISTA, is proposed to replace $\Phi^\top$ by a weight matrix to be learned from instance data to reduce iteration complexity of the original ISTA. The asymptotic linear convergence rate for LISTA is established in [11] and [34]. Several variants of LISTA were also developed using low rank or group sparsity [49], $\ell_0$ minimization [56] and learned approximate message passing [7]. The idea of LISTA has been extended to solve composite problems with linear constraints, known as the differentiable linearized alternating direction method of multipliers (D-LADMM) [55]. These LOA methods, however, still employ handcrafted regularization and require closed form solution of the proximal operator of the regularization term.

To improve reconstruction quality, the other group of LOAs follows a different approach by replacing the lower-level minimization in (1) with a DNN whose structure is inspired by a numerical optimization algorithm for solving the minimization problem. For example, recall that the standard proximal gradient method applies a gradient descent step on the smooth function $\nabla f$ at the current iterate, and then the proximal mapping of $r$ to obtain the next iterate. In this case, an LOA can be obtained by replacing the proximal mapping of $r$ with a multilayer perceptrons (MLP), which can be learned using training data. As such, one avoids explicit formation of the regularization function $r$ for (1). This paradigm has been embedded into half quadratic splitting in DnCNN [63], ADMM in [9, 36] and primal dual methods in [2, 34, 36, 51] to solve the subproblems. To improve the generic black-box CNNs above, several LOA methods are proposed to incorporate certain prior knowledge about the solution in the design of $r$, then unroll numerical optimization algorithms as deep neural networks so as to preserve their efficient structures with proven efficiency, such as the ADMM-Net [57], Variational Network [25] and ISTA-Net [61]. These methods also prescribe the phase number $K$, and map each iteration of the corresponding numerical algorithm to one phase of the network, and learn specific components of the phases in the network using training data.

Despite of the promising performance in a variety of applications, the LOAs are only related to the original optimization algorithms superficially. These LOAs themselves do not follow any provably convergent algorithm or correspond the solution of any properly defined variational problem. Moreover, certain acceleration techniques proven to be useful for numerical optimization algorithms are not effective in their LOA counterparts. For example, the acceleration approach based on momentum [40] can significantly improve iteration complexity of traditional (proximal) gradient descent methods, but does not have noticeable improvement when deployed in LOAs. This can be observed by the similar performance of ISTA-Net [61] and FISTA-Net [63]. One possible reason is that the LOA has nonconvex components, for which a linear combination of past iterates is potentially a worse extrapolation point in optimization [32].

In parallel to the development of LOAs, performance of deep networks for image reconstruction is continuously being improved due to various network engineering and training techniques these years. For example, ISTA-Net$^+$ [61] employs the residual network structure [26] and results in substantially increased reconstruction accuracy over ISTA-Net. The residual structure is also shown to improve network performance in a
number of recent work, such as ResNet-v2 [27], WRN [59], and ResNeXt [54]. In image compression and
reconstruction, the learnable sampling module is always implemented as a single convolutional layer without
activation [46, 47, 62, 64, 66]. Efficient block compressed sensing for high-dimensional data [21] can be achieved
by controlling the convolution kernel size and stride [47]. Joint reconstruction to reduce blocky effects in
image compressive sensing is proposed in [47], and the learned sampling operator is shown to automatically
satisfy the orthogonal property in [62]. A multi-channel method is proposed in [66] to elaborate manage
the sensing resources by assigning different sampling ratio to image blocks. To obtain any desired sampling
ratio, a scalable sampling and reconstruction is achieved through a greedy measurement based selection
algorithm in [46].

2.2 Nonsmooth and nonconvex optimization

Nonsmooth nonconvex optimization has been extensively studied in recent years. One of the most common
methods is the proximal gradient method (also known as the forward-backward splitting or FBS) [4, 20, 39].
Several variants, including the accelerated proximal gradient method [33] and the FBS with an inertial
force [8, 42], are proposed to improve the convergence rate. Iteratively reweighted algorithms are developed to
iteratively solve a proximal operator problem [22, 43, 65]. These algorithms are effective when the nonsmooth
components involved in the subproblems are simple, i.e., the associated proximal operators have closed-form
or are easy to solve.

There are also a number of optimization algorithms developed for certain structured nonsmooth non-
convex problems. For instance, for a class of composite optimization problems involving \( h(c(x)) \), where \( h \)
is convex but nonsmooth, and \( c \) is smooth but (possibly) nonconvex, several linearized proximal type algo-
rithms are proposed such that \( c \) is approximated by its linearization. This renders a convex subproblem in
each iteration, which can be solved with exact [17, 30, 44] or inexact [23] function and gradient evaluations.

If the problem nonconvexity is due to the difference of convex (DC) functions, a number of optimization
methods known as DC algorithms (DCA) are developed [13, 11, 48, 53]. DCA approximates a nonconvex DC
problem by a sequence of convex ones, such that each iteration only involves a convex optimization problem.
Several DC decomposition and approximation methods have been introduced for different applications [15,
50]. Recently, the proximal linearized algorithms for DC programming are proposed to iteratively solve the
subproblem, where one of the convex components is replaced by its linear approximation together with a
proximal term [13, 48]. In [53], extrapolation is integrated into proximal linearized algorithm for possible
acceleration of the proximal DCA. In [41], an inexact generalized proximal linearized algorithms is developed,
where the Euclidean distance is replaced with a quasi distance in the proximal operator, and the proximal
point is replaced with an approximate proximal point. However, the subproblem of DCA may not have
closed-form solution and thus can still require inner iterations to solve.

To solve general nonconvex and nonsmooth problems, a common approach is using the smoothing tech-
nique, possibly in combination with gradient descent and line search strategy; see [6, 10, 45, 60] and the
references therein. The main idea of the smoothing technique to construct a class of smooth nonconvex
problems (e.g., using convolution) to approximate the original problem, where the approximation accuracy
(smoothing level) is controlled by a smoothing parameter. Then one can apply gradient descent or projected
gradient descent with line search to solve the approximate problem with a fixed smoothing level; then reduce
the smoothing parameter and solve the problem again, and so on.

The descent algorithm developed in this work for nonsmooth and nonconvex optimization is largely
inspired by [10]. However, unlike [10], our goal is to construct a deep image reconstruction network in the
framework of [1], where (part of) the objective function is unknown and needs to be learned from the training
data, such that the trained network has convergence guarantee in theory and compares to the state-of-the-art
favorably in reconstruction quality in practice.

3 LDA and Convergence Analysis

In this section, we propose a novel learnable descent algorithm (LDA) to solve the nonsmooth and nonconvex
optimization problem for image reconstruction:

\[
\min_{x \in X} \phi(x) := f(x) + r(x),
\]

(2)
where \( x \) is the image to be reconstructed, \( \mathcal{X} \) is the admissible set of \( x \), e.g., \( \mathcal{X} = \mathbb{R}^n \), \( n \) is the number of pixels in \( x \), \( f \) stands for the data fidelity term of \( x \), and \( r \) represents the regularization term to be learned.

We leverage the sparse selection property of \( l_1 \) norm and parametrize the regularization term \( r \) as the composition of the \( l_{2,1} \) norm and a feature extraction operator \( g(x) \) to be learned. Specifically, \( g: \mathbb{R}^n \rightarrow \mathbb{R}^{md} \) such that \( g(x) = (g_1(x), \ldots, g_m(x)) \), where \( g_i(x) \in \mathbb{R}^d \) is the \( i \)-th feature vector of \( x \) for \( i = 1, \ldots, m \). That is, we set \( r \) in (3) to

\[
r(x) := \|g(x)\|_{2,1} = \sum_{i=1}^m \|g_i(x)\|. \tag{3}
\]

In this paper, \( g \) is realized by a deep neural network whose parameters are learned from training data. The regularization \( r \) in (3) can be interpreted as follows: we learn a smooth nonlinear mapping \( g \) to extract sparse features of \( x \), and apply the \( l_{2,1} \)-norm which has proven to be a robust and effective sparse feature regularization. In addition, we make several assumptions on \( f \) and \( g \) throughout this work.

- **Assumption 1 (A1)** \( f \) is differentiable and (possibly) nonconvex, and \( \nabla f \) is \( L_f \)-Lipschitz continuous.
- **Assumption 2 (A2)** Every component of \( g \) is differentiable and (possibly) nonconvex, \( \nabla g \) is \( L_g \)-Lipschitz continuous, and \( \sup_{x \in X} \|\nabla g(x)\| \leq M \) for some constant \( M > 0 \).
- **Assumption 3 (A3)** \( \phi \) is coercive, and \( \phi^* = \min_{x \in X} \phi(x) > -\infty \).

**Remarks.** The assumptions (A1)–(A3) are mild for imaging applications. The Lipschitz continuity of \( \nabla f \) and \( \nabla g \) is standard in optimization and most imaging applications; the smoothness of \( g \) and boundedness of \( \nabla g \) are satisfied for all standard deep neural networks with smoothly differentiable activation functions such as sigmoid, tanh, and elu; and the coercivity of \( \phi \) generally holds in image reconstruction, e.g., the DC component providing overall image intensity information (e.g., \( \|x\|_1 \)) is contained in the data, and deviation from this value makes \( f \) value tend to infinity.

Other than the requirement in A2, the design of network architecture and the choice of activation functions in \( g \) are rather flexible. A typical choice of \( g \) is a convolutional neural network (CNN), which maps an input image \( x \in \mathbb{R}^n \) (gray-scale image with \( n \) pixels) to a collection of \( m \) feature vectors \( \{g_i(x) : 1 \leq i \leq m \} \subset \mathbb{R}^d \).

### 3.1 Smooth Approximation of Nonsmooth Regularization

To tackle the nonsmooth and nonconvex regularization term \( r(x) \) in (3), we first employ Nesterov’s smoothing technique [38] to smooth the \( l_{2,1} \) norm in (3) for any fixed \( g(x) \):

\[
r(x) = \max_{y \in \mathcal{Y}} \langle g(x), y \rangle, \tag{4}
\]

where \( y \in \mathcal{Y} \) is the dual variable, \( \mathcal{Y} \) is the dual space defined by

\[
\mathcal{Y} := \{ y = (y_1, \ldots, y_m) \in \mathbb{R}^{md} \mid y_i = (y_{i1}, \ldots, y_{id}) \in \mathbb{R}^d, \|y_i\| \leq 1, i \in [m] \}.
\]

For any \( \varepsilon > 0 \), we consider the smooth version \( r_\varepsilon \) of \( r \) by perturbing the dual form (4) as follows:

\[
r_\varepsilon(x) = \max_{y \in \mathcal{Y}} \langle g(x), y \rangle - \frac{\varepsilon}{2} \|y\|^2. \tag{5}
\]

Then one can readily show that

\[
r_\varepsilon(x) \leq r(x) \leq r_\varepsilon(x) + \frac{m\varepsilon}{2}, \quad \forall x \in \mathbb{R}^n. \tag{6}
\]

Note that the perturbed dual form in (5) has a closed form solution: denoting

\[
y_\varepsilon^* = \arg \max_{y \in \mathcal{Y}} \langle g(x), y \rangle - \frac{\varepsilon}{2} \|y\|^2, \tag{7}
\]
then solving (7), we obtain the closed form of \( y^*_i = ((y^*_i)_1, \ldots, (y^*_i)_m) \) where

\[
(y^*_i)_i = \begin{cases} 
\frac{1}{\| g_i(x) \|} g_i(x), & \text{if } \| g_i(x) \| \leq \varepsilon, \\
\text{otherwise,} & \text{for } i \in [m].
\end{cases}
\]

Plugging (8) back into (5), we have

\[
r_\varepsilon(x) = \sum_{i \in I_0} \frac{1}{2\varepsilon} \| g_i(x) \|^2 + \sum_{i \in I_1} \left( \| g_i(x) \| - \frac{\varepsilon}{2} \right),
\]

where the index set \( I_0 \) and its complement \( I_1 \) at \( x \) for the given \( g \) and \( \varepsilon \) are defined by

\[
I_0 = \{ i \in [m] \mid \| g_i(x) \| \leq \varepsilon \}, \quad I_1 = [m] \setminus I_0.
\]

Moreover, it is easy to show from (9) that

\[
\nabla r_\varepsilon(x) = \nabla g(x)^T y^*_i = \sum_{i \in I_0} \nabla g_i(x)^T \frac{g_i(x)}{\varepsilon} + \sum_{i \in I_1} \nabla g_i(x)^T \frac{g_i(x)}{\| g_i(x) \|},
\]

where \( \nabla g_i(x) \in \mathbb{R}^{d \times n} \) is the Jacobian of \( g_i \) at \( x \).

The smoothing technique above yields a smooth approximation of the nonsmooth function \( r(x) \), which allows for rigorous analysis of iteration complexity and provable asymptotic convergence to the original nonsmooth problem (2), as we will show in Section 3.3.

### 3.2 Proposed Descent Algorithm

In this subsection, we propose a novel descent type algorithm for solving the minimization problem (2) with the regularization \( r \) defined in (3). The idea is to apply a modified gradient descent algorithm to minimize the objective function \( \phi \) with the nonsmooth \( r \) replaced by the smooth \( r_\varepsilon \) as follows:

\[
\phi_\varepsilon(x) := f(x) + r_\varepsilon(x),
\]

with \( \varepsilon \) automatically decreasing to 0 as the iteration progresses. Note that \( \phi_\varepsilon \) in (11) is differentiable since both \( \nabla f \) and \( \nabla r_\varepsilon \) (defined in (10)) exist. Moreover, \( \phi_\varepsilon(x) \leq \phi(x) \leq \phi_\varepsilon(x) + \frac{\varepsilon}{2} \) for any \( x \in \mathcal{X} \) due to (6).

In light of the substantial improvement in practical performance by ResNet [26], we choose to split \( f \) and \( r_\varepsilon \) and perform two residual type updates as follows: In the \( k \)-th iteration with \( \varepsilon = \varepsilon_k > 0 \), we first compute

\[
z_{k+1} = x_k - \alpha_k \nabla f(x_k),
\]

where \( \alpha_k \) is the step size to be specified later. Then we compute two candidates for \( x_{k+1} \), denoted by \( u_{k+1} \) and \( v_{k+1} \), as follows:

\[
u_{k+1} = \arg \min_x (\nabla f(x_k), x - x_k) + \frac{1}{2\alpha_k} \| x - x_k \|^2 + (\nabla r_\varepsilon_k(z_{k+1}), x - z_{k+1}) + \frac{1}{2\beta_k} \| x - z_{k+1} \|^2,
\]

\[
u_{k+1} = \arg \min_x (\nabla f(x_k), x - x_k) + (\nabla r_\varepsilon_k(x_k), x - x_k) + \frac{1}{2\alpha_k} \| x - x_k \|^2,
\]

where \( \beta_k \) is another step size along with \( \alpha_k \). Note that both minimization problems in (13a) and (13b) have closed form solutions:

\[
u_{k+1} = \frac{\alpha_k}{2\alpha_k + \beta_k} \nabla r_\varepsilon_k(x_k)
\]

\[
u_{k+1} = \frac{1}{\alpha_k + \beta_k} \nabla r_\varepsilon_k(x_k)
\]

where \( \nabla r_\varepsilon_k \) is defined in (10) and \( \tau_k = \frac{\alpha_k \beta_k}{\alpha_k + \beta_k} \). Then we choose between \( u_{k+1} \) and \( v_{k+1} \) that has the smaller function value \( \phi_{\varepsilon_k} \) to be the next iterate \( x_{k+1} \):

\[
x_{k+1} = \begin{cases} 
u_{k+1} \text{ if } \phi_{\varepsilon_k}(u_{k+1}) \leq \phi_{\varepsilon_k}(v_{k+1}), \\
u_{k+1} \text{ otherwise.}
\end{cases}
\]
Algorithm 1 Learnable Descent Algorithm (LDA) for the Nonsmooth Nonconvex Problem \(^{(2)}\)

1: **Input:** Initial \(x_0\), \(0 < \gamma < 1\), and \(\varepsilon_0, \sigma > 0\). Maximum iteration \(K\) or tolerance \(\varepsilon_{\text{tol}} > 0\).

2: for \(k = 0, 1, 2, \ldots, K\) do

3: \(z_{k+1} = x_k - \alpha_k \nabla f(x_k)\)

4: \(u_{k+1} = z_{k+1} - r_{\varepsilon_k}(z_{k+1})\)

5: \(v_{k+1} = z_{k+1} - \alpha_k \nabla r_{\varepsilon_k}(x_k)\)

6: \(x_{k+1} = \begin{cases} u_{k+1} & \text{if } \phi_{\varepsilon_k}(u_{k+1}) \leq \phi_{\varepsilon_k}(v_{k+1}) \\ v_{k+1} & \text{otherwise} \end{cases}\)

7: If \(\|\nabla \phi_{\varepsilon_k}(x_{k+1})\| < \sigma \gamma \varepsilon_k\), set \(\varepsilon_{k+1} = \gamma \varepsilon_k\); otherwise, set \(\varepsilon_{k+1} = \varepsilon_k\).

8: If \(\sigma \varepsilon_k < \varepsilon_{\text{tol}}\), terminate.

9: end for

10: **Output:** \(x_{k+1}\).

This algorithm is summarized in Algorithm \(^{(1)}\). Line 7 of Algorithm \(^{(1)}\) presents a reduction criterion. That is, if the reduction criterion \(\|\nabla \phi_{\varepsilon_k}(x_{k+1})\| < \sigma \gamma \varepsilon_k\) is satisfied, then the smoothing parameter \(\varepsilon_k\) is shrunk by \(\gamma \in (0, 1)\).

In Algorithm \(^{(1)}\) \(u_{k+1}\) in (13a) can be considered as the convex combination of two successive residue-type updates: the first update is \(z_{k+1} = x_k - \alpha_k \nabla f(x_k)\) as defined by (12)—a gradient descent of \(f\) at \(x_k\); the second is \(p_{k+1} = z_{k+1} - \beta_k \nabla r_{\varepsilon_k}(z_{k+1})\)—another gradient descent of \(r_{\varepsilon_k}\) at \(z_{k+1}\); and finally \(u_{k+1} = \frac{1}{\alpha_k + \beta_k} z_{k+1} + \frac{\alpha_k}{\alpha_k + \beta_k} p_{k+1}\)—the convex combination of \(z_{k+1}\) and \(p_{k+1}\). In this case, \(f\) and \(r_{\varepsilon_k}\) are separated so they each can participate in a residual-type update, which is proven very effective for imaging applications \(^{(26)}\). Note that the \(v_{k+1}\) in (13b) is the standard gradient descent of \(\phi_{\varepsilon_k}\) at \(x\) to safeguard the convergence of the algorithm.

### 3.3 Convergence and Complexity Analysis

In this subsection, we provide a comprehensive convergence analysis with iteration complexity of the proposed Algorithm \(^{(1)}\) LDA. Since the objective function in (2) is nonsmooth and nonconvex, we adopt the notion of Clarke subdifferential \(^{(12)}\) (also called the limiting subdifferential or simply subdifferential) to characterize the optimality of solutions.

**Definition 3.1** (Clarke subdifferential). Suppose that \(f : \mathbb{R}^n \to (-\infty, +\infty]\) is locally Lipschitz. The Clarke subdifferential of \(f\) at \(x\) is defined as

\[
\partial f(x) := \left\{ w \in \mathbb{R}^n \mid \langle w, v \rangle \leq \limsup_{z \to x, t \downarrow 0} \frac{f(z + tv) - f(z)}{t}, \quad \forall v \in \mathbb{R}^n \right\}.
\]

**Definition 3.2** (Clarke stationary point). For a locally Lipschitz function \(f\), a point \(x \in \mathbb{R}^n\) is called a Clarke stationary point of \(f\) if \(0 \in \partial f(x)\).

Note that for a differentiable function \(f\), there is \(\partial f(x) = \{\nabla f(x)\}\). For the nondifferentiable (nonsmooth) function \(r\) defined in \(^{(3)}\), we can also compute its Clarke subdifferential as in the following lemma.

**Lemma 3.1.** Let \(r(x)\) be defined in \(^{(3)}\), then the Clarke subdifferential of \(r\) at \(x\) is

\[
\partial r(x) = \left\{ \sum_{i \in I_0} \nabla g_i(x)^\top w_i + \sum_{i \in I_1} \nabla g_i(x)^\top \frac{g_i(x)}{\|g_i(x)\|} \mid w_i \in \mathbb{R}^d, \|\Pi(w; C(\nabla g_i(x)))\| \leq 1, \forall i \in I_0 \right\},
\]

where \(I_0 = \{i \in [m] \mid \|g_i(x)\| = 0\}\), \(I_1 = [m] \setminus I_0\), and \(\Pi(w; C(A))\) is the projection of \(w\) onto the column space of \(A\).

**Proof.** We observe that \(r(x) = \sum_{i=1}^{m} r_i(x)\) where \(r_i(x) = \|g_i(x)\|\). Hence we can consider the Clarke subdifferential of each \(r_i(x)\).
If $i \in I_1$, then it is clear that $r_i(x)$ is differentiable and $\partial r_i(x) = \nabla g_i(x)^\top \frac{g_i(x)}{\|g_i(x)\|}$. If $i \in I_0$, then for any $v$, there is

$$
\left| \frac{r_i(z + tv) - r_i(z)}{t} - \nabla g_i(x)v \right| = \left| \frac{\|g_i(z + tv)\| - \|g_i(z)\|}{t} - \|\nabla g_i(x)v\| \right|
\leq \frac{\|g_i(z + sv) - g_i(z)\|}{t} \frac{1}{t} \int_0^t \nabla g_i(z + sv)v \, ds - \nabla g_i(z)^\top v \right|
\leq \frac{1}{t} \int_0^t \|\nabla g_i(z + sv)^\top v - \nabla g_i(z)^\top v + \nabla g_i(z)^\top v - \nabla g_i(z)^\top v \| \, ds
\leq \frac{1}{t} \int_0^t M(\|v\|^2 s + \|z - x\|\|v\|) \, ds = \frac{t}{2} M(\|v\| + 2\|z - x\|\|v\|) \to 0
$$
as $(z, t) \to (x, 0)$, which implies that

$$
\|\nabla g_i(x)v\| = \lim_{(z, t) \to (x, 0)} \frac{r_i(z + tv) - r_i(z)}{t} = \lim_{z \to x, t \to 0} \frac{r_i(z + tv) - r_i(z)}{t}.
$$

Therefore, for any $w \in \mathbb{R}^d$ satisfying $\|\Pi(w; C(\nabla g_i(x)))\| \leq 1$, we have

$$
\langle \nabla g_i(x)^\top w, v \rangle = \langle w, \nabla g_i(x)v \rangle = \langle \Pi(w; C(\nabla g_i(x))), \nabla g_i(x)v \rangle \leq \|\nabla g_i(x)v\| = \lim_{z \to x, t \to 0} \frac{r_i(z + tv) - r_i(z)}{t}
$$

where the second equality is due to $\nabla g_i(x)v \in C(\nabla g_i(x))$. On the other hand, for any $w \in \mathbb{R}^d$ satisfying $\|\Pi(w; C(\nabla g_i(x)))\| > 1$, there exists $v \in \mathbb{R}^n$, such that $\nabla g_i(x)v = \Pi(w; C(\nabla g_i(x)))$ and

$$
\langle \nabla g_i(x)^\top w, v \rangle = \langle \Pi(w; C(\nabla g_i(x))), \nabla g_i(x)v \rangle = \|\nabla g_i(x)v\|^2 > \|\nabla g_i(x)v\| = \lim_{z \to x, t \to 0} \frac{r_i(z + tv) - r_i(z)}{t}.
$$

Therefore, by Definition 3.1 we obtain the Clarke subdifferential $\partial r(x)$ as in (16).

We immediately have the subdifferential $\partial \phi$ due to (16) and the differentiability of $f$:

$$
\partial \phi(x) = \partial r(x) + \nabla f(x).
$$

The following lemma also provides the Lipschitz constant of $\nabla r_\varepsilon$.

**Lemma 3.2.** The gradient $\nabla r_\varepsilon$ of $r_\varepsilon$ defined in (5) is Lipschitz continuous with constant $\sqrt{mL_\varepsilon + \frac{M^2}{\varepsilon}}$.

**Proof.** For any $x_1, x_2 \in X$, we first define $y_1$ and $y_2$ as follows,

$$
y_1 = \arg\max_{y \in Y} \langle g(x_1), y \rangle - \frac{\varepsilon}{2}\|y\|^2,
$$

$$
y_2 = \arg\max_{y \in Y} \langle g(x_2), y \rangle - \frac{\varepsilon}{2}\|y\|^2,
$$

which are well defined since the maximization problems have unique solution. Due to the concavity of the problems above (in $y$) and the optimality conditions of $y_1$ and $y_2$, we have

$$
\langle g(x_1) - \varepsilon y_1, y_2 - y_1 \rangle \leq 0,
$$

$$
\langle g(x_2) - \varepsilon y_2, y_1 - y_2 \rangle \leq 0.
$$

Adding the two inequalities above yields

$$
\langle g(x_1) - g(x_2) - \varepsilon (y_1 - y_2), y_2 - y_1 \rangle \leq 0,
$$

which, together with the Cauchy-Schwarz inequality, implies

$$
\varepsilon \|y_2 - y_1\|^2 \leq \langle g(x_1) - g(x_2), y_1 - y_2 \rangle \leq \|g(x_1) - g(x_2)\| \cdot \|y_1 - y_2\|.
$$
Therefore, \( \varepsilon \|y_1 - y_2\| \leq \|g(x_1) - g(x_2)\| \). Recall that \( \nabla r\varepsilon(x_j) = \nabla g(x_j)^T y_j \) for \( j = 1, 2 \). Therefore, we have

\[
\|\nabla r\varepsilon(x_1) - \nabla r\varepsilon(x_2)\| = \|\nabla g(x_1)^T y_1 - \nabla g(x_2)^T y_2\|
\]

\[
= \left\| \left( \nabla g(x_1)^T y_1 - \nabla g(x_2)^T y_1 \right) + \left( \nabla g(x_2)^T y_1 - \nabla g(x_2)^T y_2 \right) \right\| 
\]

\[
\leq \left\| \left( \nabla g(x_1) - \nabla g(x_2) \right)^T y_1 \right\| + \|\nabla g(x_2)\| \|y_1 - y_2\|
\]

\[
\leq \|\nabla g(x_1) - \nabla g(x_2)\| \cdot \|y_1\| + \frac{1}{\varepsilon} \cdot \|\nabla g(x_2)\| \cdot \|g(x_1) - g(x_2)\|
\]

\[
\leq L_g \|x_1 - x_2\| \cdot \|y_1\| + \frac{M}{\varepsilon} \cdot \|\nabla g(x_2)\| \cdot \|x_1 - x_2\|,
\]

where the last inequality is due to the \( L_g \)-Lipschitz continuity of \( \nabla g \) for the first term, and \( \|g(x_1) - g(x_2)\| = \|\nabla g(x)(x_1 - x_2)\| \) for some \( \bar{x} \) due to the mean value theorem and that \( \|\nabla g(x)\| \leq \sup_{x \in X} \|\nabla g(x)\| \leq M \) for the second term. Since \( \max_{y \in Y} \|y\| = \sqrt{m} \), we have

\[
\|\nabla r\varepsilon(x_1) - \nabla r\varepsilon(x_2)\| \leq \|\nabla g(x_1)^T y_1 - \nabla g(x_2)^T y_2\| \leq \left( \sqrt{m} L_g + \frac{M^2}{\varepsilon} \right) \|x_1 - x_2\|,
\]

which completes the proof. \( \square \)

Now we return to Algorithm 1. We first consider its behavior if a constant \( \varepsilon > 0 \) is used, i.e., an iterative scheme that only executes its Lines 3–6.

**Lemma 3.3.** Let \( \varepsilon, \eta > 0 \), \( \delta_1 \geq \delta_2 \geq 1 \) and \( x_0 \in X \) be arbitrary. Suppose \( \{x_k\} \) is the sequence generated by repeating Lines 3–6 of Algorithm 1 with \( \varepsilon = \varepsilon_k \) and step sizes \( \frac{1}{\delta_1 \varepsilon_k} \leq \alpha_k \leq \frac{1}{\delta_2 \varepsilon_k} \) for all \( k \geq 0 \), where \( L_\varepsilon = L_f + \sqrt{m} L_g + \frac{M^2}{\varepsilon} \), and \( \phi^* := \min_{x \in X} \phi(x) \). Then the following statements hold:

1. \( \|\nabla \phi(\varepsilon(x_k))\| \to 0 \) as \( k \to \infty \).

2. \( \min\{k \in \mathbb{N} \mid \|\nabla \phi(\varepsilon(x_{k+1}))\| \leq \eta\} \leq \frac{\delta_1 \delta_2 L_\varepsilon (2\varepsilon(x_0) - 2\phi^* + m\varepsilon)}{(\delta_2 - 1)m^2} \).

**Proof.** 1. Due to the optimality condition of \( v_{k+1} \) in (13b), we have

\[
\langle \nabla \phi(\varepsilon(x_k)), v_{k+1} - x_k \rangle + \frac{1}{2\alpha_k} \|v_{k+1} - x_k\|^2 \leq 0.
\]

In addition, \( \nabla \phi(\varepsilon) \) is \( L_\varepsilon \)-Lipschitz continuous due to Lemma 3.2, which implies that

\[
\phi(\varepsilon(v_{k+1})) \leq \phi(\varepsilon(x_k)) + \langle \nabla \phi(\varepsilon(x_k)), v_{k+1} - x_k \rangle + \frac{L_\varepsilon}{2} \|v_{k+1} - x_k\|^2.
\]

Combining (18), (19) and \( v_{k+1} = x_k - \alpha_k \nabla \phi(\varepsilon(x_k)) \) in (14b) yields

\[
\phi(\varepsilon(v_{k+1})) - \phi(\varepsilon(x_k)) \leq -\left( \frac{L_\varepsilon}{2\alpha_k} \right) \|v_{k+1} - x_k\|^2 = -\frac{\alpha_k (1 - \alpha_k L_\varepsilon)}{2} \|\nabla \phi(\varepsilon(x_k))\|^2 \leq 0,
\]

where we used the fact that \( \alpha_k L_\varepsilon \leq \frac{1}{\delta_1 \varepsilon_k} < 1 \) to obtain the last inequality. According to the selection rule (15), if \( \phi(\varepsilon(u_{k+1})) \leq \phi(\varepsilon(v_{k+1})) \), then \( x_{k+1} = u_{k+1} \), and \( \phi(\varepsilon(x_{k+1})) = \phi(\varepsilon(u_{k+1})) \leq \phi(\varepsilon(v_{k+1})) \); If \( \phi(\varepsilon(v_{k+1})) < \phi(\varepsilon(u_{k+1})) \), then \( x_{k+1} = v_{k+1} \), and \( \phi(\varepsilon(x_{k+1})) = \phi(\varepsilon(v_{k+1})) \). Therefore, in either case, (20) implies \( \phi(\varepsilon(x_{k+1})) - \phi(\varepsilon(x_k)) \leq \phi(\varepsilon(v_{k+1})) - \phi(\varepsilon(x_k)) \leq 0 \), and hence

\[
\phi(\varepsilon(x_{k+1})) \leq \phi(\varepsilon(v_{k+1})) \leq \phi(\varepsilon(x_k)) \leq \cdots \leq \phi(\varepsilon(x_0)),
\]

for all \( k \geq 0 \). Moreover, rearranging (20) and recalling that \( \frac{1}{\delta_1 \varepsilon_k} \leq \alpha_k \leq \frac{1}{\delta_2 \varepsilon_k} \) yield

\[
\frac{\delta_2 - 1}{2\delta_1 \delta_2 L_\varepsilon} \|\nabla \phi(\varepsilon(x_k))\|^2 \leq \frac{\alpha_k (1 - \alpha_k L_\varepsilon)}{2} \|\nabla \phi(\varepsilon(x_k))\|^2 \leq \phi(\varepsilon(x_k)) - \phi(\varepsilon(v_{k+1})) \leq \phi(\varepsilon(x_k)) - \phi(\varepsilon(x_{k+1})).
\]

(22)
Summing up (22) for $k = 0, \ldots, K$ and using the fact that $\phi_\varepsilon(x) \geq \phi(x) - \frac{\varepsilon}{2} \geq \phi^* - \frac{\varepsilon}{2}$ for every $x \in X$, we know that

$$\sum_{k=0}^{K} \left\| \nabla \phi_\varepsilon(x_k) \right\|^2 \leq \frac{2\delta_1 \delta_2 L_\varepsilon (\phi_\varepsilon(x_0) - \phi_\varepsilon(x_{K+1}))}{\delta_2 - 1} \leq \frac{\delta_1 \delta_2 L_\varepsilon (2\phi_\varepsilon(x_0) - 2\phi^* + m\varepsilon)}{\delta_2 - 1}. \tag{23}$$

Note that the right hand side is a finite constant, and hence by letting $K \to \infty$ we know that $\left\| \nabla \phi_\varepsilon(x_k) \right\| \to 0$, which proves the first statement.

2. Denote $\kappa := \min \{ k \in \mathbb{N} \mid \left\| \nabla \phi_\varepsilon(x_{k+1}) \right\| < \eta \}$, then we know that $\left\| \nabla \phi_\varepsilon(x_{k+1}) \right\| \geq \eta$ for all $k \leq \kappa - 1$. Hence we have

$$\kappa \eta^2 \leq \sum_{k=0}^{\kappa-1} \left\| \nabla \phi_\varepsilon(x_{k+1}) \right\|^2 = \sum_{k=1}^{\kappa} \left\| \nabla \phi_\varepsilon(x_k) \right\|^2 \leq \frac{\delta_1 \delta_2 L_\varepsilon (2\phi_\varepsilon(x_0) - 2\phi^* + m\varepsilon)}{\delta_2 - 1},$$

which implies the second statement.

Now we consider the complete version of Algorithm 1. The first result we have is on the monotonicity of $\phi_\varepsilon(x_k) + \frac{m\varepsilon_k}{2}$ in $k$.

**Lemma 3.4.** Suppose that the sequence $\{x_k\}$ is generated by Algorithm 1 with $\frac{1}{\delta_1 \varepsilon_k} \leq \alpha_k \leq \frac{1}{\delta_2 \varepsilon_k}$ and any initial $x_0$. Then for any $k \geq 0$ there is

$$\phi_{\varepsilon_{k+1}}(x_{k+1}) + \frac{m\varepsilon_{k+1}}{2} \leq \phi_{\varepsilon_k}(x_{k+1}) + \frac{m\varepsilon_k}{2} \leq \phi_{\varepsilon_k}(x_k) + \frac{m\varepsilon_k}{2}. \tag{24}$$

*Proof.* Due to (21), the second inequality holds immediately. So we focus on the first inequality. For any $\varepsilon > 0$ and $x$, denote

$$r_{\varepsilon,i}(x) := \begin{cases} \frac{1}{\varepsilon} \| g_i(x) \|, & \text{if } \| g_i(x) \| \leq \varepsilon, \\ \| g_i(x) \| - \frac{\varepsilon}{2}, & \text{if } \| g_i(x) \| > \varepsilon. \end{cases} \tag{25}$$

Then it is clear that $\phi_\varepsilon(x) = \sum_{i=1}^{m} r_{\varepsilon,i}(x) + f(x)$. To prove the first inequality, it suffices to show that

$$r_{\varepsilon_{k+1},i}(x_{k+1}) + \frac{\varepsilon_{k+1}}{2} \leq r_{\varepsilon_k,i}(x_{k+1}) + \frac{\varepsilon_k}{2}. \tag{26}$$

If $\varepsilon_{k+1} = \varepsilon_k$, then the two quantities above are identical and the first inequality holds. Now suppose $\varepsilon_{k+1} = \gamma \varepsilon_k < \varepsilon_k$. We then consider the relation between $\| g_i(x_{k+1}) \|$, $\varepsilon_{k+1}$ and $\varepsilon_k$ in three cases: (i) If $\| g_i(x_{k+1}) \| > \varepsilon_k > \varepsilon_{k+1}$, then by the definition in (25), there is

$$r_{\varepsilon_{k+1},i}(x_{k+1}) + \frac{\varepsilon_{k+1}}{2} = \| g_i(x_{k+1}) \| = r_{\varepsilon_k,i}(x_{k+1}) + \frac{\varepsilon_k}{2}.$$

(ii) If $\varepsilon_k \geq \| g_i(x_{k+1}) \| > \varepsilon_{k+1}$, then (25) implies

$$r_{\varepsilon_{k+1},i}(x_{k+1}) + \frac{\varepsilon_{k+1}}{2} = \frac{\| g_i(x_{k+1}) \|^2}{2\varepsilon_{k+1}} + \frac{\varepsilon_{k+1}}{2} \leq \frac{\| g_i(x_{k+1}) \|^2}{2\varepsilon_k} + \frac{\| g_i(x_{k+1}) \|}{2} = r_{\varepsilon_k,i}(x_{k+1}) + \frac{\varepsilon_k}{2}.$$  \hspace{2cm} (ii)

(iii) If $\varepsilon_k > \varepsilon_{k+1} \geq \| g_i(x_{k+1}) \|$, then we know that $\frac{|\| g_i(x_{k+1}) \|^2|}{2\varepsilon_k} + \frac{\varepsilon_{k+1}}{2}$—as a function of $\varepsilon_k$—is non-decreasing for all $\varepsilon \geq \| g_i(x_{k+1}) \|^2$, which implies (26). Therefore, in either of the three cases, (26) holds and hence

$$r_{\varepsilon_{k+1},i}(x_{k+1}) + \frac{m\varepsilon_{k+1}}{2} = \sum_{i=1}^{m} \left( r_{\varepsilon_{k+1},i}(x_{k+1}) + \frac{\varepsilon_{k+1}}{2} \right) \leq \sum_{i=1}^{m} \left( r_{\varepsilon_k,i}(x_{k+1}) + \frac{\varepsilon_k}{2} \right) = r_{\varepsilon_k}(x_{k+1}) + \frac{m\varepsilon_k}{2},$$

which implies the first inequality of (24). \qed
algorithm. Let $k_l$ be the counter of iteration when the criterion in Line 7 of Algorithm 1 is met for the $l$-th time (we set $k_0 = -1$), then we can partition the iteration counters $k = 0, 1, 2, \ldots$ into segments accordingly, such that $\varepsilon_k = \varepsilon_{k+1} = \varepsilon_0 \gamma^l$ for $k = k_1 + 1, \ldots, k_{l+1}$ in the $l$-th segment. From Lemma 3.3 we can bound the length of each segment and hence the total iteration number which is the sum of these lengths. These results are given in the following theorem.

**Theorem 3.5.** Suppose that $\{x_k\}$ is the sequence generated by Algorithm 1 with any initial $x_0$ and step size $\frac{1}{\delta_1 L x_k} \leq \alpha_k \leq \frac{1}{\delta_2 L x_k}$. Then the following statements hold:

1. The number of iterations, $k_{l+1} - k_l$, for the $l$-th segment is bounded by

$$k_{l+1} - k_l \leq c_1 \gamma^{-2l} + c_2 \gamma^{-3l},$$

where the constants $c_1$ and $c_2$ are defined by

$$c_1 = \frac{\delta_1 \delta_2 (L_f + \sqrt{m L_0}) (2\phi(x) - 2\phi^* + m \varepsilon_0)}{(\delta_1 - 1) \sigma^2 \varepsilon^2 \varepsilon^2},$$

$$c_2 = \frac{\delta_1 \delta_2 L^2 (2\phi(x) - 2\phi^* + m \varepsilon_0)}{(\delta_1 - 1) \sigma^2 \varepsilon^2 \varepsilon^2}.$$  \hfill (28)

2. The total number of iterations for Algorithm 1 to terminate with $\varepsilon_{\text{tol}} > 0$ is bounded by

$$c_1 \frac{\sigma^2 \varepsilon^2}{1 - \gamma^2 \delta_{\text{tol}}} + c_2 \frac{\sigma^3 \varepsilon^3}{1 - \gamma^3 \delta_{\text{tol}}} - \frac{c_1 \gamma^2 + c_2 \gamma^3}{(1 - \gamma^2)(1 - \gamma^3)} = O(\varepsilon^{-3}).$$

**Proof.** 1. Due to Lemma 3.4 we know that, for all $k \geq 0$, there is

$$\phi_{\varepsilon_{k+1}}(x_{k+1}) + \frac{m \varepsilon_{k+1}}{2} \leq \phi_{\varepsilon_k}(x_k) + \frac{m \varepsilon_k}{2} \leq \phi_{\varepsilon_0}(x_0) + \frac{m \varepsilon_0}{2} \leq \phi(x_0) + \frac{m \varepsilon_0}{2}$$

where we used the fact that $\phi_\varepsilon(x) \leq \phi(x_0)$ for all $\varepsilon > 0$ and $x \in X$ in the last inequality. Therefore $k_{l+1} - k_l$ satisfies the bound in Lemma 3.3 (Statement 2) with $\varepsilon = \varepsilon_{k+1} = \varepsilon_0 \gamma^l$, $\eta = \sigma \gamma \varepsilon_{k+1} = \sigma \varepsilon_0 \gamma^{l+1}$ and initial $x_{k+1}$. Namely, there is

$$k_{l+1} - k_l \leq \frac{2 \delta_1 \delta_2 (L_f + \sqrt{m L_0} + \frac{M^2}{4}) (\phi_{\varepsilon_k}(x_{k+1}) - \phi^* + \frac{m \varepsilon_k}{2})}{(\delta_1 - 1) \eta^2} \leq \frac{\delta_1 \delta_2 (L_f + \sqrt{m L_0}) (2\phi(x_{k+1}) - 2\phi^* + m \varepsilon_0)}{(\delta_1 - 1) \eta^2} + \frac{\delta_1 \delta_2 L^2 (2\phi(x_{k+1}) - 2\phi^* + m \varepsilon_0)}{(\delta_1 - 1) \eta^2} = c_1 \gamma^{-2l} + c_2 \gamma^{-3l},$$

where we used (30) to obtain $\phi_{\varepsilon_k}(x_{k+1}) + \frac{m \varepsilon_k}{2} \leq \phi(x_{k+1}) + \frac{m \varepsilon_0}{2}$ for $\varepsilon = \varepsilon_{k+1}$ in the second inequality and the definitions of $c_1$ and $c_2$ in (25) to obtain the last equality.

2. Let $\ell$ be the number of times the reduction criterion in Line 7 of Algorithm 1 is satisfied before it is terminated by Line 8. Then $\sigma \varepsilon_0 \gamma^{\ell-1} \geq \varepsilon_{\text{tol}}$. Hence we have $\ell - 1 \leq \log(\sigma \varepsilon_0 \gamma^{\ell-1}) / \varepsilon_{\text{tol}}$, which implies that the total number of iterations for Algorithm 1 to terminate with $\varepsilon_{\text{tol}}$ is

$$\sum_{l=0}^{\ell-1} (k_{l+1} - k_l) \leq \sum_{l=0}^{\ell-1} (c_1 \gamma^{-2l} + c_2 \gamma^{-3l}) \leq \frac{c_1 (\gamma^{-2(l-1)} - \gamma^2)}{1 - \gamma^2} + \frac{c_2 (\gamma^{-3(l-1)} - \gamma^3)}{1 - \gamma^3}$$

and readily reduces to (29). This completes the proof. \hfill \Box

If we set $\varepsilon_{\text{tol}} = 0$ and $K = \infty$ in Algorithm 1 then LDA will generate an infinite sequence $\{x_k\}$. We focus on the subsequence $\{x_{k_{l+1}}\}$ which selects the iterates when the reduction criterion in Line 7 is satisfied for $k = k_l$ and $\varepsilon_k$ is reduced. Then we can show that every accumulation point of this subsequence is a Clarke stationary point, as shown in the following theorem.

**Theorem 3.6.** Suppose that $\{x_k\}$ is the sequence generated by Algorithm 1 with any initial $x_0$ and step size $\frac{1}{\delta_1 L x_k} \leq \alpha_k \leq \frac{1}{\delta_2 L x_k}$, $\varepsilon_{\text{tol}} = 0$ and $K = \infty$. Let $\{x_{l+1}\}$ be the subsequence where the reduction criterion Line 7 of Algorithm 1 is met for $k = k_l$ and $l = 1, 2, \ldots$. Then the following statements hold:
1. \( \{x_{k+1}\} \) has at least one accumulation point.

2. Every accumulation point of \( \{x_{k+1}\} \) is a Clarke stationary point of [2].

Proof. 1. Due to Lemma 3.4 and \( \phi(x) \leq \phi_{\epsilon}(x) + \frac{m\epsilon}{2} \) for all \( \epsilon > 0 \) and \( x \in \mathcal{X} \), we know that

\[
\phi(x_k) \leq \phi_{\epsilon_k}(x_k) + \frac{m\epsilon_k}{2} \leq \cdots \leq \phi_{\epsilon_0}(x_0) + \frac{m\epsilon_0}{2} < \infty.
\]

Since \( \phi \) is coercive, we know that \( \{x_k\} \) is bounded. Hence \( \{x_{k+1}\} \) is also bounded and has at least one accumulation point.

2. Note that \( x_{k+1} \) satisfies the reduction criterion in Line 7 of Algorithm 1, i.e., \( \|\nabla \phi_{\epsilon_{k+1}}(x_{k+1})\| \leq \sigma \gamma \epsilon_{k+1} = \sigma \epsilon_{0} \gamma_{l+1} \rightarrow 0 \) as \( l \rightarrow \infty \). For notation simplicity, we let \( \{x_{j+1}\} \) denote any convergent subsequence of \( \{x_{k+1}\} \) and \( \epsilon_j \) the corresponding \( \epsilon_k \) used in the iteration to generate \( x_{j+1} \). Then there exists \( x \in \mathcal{X} \) such that \( x_{j+1} \rightarrow x \), \( \epsilon_j \rightarrow 0 \), and \( \nabla \phi_{\epsilon_j}(x_{j+1}) \rightarrow 0 \) as \( j \rightarrow \infty \).

Recall the Clarke subdifferential of \( \phi \) at \( \hat{x} \) is given by (17):

\[
\partial \phi(\hat{x}) = \left\{ \sum_{i \in I_0} \nabla g_i(\hat{x})^\top w_i + \sum_{i \in I_1} \nabla g_i(\hat{x})^\top \frac{g_i(\hat{x})}{\|g_i(\hat{x})\|} + \nabla f(\hat{x}) \mid \|\Pi(w_i; \mathcal{C}(\nabla g_i(\hat{x})))\| \leq 1, \forall i \in I_0 \right\}, \tag{31}
\]

where \( I_0 = \{i \in [m] \mid \|g_i(\hat{x})\| = 0\} \) and \( I_1 = [m] \setminus I_0 \). Then we know that there exists \( J \) sufficiently large, such that

\[
\epsilon_j < \frac{1}{2} \min\{\|g_i(\hat{x})\| \mid i \in I_1\} \leq \frac{1}{2} \|g_i(\hat{x})\| \leq \|g_i(x_{j+1})\|, \quad \forall j \geq J, \quad \forall i \in I_1,
\]

where we used the facts that \( \min\{\|g_i(\hat{x})\| \mid i \in I_1\} > 0 \) and \( \epsilon_j \rightarrow 0 \) in the first inequality, and \( x_{j+1} \rightarrow \hat{x} \) and the continuity of \( g_i \) for all \( i \) in the last inequality. Furthermore, we denote

\[
s_{j,i} := \begin{cases} g_i(x_{j+1}), & \text{if } \|g_i(x_{j+1})\| \leq \epsilon_j, \\ \frac{\epsilon_j}{\|g_i(x_{j+1})\|} g_i(x_{j+1}), & \text{if } \|g_i(x_{j+1})\| > \epsilon_j. \end{cases}
\]

Then we have

\[
\nabla \phi_{\epsilon_j}(x_{j+1}) = \sum_{i \in I_0} \nabla g_i(x_{j+1})^\top s_{j,i} + \sum_{i \in I_1} \nabla g_i(x_{j+1})^\top \frac{g_i(x_{j+1})}{\|g_i(x_{j+1})\|} + \nabla f(x_{j+1}). \tag{32}
\]

Comparing (31) and (32), we can see that the last two terms on the right hand side of (32) converge to those of (31), respectively, due to the facts that \( x_{j+1} \rightarrow \hat{x} \) and the the continuity of \( g_i, \nabla g_i, \nabla f \). Moreover, noting that \( \|\Pi(s_{j,i}; \mathcal{C}(\nabla g_i(\hat{x})))\| \leq \|s_{j,i}\| \leq 1 \), we can see that the first term on the right hand side of (32) also converges to the set formed by the first term of (31) due to the continuity of \( g_i \) and \( \nabla g_i \). Hence we know that

\[
\text{dist}(\nabla \phi_{\epsilon_j}(x_{j+1}), \partial \phi(\hat{x})) \rightarrow 0,
\]

as \( j \rightarrow 0 \). Since \( \nabla \phi_{\epsilon_j}(x_{j+1}) \rightarrow 0 \) and \( \partial \phi(\hat{x}) \) is closed, we conclude that \( 0 \in \partial \phi(\hat{x}) \). \( \square \)

4 Numerical Experiments

4.1 Network architecture and parameter setting

Throughout our experiments, we parameterize \( g \) in [3] as a simple 4-layer convolutional neural network with component-wise activation function \( \sigma \) and no bias as follows:

\[
\begin{aligned}
\text{For any } x, \text{ compute } g(x) &= h_1, \\
\text{where } h_0 &= x, \text{ and } \\
h_l &= \sigma(W_{l-1}h_{l-1}), \quad l = 1, 2, 3, 4, \\
\text{and } \sigma(x) &= \begin{cases} 0, & \text{if } x \leq -\delta, \\ \frac{1}{2\delta^2}x^2 + \frac{1}{2}x + \frac{\delta}{4}, & \text{if } -\delta < x < \delta, \\ x, & \text{if } x \geq \delta, \end{cases} \quad (33)
\end{aligned}
\]

\[
\text{12}
\]
where \( \delta = 0.01 \) in our experiment. In (33), \( \mathbf{W}_l \) represents the convolution in the \( l \)-th layer. We set the kernel size to \( 3 \times 3 \times d \) for all layers, where \( d = 32 \) is the depth of the convolution kernel. In our experiments, we set stride to 1, and use zero-padding to preserve image size. Then \( \mathbf{W}_0 \) can be interpreted as a \( dn \times n \) matrix with \( 3^2 \times 32 \) learnable parameters and \( \mathbf{W}_l \) as \( dn \times dn \) for \( l = 1, 2, 3 \) each with \( 3^2 \times 32^2 \) learnable parameters. In this case \( m = n \) is the number of pixels in the image. Note that \( \mathbf{g} \) satisfies Assumption A2 due to the boundedness of \( \sigma' \) and the fixed \( \mathbf{W}_l \) once learned. The regularization is \( \mathcal{R}(\mathbf{x}) = \| \mathbf{g}(\mathbf{x}) \|_{2,1} \) as in (3), and \( r_{\varepsilon} \) and \( \nabla r_{\varepsilon} \) are given in (9) and (10), respectively.

During training, we prescribe the iteration number \( K = 15 \) for Algorithm 4, which seems to reach a good compromise between network depth and performance in practice. We adopt a warm start strategy by first training LDA with \( K = 3 \) for 500 epochs, and then add 2 more phases and train the network for another 200 epochs, and so on, until we finish with \( K = 15 \). The step sizes \( \alpha_k \) and \( \tau_k \) are also to be learned and allowed to vary across different phases. The threshold \( \varepsilon_k \) is updated according to Algorithm 4, where the starting \( \varepsilon_0 \) is to be learned. We let \( \theta \) denote the set of trainable parameters of LOA in Algorithm 4, including the convolutions \( \{ \mathbf{W}_l \}_{l=0}^{3} \), the step sizes \( \{ \alpha_k, \tau_k \}_{k=0}^{K} \) and the starting \( \varepsilon_0 \).

Given \( N \) training data pairs \( \{(\mathbf{b}^{(s)}, \hat{x}^{(s)})\}_{s=1}^{N} \), where each \( \hat{x}^{(s)} \) is the ground truth data and \( \mathbf{b}^{(s)} \) is the measurement of \( x^{(s)} \), we solve \( \theta \) by minimizing the loss function in (1) using the Adam Optimizer with \( \beta_1 = 0.9 \) and \( \beta_2 = 0.999 \) and Xavier Initializer implemented in TensorFlow 1. All the experiments are performed on a desktop computer with Intel i7-6700K CPU at 3.40 GHz, 16 GB of memory, and an Nvidia GTX-1080Ti GPU of 11GB graphics card memory.

### 4.2 Experimental results on image reconstruction

#### 4.2.1 Reconstruction on natural image compressed sensing

We first consider the natural image block compressed sensing (block CS) problem [21] to recover images (image patches) from compressed data. In block CS, an image is partitioned into small blocks of size \( 3 \times 3 \) to \( 8 \times 8 \), each of which (treated as a vector \( \mathbf{x} \in \mathbb{R}^m \)) is left-multiplied by a prescribed sensing matrix \( \mathbf{A} \in \mathbb{R}^{m \times n} \) to obtain the compressed data \( \mathbf{b} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m \), where \( c \in (0, 1) \) is the compression ratio (CS ratio) [14][19][21]. The flowchart of this process, including the compressed sensing part using the prescribed sampling matrix \( \mathbf{A} \) and the reconstruction by a \( K \)-phase LDA network, is shown in the top panel of Figure 4.

We test the proposed Algorithm 4 LDA on 91 Images for training and Set11 for testing [29]. The training set \( \mathcal{D} \) consists of \( N = 88,912 \) pairs of form \( (\mathbf{b}, \mathbf{x}) \in \mathbb{R}^{m \times n} \), where \( \mathbf{x} \) is randomly cropped from the images. The experiments on three different CS ratios \( c = 10\%, 25\%, 50\% \) are performed. The matrix \( \mathbf{A} \) is set to a random Gaussian matrix whose rows are orthogonalized and the initial \( \mathbf{x}_0 \) is set to be \( \mathbf{x}_0 = \mathbf{Q}\mathbf{b} \), where \( \mathbf{Q} = \mathbf{X}\mathbf{B}\mathbf{B}^{-1} \) and \( \mathbf{X} = [\hat{x}^{(1)}, ..., \hat{x}^{(N)}], \mathbf{B} = [\mathbf{b}^{(1)}, ..., \mathbf{b}^{(N)}] \), which follow [61]. We follow the same criterion when generating the testing data pairs from Set11. All the testing results are evaluated on the average Peak Signal-to-Noise Ratio (PSNR) of the reconstruction quality. We compare with two classical image reconstruction methods, i.e., TVAL3 [31] and D-AMP [37], and three state-of-the-art methods based on deep learning approaches, i.e., IRCNN [63], ReconNet [29] and ISTA-Net+ [61]. The comparison results tested on Set11 [29] in Table 4, where the results of the first five methods are quoted from [61]. The number of learnable parameters in the networks are also shown in the last column of Table 4. In general, a network has higher capacity and yields lower reconstruction error with more parameters (e.g., ISTA-Net+ with varying parameters across different phases yields higher PSNR than that with parameters shared by all phases), but may also suffer the issue of parameter overfitting. As LDA uses the same set of network parameters in all phases, except the step size which is different in each phase but is only a scalar to be learned, it requires much fewer parameters than IRCNN and ISTA-Net+. From Table 4, we can see that the proposed LDA obtained higher accuracy in reconstruction while using relatively small amount of network parameters.

#### 4.2.2 Joint compression and reconstruction of natural images

We test LDA Algorithm 4 on the problem of joint image compression and reconstruction, which is considered in several recent CS image reconstruction work [46][62][64][66]. In this experiment, we prescribe the CS ratio \( c \in (0, 1) \) and let the compressed sensing matrix \( \mathbf{A} \in \mathbb{R}^{m \times n} \) be learned together with the reconstruction network. More precisely, we let a ground truth image (patch) \( \hat{x} \) first pass a linear layer \( \mathbf{b} = \mathbf{A}\hat{x} \), where \( \mathbf{A} \) is
Figure 1: The flowchart of the block compressed sensing natural image reconstruction. An image is partitioned into patches of size $n$, each of which, denoted by $\hat{x}$, is compressed by the sampling matrix $A$ into data $b = A\hat{x} \in \mathbb{R}^{cn}$. Top: The compressed data $b$ is obtained using a prescribed sampling matrix $A$ and is mapped to $x_0$ as the initial value of the $K$-phase LDA reconstruction network; Middle: The sampling matrix $A$ is jointly learned with the network parameters by appending $b = A\hat{x} \in \mathbb{R}^{cn}$ as a linear layer before the LDA; Bottom: The detailed illustration of $k$th-phase Recovery Net.

also to be learned. Here $A$ can be implemented as a convolutional operation with $cn$ kernels of size $\sqrt{n} \times \sqrt{n}$ and stride $\sqrt{n} \times \sqrt{n}$, and hence once applied to an image patch it returns a $cn$-vector. The sampling layer is followed by an initialization layer $x_0 = \tilde{A}b$, where $\tilde{A} \in \mathbb{R}^{n \times cn}$ is implemented as transposed convolutional operation [18]. Then $x_0$ is served as the input of LDA. Moreover, we add $(1/N) \cdot \sum_{s=1}^{N} \| A\tilde{A}x^{(s)} - x^{(s)} \|^2$ with weight 0.01 to the loss function in (1), such that $A$ and $\tilde{A}$ are learned jointly with the network parameters during training.

The training dataset in our experiment consists of 89,600 image patches of size $96 \times 96$, where all these patches are the luminance components randomly cropped from images in BSD500 training and testing set (200 + 200 images) [3]. Each image patch consists of 9 non-overlapping blocks of size $n = 32 \times 32 = 32^2$, where each block can be sampled independently by $A$. We use Set11 for testing. For comparison, we also test four recent methods in this experiment: CS-Net [47], SCS-Net [46], BCS-Net [66], AMP-Net [64]. All the compared methods are applied to Set11, and the average PSNR are shown in Table 2. Table 2 also shows the number of learnable parameters of the reconstruction network part of each method. In addition to these parameters, all methods also need to learn the sampling matrix $A$ with $cn \times n = 104,448$ variables when $c = 0.1$ and another 104,448 variables of $\tilde{A}$ for initialization, except that BCS-Net requires over 2.2M parameters for sampling and initialization. BCS-Net learns a set of sampling matrices with different rates and dynamically assigns the sampling resource depending on the embedded saliency information of each block [66]. From Table 2 we can see that LDA outperforms all these state-of-the-art methods by a large margin, but only need a fraction of the amount of learnable parameters compared to most methods.
Table 1: Average PSNR (dB) of reconstructions obtained by the compared methods and the proposed LDA on Set11 dataset with CS ratios 10%, 25% and 50% and the number of learnable network parameters (#Param) using a prescribed compressed sensing matrix A. Subscript * indicates that network parameters are shared across different phases.

<table>
<thead>
<tr>
<th>Method</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
<th>#Param</th>
</tr>
</thead>
<tbody>
<tr>
<td>TVAL3 [31]</td>
<td>22.99</td>
<td>27.92</td>
<td>33.55</td>
<td>NA</td>
</tr>
<tr>
<td>D-AMP [37]</td>
<td>22.64</td>
<td>28.46</td>
<td>35.92</td>
<td>NA</td>
</tr>
<tr>
<td>IRCNN [63]</td>
<td>24.02</td>
<td>30.07</td>
<td>36.23</td>
<td>185,472</td>
</tr>
<tr>
<td>ReconNet [29]</td>
<td>24.28</td>
<td>25.60</td>
<td>31.50</td>
<td>22,914</td>
</tr>
<tr>
<td>ISTA-Net* [61]</td>
<td>26.51</td>
<td>32.08</td>
<td>37.59</td>
<td>37,450</td>
</tr>
<tr>
<td>ISTA-Net+ [61]</td>
<td>26.64</td>
<td>32.57</td>
<td>38.07</td>
<td>336,978</td>
</tr>
<tr>
<td>LDA</td>
<td>27.42</td>
<td>32.92</td>
<td>38.50</td>
<td>27,967</td>
</tr>
</tbody>
</table>

Table 2: Average PSNR (dB) of reconstructions obtained by the compared methods and the proposed LDA on Set11 dataset with CS ratios 10%, 30% and the number of parameters (#Param) in the reconstruction part of the network using jointly learned compressed sensing matrix A.

<table>
<thead>
<tr>
<th>Method</th>
<th>10%</th>
<th>30%</th>
<th>#Param</th>
</tr>
</thead>
<tbody>
<tr>
<td>CS-Net [47]</td>
<td>28.10</td>
<td>33.86</td>
<td>370,560</td>
</tr>
<tr>
<td>SCS-Net [46]</td>
<td>28.48</td>
<td>34.62</td>
<td>587,520</td>
</tr>
<tr>
<td>BCS-Net [66]</td>
<td>29.43</td>
<td>35.60</td>
<td>1,117,440</td>
</tr>
<tr>
<td>AMP-Net [64]</td>
<td>29.45</td>
<td>35.90</td>
<td>229,254</td>
</tr>
<tr>
<td>LDA</td>
<td>30.03</td>
<td>36.47</td>
<td>27,967</td>
</tr>
</tbody>
</table>

4.2.3 Magnetic resonance image reconstruction

In this experiment, we consider the reconstruction problem in compressed sensing magnetic resonance imaging (CS-MRI). In CS-MRI, we set $A = PF$, where $P$ is a binary selection matrix representing the Fourier space ($k$-space) sampling trajectory, and $F$ is the discrete Fourier transform. The ground truth image is shown in Figure 5(a). We use radial mask $P$ with three different sampling ratios 10%, 20%, and 30% in this experiments. The one with 20% sampling ratio is shown in Figure 5(e). We randomly select 150 2D images from the brain MRI datasets [5], then extract the main center region of interests (size $190 \times 190$) of every image as the ground truth images $\hat{x}$, and set the data to $b = Ax$. Then we randomly select 100 images for training and use the other 50 for testing.

During the training, for each of the sampling ratios 10%, 20%, and 30%, we train LDA for phase numbers $K = 3, 5, \ldots, 11$, and the PSNR obtained for each case is shown in the middle panel of Figure 6. For comparison, we also apply ISTA-Net+ [61] to the same data. We use $x_0 = 0$ as the initial for both ISTA-Net+ and LDA. The quantitative comparison results are shown in Table 3, where the PSNR, the relative error (RelErr) of the reconstruction $x$ to the ground truth $\hat{x}$ defined by $\|x - \hat{x}\| / \|\hat{x}\|$, and the structural similarity index (SSIM) [52] are provided for each of the three sampling ratios. These results show that LDA generates more accurate images using only 8% of the amount of learnable parameters of ISTA-Net+.

Figure 5 shows the pointwise absolute error of the reconstructed images obtained by ISTA-Net+ and LDA under these three sampling ratios, where brighter pixels indicate larger errors. From Figure 5 it can be seen that LDA attains much lower error and hence better reconstruction quality.

4.3 Experimental results on convergence and learned feature map

4.3.1 Comparison with standard gradient descent

The proposed LDA performs two residual-type updates, one on the data fidelity $f$ and the other on the regularization $r$ (and the smoothed version $r_\varepsilon$), which is motivated by the effectiveness of the ResNet structure. In this experiment, we also unroll the standard gradient descent iteration by turning off the $u$ step
Table 3: Average PSNR (dB), RelErr, and SSIM of the reconstructions obtained by ISTA-Net+ and LDA on CS-MRI dataset with sampling ratios 10%, 20%, and 30%.

<table>
<thead>
<tr>
<th>Method</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PSNR</td>
<td>RelErr</td>
<td>SSIM</td>
</tr>
<tr>
<td>ISTA-Net+</td>
<td>32.62</td>
<td>0.0950</td>
<td>0.9312</td>
</tr>
<tr>
<td>LDA</td>
<td>34.20</td>
<td>0.0790</td>
<td>0.9462</td>
</tr>
</tbody>
</table>

of LDA, and an accelerated inertial version by setting $x_{k+1} = x_k - \alpha_k \nabla f(x_k) + \theta_k (x_k - x_{k-1})$ where $\theta_k$ is also learned. We call these two networks GD-Net and AGD-Net, respectively. We test all three methods following the same experiment setting in Section 4.2.1 and show the average PSNR versus phase (iteration) number of these methods in Figure 6. As we can see, LDA achieves a much higher PNSR than both GD-Net and AGD-Net, where the latter perform very similarly. In particular, although AGD has improved iteration complexity in the standard convex optimization setting, its network version does not seem to inherit the effectiveness for deep learning applications. Similar comparison has been made for ISTA-Net and FISTA-Net, which are based on ISTA and FISTA with the latter algorithm provably having improved iteration complexity, but their deep network versions have nearly identical performance [63]. This is also partly due to the nonconvexity of the learned objective function, for which inertial gradient descent may produce improper extrapolation and do not improve efficiency.

4.3.2 Convergence behavior of LDA

As in the standard approach of deep neural network training, we set the phase (iteration) number to $K = 15$ in LDA in the experiments above. On the other hand, we proved that the iterates generated by LDA converge to a Clarke stationary point in Section 3.3. This provides a theoretical guarantee that the LDA is indeed minimizing an objective function where the regularization is learned, and LDA is expected to perform stably even beyond the trained phases. To demonstrate this stability empirically, we continue to run LDA for another 15 phases using the feature map $g(x)$ learned in the first 15 phases and step sizes $\alpha_k$ computed by the standard line search tracking such that $\phi_{\varepsilon_k}(v_{k+1}) - \phi_{\varepsilon_k}(x_k) \leq -\tau \|v_{k+1} - x_k\|^2$ for $\tau = 0.35$ with step size reduction rate 0.5. The objective function value $\phi_{\varepsilon_k}(x_k)$ versus iteration $k$ is shown in the right panel of Figure 6. As we can see, LDA continuous to reduce function value during these extended iteration stably even beyond the trained phases, as shown in our convergence analysis in Section 3.3.

4.3.3 Learned feature map in LDA

A main advantage of LDA is that the feature map $g$ in the regularization can be learned adaptively from the training data such that the network is more interpretable. This data-driven approach yields automated design of feature maps which are often more complex and efficient, rather than the manually crafted features in the classical image reconstruction models.

In this experiment, we plot the norm of the gradient (as a 2D vector computed by forward finite difference) at every pixel $i$ which is used as the feature of the TV based image reconstruction and also $\|g_i(x)\|$ at pixel $i$ of the regularization learned in LDA in Figure 7. We can see that the learned feature map $g$ captures more important structural details of the images, such as the antennae of the butterfly, the lip, chin, and shoulder of Lena, and the bill of the parrot. These details are crucial in species detection and facial recognition, which seem to be accurately recovered using the learned feature map $g$ but are heavily blurred or completely missing from the simple gradient image used in TV regularization. This also explains the better image quality obtained by LDA compared to the classical TV based image reconstruction methods.
Figure 2: Block CS reconstruction of butterfly image with CS ratio 10% obtained by CS-Net, SCS-Net and the proposed LDA. Images in the bottom row zoom in the corresponding ones in the top row. PSNR are shown in the parentheses.

Figure 3: Block CS reconstruction of Lena image with CS ratio 10% obtained by CS-Net, SCS-Net and the proposed LDA. Images in the bottom row zoom in the corresponding ones in the top row. PSNR are shown in the parentheses.
Figure 4: Block CS reconstruction of parrot image with CS ratio 10% obtained by CS-Net, SCS-Net and the proposed LDA. Images in the bottom row zoom in the corresponding ones in the top row. PSNR are shown in the parentheses.

Figure 5: Pointwise absolute error of the brain MR image reconstruction with 10%, 20% and 30% sampling ratio using ISTA-Net+ and the proposed LDA under the same color scale. Brighter pixel indicates larger value. PSNR are shown in the parentheses. Ground truth reference image and a radial mask with 20% sampling ratio are shown in (a) and (e) respectively.
Figure 6: Left: PSNR of reconstructions obtained by LDA versus phase number $K$ on three CS ratios 10%, 20% and 30% for the brain MR image. Middle: PSNR of reconstructions obtained by GD-Net, AGD-Net, and LDA versus phase number $K$ on the block CS image reconstruction with CS ratio 10%. Right: Function value $\phi_{\epsilon_k}(x_k)$ of LDA versus iteration number, where the last 15 iterations are obtained by continue running LDA using the feature map $g$ learned in the first 15 iterations.

Figure 7: The norm of the gradient at every pixel in TV based image reconstruction (top row) and the norm of the feature map $g$ at every pixel learned in LDA (bottom row). Important details, such as the antennae of the butterfly, the lip, chin, and shoulder of Lena, and the bill of the parrot, are faithfully recovered by LDA.
5 Conclusion

We proposed a general learning based framework for solving nonsmooth and nonconvex image reconstruction problems, where the regularization function is modeled as the composition of the $l_{2,1}$ norm and a smooth but nonconvex feature mapping parametrized as a deep convolutional neural network. We developed a provably convergent descent-type algorithm to solve the nonsmooth nonconvex minimization problem by leveraging the Nesterov’s smoothing technique and the idea of residual learning, and learn the network parameters such that the outputs of the algorithm match the references in training data. Our method is versatile as one can employ various modern network structures into the regularization, and the resulting network inherits the guaranteed convergence of the algorithm. The proposed network is applied to a variety of real-world image reconstruction problems, and the numerical results demonstrate the outstanding performance and efficiency of our method.

References


