3(d). $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

Solution Using the fact that $\sqrt{a} - \sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$, we can see that $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1}$

Since $\frac{1}{n} \to 0$, we know $x_n \to \frac{1}{2}$.

To prove $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$ using the definition of limits, we need to show: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|x_n - \frac{1}{2}| < \varepsilon$ for all $n \geq N$

To show this, first note that

$$\left|\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} - \frac{1}{2}\right| = \left|\frac{2\sqrt{n} - (\sqrt{n+1}+\sqrt{n})}{2(\sqrt{n+1}+\sqrt{n})}\right| = \left|\frac{\sqrt{n+1}-\sqrt{n}}{2(\sqrt{n+1}+\sqrt{n})}\right|$$

$$= \left|\frac{1}{2(\sqrt{n+1}+\sqrt{n})^2}\right| < \left|\frac{1}{2(2\sqrt{n})^2}\right| \quad (\because \sqrt{n+1} > \sqrt{n})$$

$$= \left|\frac{1}{8n}\right| = \frac{1}{8n}$$

Therefore, $\forall \varepsilon > 0$, just set $N = \left\lceil \frac{1}{8\varepsilon} \right\rceil$, then $N \geq \frac{1}{8\varepsilon}$

and $|x_n - \frac{1}{2}| = \left|\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} - \frac{1}{2}\right| < \frac{1}{8n} \leq \frac{1}{8N} \leq \varepsilon$ for all $n \geq N$

By the definition of limits, this implies $\lim\limits_{n\to+\infty} x_n = \frac{1}{2}$.