# MATH 4211/6211 – Optimization

# Convex Optimization Problems

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**Definition.** A set $\Omega \subset \mathbb{R}^n$ is called *convex* if for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, there is $\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in \Omega$ for all $\alpha \in [0, 1]$

**Definition.** A function $f : \Omega \to \mathbb{R}$, where $\Omega$ is a convex set, is called *convex* if for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha \in [0, 1]$, there is

$$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}).$$

Moreover, $f$ is called *strictly convex* if for any distinct $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha \in (0, 1)$, there is

$$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) < \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y})$$

A function $f$ is called (strictly) concave if $-f$ is (strictly) convex.

There is an alternative definition based on the convexity of the epigraph of $f$.

**Definition.** The *graph* of $f : \Omega \to \mathbb{R}$ is defined by

$$\{[\boldsymbol{x}; f(\boldsymbol{x})] \in \mathbb{R}^{n+1} : \boldsymbol{x} \in \Omega\}$$

**Definition.** The *epigraph* of $f : \Omega \to \mathbb{R}$ is defined by

$$\mathrm{epi}(f) := \{[\boldsymbol{x}; \beta] \in \mathbb{R}^{n+1} : \boldsymbol{x} \in \Omega, \beta \geq f(\boldsymbol{x})\}$$

**Definition.** A function $f : \Omega \to \mathbb{R}$, where $\Omega$ is a convex set, is called *convex* if $\mathrm{epi}(f)$ is a convex set.

**Example.** Let $f(\boldsymbol{x}) = x_1 x_2$ be defined on $\Omega := \{\boldsymbol{x} : \boldsymbol{x} \geq \mathbf{0}\}$. Is $f$ convex?

**Solution.** $f$ is *not* convex. The set $\Omega \subset \mathbb{R}^2$ is convex. But if we choose $\boldsymbol{x} = [1; 2]$ and $\boldsymbol{y} = [2; 1]$, then

$$\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y} = [2 - \alpha; 1 + \alpha].$$

On the one hand

$$f(\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) = 2 + \alpha - \alpha^2.$$

On the other hand,

$$\alpha f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{y}) = 2.$$

Choosing $\alpha = 1/2$ yields

$$f(\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) > \alpha f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{y})$$

which means that $f$ is not convex.

There are several *sufficient and necessary* conditions for the convexity of $f$.

**Theorem.** If $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1$ and $\Omega$ is convex, then $f$ is convex on $\Omega$ iff for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$,

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x})$$

**Proof.** ($\Rightarrow$) Suppose $f$ is convex, then for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha \in (0, 1]$,

$$f((1-\alpha)\boldsymbol{x} + \alpha \boldsymbol{y}) \leq (1-\alpha) f(\boldsymbol{x}) + \alpha f(\boldsymbol{y})$$

Rearrange terms to obtain

$$\frac{f(\boldsymbol{x} + \alpha(\boldsymbol{y} - \boldsymbol{x})) - f(\boldsymbol{x})}{\alpha} \leq f(\boldsymbol{y}) - f(\boldsymbol{x})$$

Taking the limit as $\alpha \to 0$ yields

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x})$$

**Proof (cont.)** ($\Leftarrow$) For any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha \in [0, 1]$, define $\boldsymbol{x}_\alpha = \alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}$. Then

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_\alpha) + \nabla f(\boldsymbol{x}_\alpha)^\top (\boldsymbol{x}_\alpha - \boldsymbol{x})$$
$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}_\alpha) + \nabla f(\boldsymbol{x}_\alpha)^\top (\boldsymbol{x}_\alpha - \boldsymbol{y})$$

Multiplying the two inequalities by $\alpha$ and $1 - \alpha$ respectively, and adding together yields

$$f(\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{y})$$

**Theorem.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^2$ and $\Omega$ be convex, then $f$ is convex on $\Omega$ iff $\nabla^2 f(\boldsymbol{x}) \succeq \mathbf{0}$ for all $\boldsymbol{x} \in \Omega$.

**Proof.** ($\Rightarrow$) If not, then exist $\boldsymbol{x} \in \Omega$ and $\boldsymbol{d} \in \mathbb{R}^n$, such that

$$\boldsymbol{d}^\top \nabla^2 f(\boldsymbol{x}) \boldsymbol{d} < 0$$

Since $\nabla^2 f(\boldsymbol{x})$ is continuous, there exists $s > 0$ sufficiently small, such that for $\boldsymbol{y} = \boldsymbol{x} + s\boldsymbol{d} \in \Omega$, there is

$$\begin{aligned} f(\boldsymbol{y}) &= f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x}) + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{x})^\top \nabla^2 f(\boldsymbol{x} + t\boldsymbol{d})(\boldsymbol{y} - \boldsymbol{x}) \\ &< f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x}) \end{aligned}$$

for some $t \in (0, s)$ since $(\boldsymbol{y} - \boldsymbol{x})^\top \nabla^2 f(\boldsymbol{x} + t\boldsymbol{d})(\boldsymbol{y} - \boldsymbol{x}) = s^2 \boldsymbol{d}^\top \nabla^2 f(\boldsymbol{x} + t\boldsymbol{d})\boldsymbol{d} < 0$. Hence $f$ is not convex, a contradiction.

**Proof (cont.)** ($\Leftarrow$) For any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, there is

$$\begin{aligned} f(\boldsymbol{y}) &= f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x}) + \frac{1}{2} (\boldsymbol{y} - \boldsymbol{x})^\top \nabla^2 f(\boldsymbol{x} + t\boldsymbol{d})(\boldsymbol{y} - \boldsymbol{x}) \\ &\geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x}) \end{aligned}$$

where $\boldsymbol{d} := \boldsymbol{y} - \boldsymbol{x}$ and $t \in (0, 1)$. Note that we used the fact that $\nabla^2 f(\boldsymbol{x} + t\boldsymbol{d}) \succeq \boldsymbol{0}$. Hence $f$ is convex.

**Examples.** Determine if any of the following functions is convex.

$$f_1(x) = -8x^2$$
$$f_2(\boldsymbol{x}) = 4x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + x_1x_3 - 3x_1 - 2x_2 + 15$$
$$f_3(\boldsymbol{x}) = 2x_1x_2 - x_1^2 - x_2^2$$

**Solution.** $f_1''(x) = -16 < 0$, so $f_1$ is concave.

For $f_2$, we have

$$\nabla^2 f_2 = \begin{bmatrix} 8 & 6 & 1 \\ 6 & 6 & 0 \\ 1 & 0 & 10 \end{bmatrix}$$

whose leading principal minors are $8, 12, 114$. Hence $f_2$ is convex.

For $f_3$, we have

$$\nabla^2 f_3 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

whose eigenvalues are $-4$ and $0$, hence $f_3$ is negative semidefinite.

**Theorem.** Suppose $f : \Omega \to \mathbb{R}$ is convex. Then $\boldsymbol{x}$ is a global minimizer of $f$ on $\Omega$ iff it is a local minimizer of $f$.

**Proof.** The necessity is trivial. Suppose $\boldsymbol{x}$ is a local minimizer, then $\exists\, r > 0$ such that $f(\boldsymbol{x}) \le f(\boldsymbol{z})$ for all $\boldsymbol{z} \in B(\boldsymbol{x}, r)$. If $\exists\, \boldsymbol{y}$, such that $f(\boldsymbol{x}) > f(\boldsymbol{y})$, then let $\alpha = \frac{r}{\|\boldsymbol{y}-\boldsymbol{x}\|}$ and

$$\boldsymbol{x}_\alpha = (1-\alpha)\boldsymbol{x} + \alpha\boldsymbol{y} = \boldsymbol{x} + \frac{r}{\|\boldsymbol{y}-\boldsymbol{x}\|}(\boldsymbol{y}-\boldsymbol{x}).$$

Then $\boldsymbol{x}_\alpha \in B(\boldsymbol{x}, r)$ and

$$f(\boldsymbol{x}_\alpha) \ge f(\boldsymbol{x}) > (1-\alpha)f(\boldsymbol{x}) + \alpha f(\boldsymbol{y}),$$

which is a contradiction. Hence $\boldsymbol{x}$ must be a global minimizer.

**Lemma.** Suppose $f : \Omega \to \mathbb{R}$ is convex. Then the *sub-level set* of $f$

$$\Gamma_c = \{\boldsymbol{x} \in \Omega : f(\boldsymbol{x}) \le c\}$$

is empty or convex for any $c \in \mathbb{R}$.

**Proof.** If $\boldsymbol{x}, \boldsymbol{y} \in \Gamma_c$, then $f(\boldsymbol{x}), f(\boldsymbol{y}) \le c$. Since $f$ is convex, we have

$$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) \le c$$

i.e., $\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in \Gamma_c$ for all $\alpha \in [0, 1]$. Hence $\Gamma_c$ is a convex set.

**Corollary.** Suppose $f : \Omega \to \mathbb{R}$ is convex. Then the set of all global minimizers of $f$ over $\Omega$ is convex.

**Proof.** Let $f^* = \min_{\boldsymbol{x} \in \Omega} f(\boldsymbol{x})$. Then $\Gamma_{f^*}$ is the set of all global minimizers. By the lemma above, we knwo $\Gamma_c$ is a convex set.

**Lemma.** Suppose $f : \Omega \to \mathbb{R}$ is convex and $\mathcal{C}^1$. Then $\boldsymbol{x}^*$ is a global minimizer of $f$ over $\Omega$ iff

$$\nabla f(\boldsymbol{x}^*)^\top(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \quad \forall\, \boldsymbol{x} \in \Omega.$$

**Proof.** ($\Rightarrow$) If not, then $\exists\, \boldsymbol{x} \in \Omega$, s.t.

$$\nabla f(\boldsymbol{x}^*)^\top(\boldsymbol{x} - \boldsymbol{x}^*) < 0$$

Denote $\boldsymbol{x}_\alpha = (1-\alpha)\boldsymbol{x}^* + \alpha\boldsymbol{x} = \boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)$ for $\alpha \in (0,1)$. Since $f \in \mathcal{C}^1$, we know there exists $\alpha$ small enough, s.t.

$$\nabla f(\boldsymbol{x}_{\alpha'})^\top(\boldsymbol{x} - \boldsymbol{x}^*) < 0, \quad \forall\, \alpha' \in (0,\alpha)$$

**Proof (cont.)** Moreover, there exists $\alpha' \in (0, \alpha)$ s.t.

$$\begin{aligned} f(\boldsymbol{x}_\alpha) &= f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}_{\alpha'})^\top (\boldsymbol{x}_\alpha - \boldsymbol{x}^*) \\ &= f(\boldsymbol{x}^*) + \alpha \nabla f(\boldsymbol{x}_{\alpha'})^\top (\boldsymbol{x} - \boldsymbol{x}^*) \\ &< f(\boldsymbol{x}^*) \end{aligned}$$

which contradicts to $\boldsymbol{x}^*$ being a global minimizer.

($\Leftarrow$) For all $\boldsymbol{x} \in \Omega$, there is

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}^*)^\top (\boldsymbol{x} - \boldsymbol{x}^*) \geq f(\boldsymbol{x}^*)$$

Hence $\boldsymbol{x}^*$ is a global minimizer.

**Theorem.** Suppose $f : \Omega \to \mathbb{R}$ is convex and $\mathcal{C}^1$. Then $\boldsymbol{x}^*$ is a global minimizer of $f$ over $\Omega$ iff for any feasible direction $\boldsymbol{d}$ at $\boldsymbol{x}^*$ there is

$$\boldsymbol{d}^\top \nabla f(\boldsymbol{x}^*) \geq 0.$$

**Proof.** ($\Rightarrow$) Let $\boldsymbol{d}$ be feasible, then $\exists\, \boldsymbol{x} \in \Omega$ s.t. $\boldsymbol{x} - \boldsymbol{x}^* = \alpha \boldsymbol{d}$ for some $\alpha > 0$. Hence by the Lemma above, we have

$$\nabla f(\boldsymbol{x}^*)^\top (\boldsymbol{x} - \boldsymbol{x}^*) = \alpha \nabla f(\boldsymbol{x}^*)^\top \boldsymbol{d} \geq 0.$$

So $\nabla f(\boldsymbol{x}^*)^\top \boldsymbol{d} \geq 0$.

($\Leftarrow$) For any $\boldsymbol{x} \in \Omega$, we know $\boldsymbol{x}_\alpha = (1-\alpha)\boldsymbol{x}^* + \alpha \boldsymbol{x} \in \Omega$ for all $\alpha \in (0,1)$. Hence $\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{x}^* = (\boldsymbol{x}_\alpha - \boldsymbol{x}^*)/\alpha$ is a feasible direction. Therefore

$$\nabla f(\boldsymbol{x}^*)^\top (\boldsymbol{x} - \boldsymbol{x}^*) = \nabla f(\boldsymbol{x}^*)^\top \boldsymbol{d} \geq 0.$$

As $\boldsymbol{x} \in \Omega$ is arbitrary, we know $\boldsymbol{x}^*$ is a global minimizer.

**Corollary.** Suppose $f : \Omega \to \mathbb{R}$ is convex and $\mathcal{C}^1$. If $\boldsymbol{x}^* \in \Omega$ is such that

$$\nabla f(\boldsymbol{x}^*) = \mathbf{0},$$

then $\boldsymbol{x}^*$ is a global minimizer of $f$.

**Proof.** For any feasible $\boldsymbol{d}$ there is $\nabla f(\boldsymbol{x}^*)^\top \boldsymbol{d} = 0$. Hence $\boldsymbol{x}^*$ is a global minimizer.

**Theorem.** Let $f : \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathcal{C}^1$ be convex, and $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}\}$ where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\Omega$ is convex. Then $\boldsymbol{x}^* \in \Omega$ is a global minimizer of $f$ over $\Omega$ iff there exists $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that

$$\nabla f(\boldsymbol{x}^*) + D\boldsymbol{h}(\boldsymbol{x}^*)^\top \boldsymbol{\lambda}^* = \boldsymbol{0}.$$

**Proof.** ($\Rightarrow$) By the KKT condition.

($\Leftarrow$) Note that $f$ being convex implies

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}^*)^\top (\boldsymbol{x} - \boldsymbol{x}^*), \quad \forall \boldsymbol{x} \in \Omega$$

Also note that $\nabla f(\boldsymbol{x}^*) = -D\boldsymbol{h}(\boldsymbol{x}^*)^\top \boldsymbol{\lambda}^*$, we know

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) - \boldsymbol{\lambda}^{*\top} D\boldsymbol{h}(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*)$$

**Proof (cont.)** For any $\boldsymbol{x} \in \Omega$, we know $\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*) \in \Omega$ for all $\alpha \in (0, 1)$. Hence $\boldsymbol{h}(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) = \boldsymbol{0}$ and

$$D\boldsymbol{h}(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) = \lim_{\alpha \to 0} \frac{\boldsymbol{h}(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) - \boldsymbol{h}(\boldsymbol{x}^*)}{\alpha} = 0$$

Hence $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*)$ for all $\boldsymbol{x} \in \Omega$. Therefore $\boldsymbol{x}^*$ is a global minimizer.

**Theorem.** Let $f : \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathcal{C}^1$ be convex, and

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0},\ \boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}\}$$

where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$ are $\mathcal{C}^1$ and such that $\Omega$ is convex. Then $\boldsymbol{x}^* \in \Omega$ is a global minimizer of $f$ over $\Omega$ iff there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m, \boldsymbol{\mu}^* \in \mathbb{R}^p_+$ such that

$$\begin{aligned} \nabla f(\boldsymbol{x}^*)^\top + \boldsymbol{\lambda}^{*\top} D\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\top} D\boldsymbol{g}(\boldsymbol{x}^*) &= \boldsymbol{0}^\top, \\ \boldsymbol{g}(\boldsymbol{x}^*)^\top \boldsymbol{\mu}^* &= 0. \end{aligned}$$

**Proof.** ($\Rightarrow$) By the KKT condition.

**Proof (cont.)** ($\Leftarrow$) Note that $f$ being convex implies

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}^*)^\top (\boldsymbol{x} - \boldsymbol{x}^*), \quad \forall\, \boldsymbol{x} \in \Omega.$$

Also note that $\nabla f(\boldsymbol{x}^*) = -D\boldsymbol{h}(\boldsymbol{x}^*)^\top \boldsymbol{\lambda}^* - D\boldsymbol{g}(\boldsymbol{x}^*)^\top \boldsymbol{\mu}^*$, we know

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) - {\boldsymbol{\lambda}^*}^\top D\boldsymbol{h}(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) - {\boldsymbol{\mu}^*}^\top D\boldsymbol{g}(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*).$$

For any $\boldsymbol{x} \in \Omega$, we know $\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*) \in \Omega$ for all $\alpha \in (0, 1)$. Hence $\boldsymbol{h}(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) = \boldsymbol{0}$ and

$$D\boldsymbol{h}(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) = \lim_{\alpha \to 0} \frac{\boldsymbol{h}(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) - \boldsymbol{h}(\boldsymbol{x}^*)}{\alpha} = 0.$$

**Proof (cont.)** Moreover $\boldsymbol{g}(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) \le \mathbf{0}$, and hence $\boldsymbol{\mu}^* \ge \mathbf{0}$ implies

$$\boldsymbol{\mu}^{*\top} \boldsymbol{g}(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) \le 0.$$

Therefore, we have

$$\boldsymbol{\mu}^{*\top} D\boldsymbol{g}(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) = \lim_{\alpha \to 0} \frac{\boldsymbol{\mu}^{*\top} \boldsymbol{g}(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) - \boldsymbol{\mu}^{*\top} \boldsymbol{g}(\boldsymbol{x}^*)}{\alpha} \le 0$$

Hence we obtain

$$f(\boldsymbol{x}) \ge f(\boldsymbol{x}^*), \quad \forall\, \boldsymbol{x} \in \Omega.$$

Therefore $\boldsymbol{x}^*$ is a global minimizer.

**Example.** Suppose we can deposit $x_i \geq 0$ amount of money into a bank account (with initial balance 0) at the beginning of the $i$th month for $i = 1, \ldots, n$. The monthly interest rate is $r > 0$. If the total amount we can deposit is $D$, then find the way to maximize the total balance including the interests at the end of the $n$th month.

**Intuition.** We should deposit all money $D$ in the first month.

**Solution.** Define $\boldsymbol{c} = -[(1+r)^n; \ldots; (1+r)] \in \mathbb{R}^n$. Then this is an LP (which is a convex program):

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{e}^\top \boldsymbol{x} = D \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{e} = [1; \ldots; 1] \in \mathbb{R}^n$.

**Solution (cont.)** To show our intuition is correct, it suffices to show that $\boldsymbol{x}^* = [D; 0; \ldots; 0] \in \mathbb{R}^n$ satisfies the KKT condition:

$$\begin{aligned} \boldsymbol{c} + \lambda^* \boldsymbol{e} - \boldsymbol{\mu}^* &= \boldsymbol{0} \\ \boldsymbol{e}^\top \boldsymbol{x}^* &= D \\ \boldsymbol{x}^* &\geq \boldsymbol{0} \\ \boldsymbol{\mu}^* &\geq \boldsymbol{0} \\ {\boldsymbol{\mu}^*}^\top \boldsymbol{x}^* &= 0 \end{aligned}$$

for some $\lambda \in \mathbb{R}$ and $\boldsymbol{\mu}^* \in \mathbb{R}^n$.

Let $\lambda^* = (1+r)^n$ and $\boldsymbol{\mu}^* = \boldsymbol{c} + \lambda^* \boldsymbol{e}$, then it is easy to verify that $(\boldsymbol{x}^*, \lambda^*, \boldsymbol{\mu}^*)$ satisfies the KKT condition. This implies that $\boldsymbol{x}^*$ is a global minimizer.