Section 3

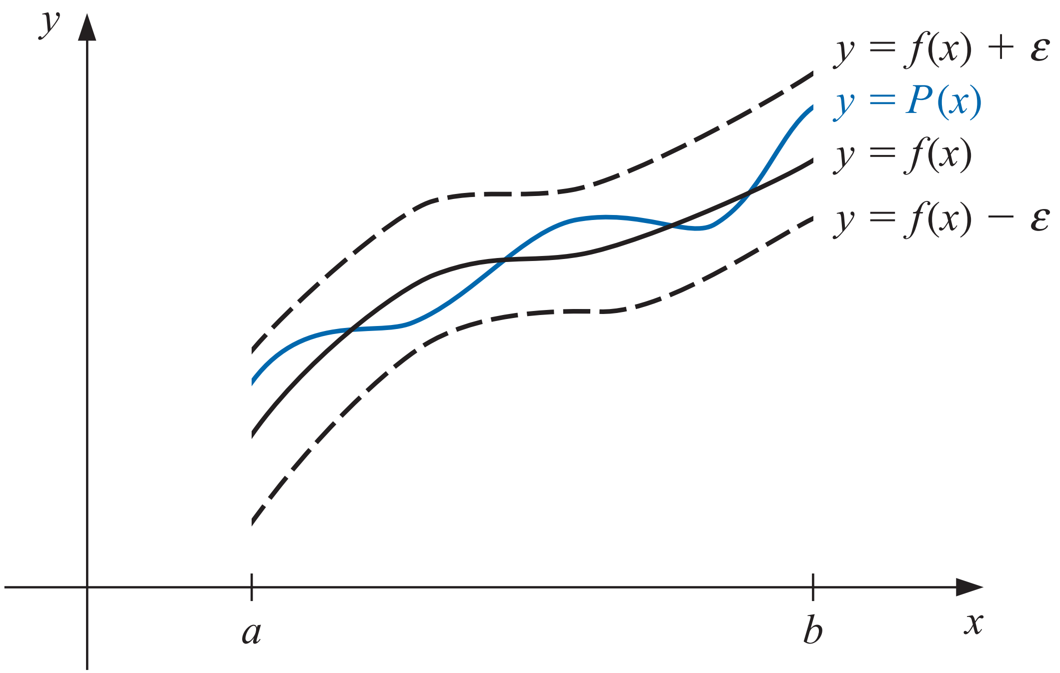
Interpolation and Polynomial Approximation

Interpolation

Given data points $\{(x_i, y_i) : i = 1, \ldots, n\}$, can we find a function to "fit" the data?

Theorem (Weierstrass approximation theorem)

Suppose $f \in C[a, b]$, *then* $\forall \epsilon > 0$, $\exists$ *a polynomial* $P(x)$ *such that* $|f(x) - P(x)| < \epsilon$, $\forall x \in [a, b]$.



Polynomial interpolation

So polynomials could work. But how to find the polynomial?

First Try: Taylor's polynomial

For any given function $f(x)$ and a point x_0, we approximate $f(x)$ by the Taylor's polynomial $P_n(x)$:

$$f(x) \approx P_n(x) := f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

Polynomial interpolation

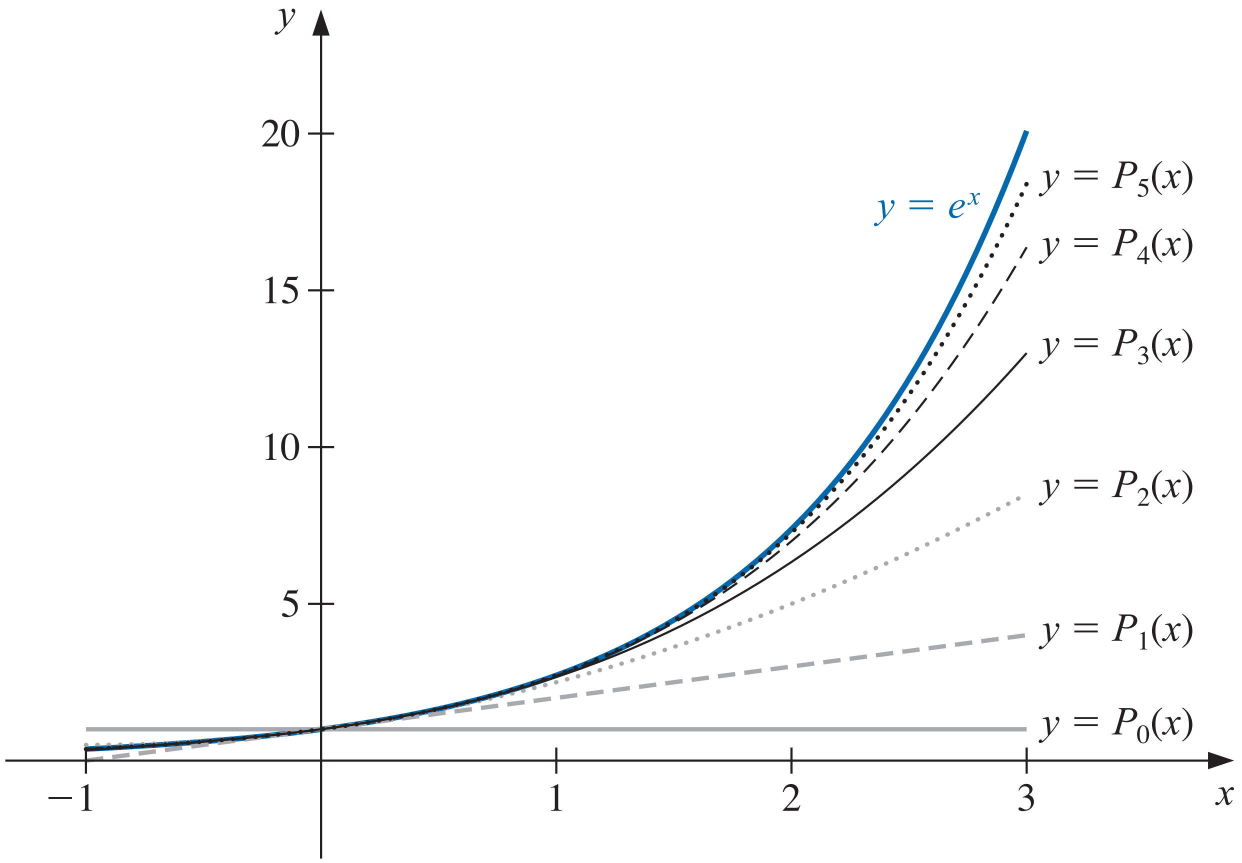
Example (Problem with Taylor's polynomial)

Let $f(x) = e^x$ and $x_0 = 0$. See how Taylor's polynomial behaves.

Solution. Taylor's polynomial $P_n(x) = 1 + x + \cdots + \frac{1}{n!}x^n$.

However, no matter how large we choose n, $P_n(x)$ is far from $f(x)$ where x is slightly large.

Issue with Taylor's polynomial approximation



Example

Example (Problem with Taylor's polynomial)

Let $f(x) = \frac{1}{x}$ and $x_0 = 1$. See how Taylor's polynomial behaves.

Solution. We know $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$. Then Taylor's polynomial is

$$P_n(x) = \sum_{i=0}^{n} (-1)^n (x-1)^n = 1-(x-1)+(x-1)^2+\cdots+(-1)^n(x-1)^n$$

Suppose we use $P_n(x)$ to approximate f at $x = 3$, we get

$P_0(3)$	$P_1(3)$	$P_2(3)$	$P_3(3)$	$P_4(3)$	$P_5(3)$	$P_6(3)$	$P_7(3)$
1	-1	3	-5	11	-21	43	-85

But the true value is $f(3) = \frac{1}{3}$.

Lagrange interpolating polynomial

We should not use Taylor's polynomial since it only approximates well locally.

Suppose we have two points (x_0, y_0) and (x_1, y_1), then best use a straight line to interpolate. Define two linear polynomials:

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

So L_0 and L_1 are polynomials of degree 1, and

$$L_0(x_1) = 0, \quad L_0(x_0) = 1, \quad L_1(x_0) = 0, \quad L_1(x_1) = 1$$

Now set $P(x) = f(x_0)L_0(x) + f(x_1)L_1(x)$, then $P(x)$ coincides $f(x)$ at x_0 and x_1.

Example

Recall that the polynomial we derived is

$$P(x) = f(x_0)L_0(x) + f(x_1)L_1(x) = \frac{x - x_0}{x_1 - x_0} f(x_0) + \frac{x - x_1}{x_0 - x_1} f(x_1)$$

$P(x)$ is called the **Lagrange interpolating polynomial** of f given values at x_0 and x_1.

Example (Linear Lagrange interpolating polynomial)

Use linear Lagrange interpolating polynomial of f where $f(2) = 5$ and $f(4) = 1$.

Solution. $P(x) = -x + 6$.

Lagrange interpolating polynomial

Given $n+1$ points $\left\{(x_i, f(x_i)) : 0 \leq i \leq n\right\}$. For each i, define:

$$L_{n,k} = \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$$

for $k = 0, 1, \dots, n$. Then it is easy to verify

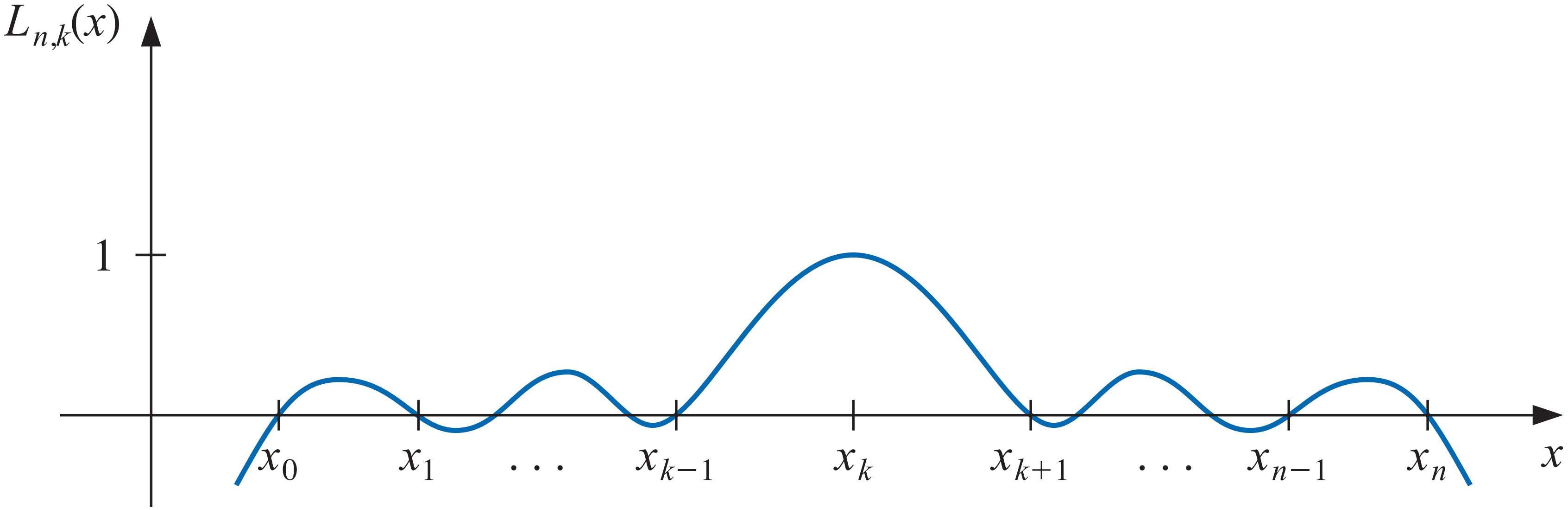
$$L_{n,k}(x) = \begin{cases} 1 & \text{if } x = x_k \\ 0 & \text{if } x = x_j, \text{ where } j \neq k \end{cases}$$

Then the n**th Lagrange interpolating polynomial** of f is

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

Lagrange interpolating polynomial

Illustration of $L_{n,k}(x)$:



Lagrange interpolating polynomial

The n**th Lagrange interpolating polynomial** of f at $x_0, \dots, x_n$ is

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

Properties:

- $P(x)$ is a polynomial of degree n
- $P(x_k) = f(x_k)$ for all $k = 0, \dots, n$.

Example

Example (Lagrange interpolating polynomial)

Let $f(x) = \frac{1}{x}$, $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$. Find the 2nd Lagrange interpolating polynomial $P(x)$ of $f(x)$ and compute $P(3)$.

Solution. First we compute $L_{2,k}$ for $k = 0, 1, 2$:

$$L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2.75)(x - 4)}{(2 - 2.75)(2 - 4)}$$

$$L_{2,1}(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 2)(x - 4)}{(2.75 - 2)(2.75 - 4)}$$

$$L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 2)(x - 2.75)}{(4 - 2)(4 - 2.75)}$$

Then the 2nd Lagrange interpolating polynomial is

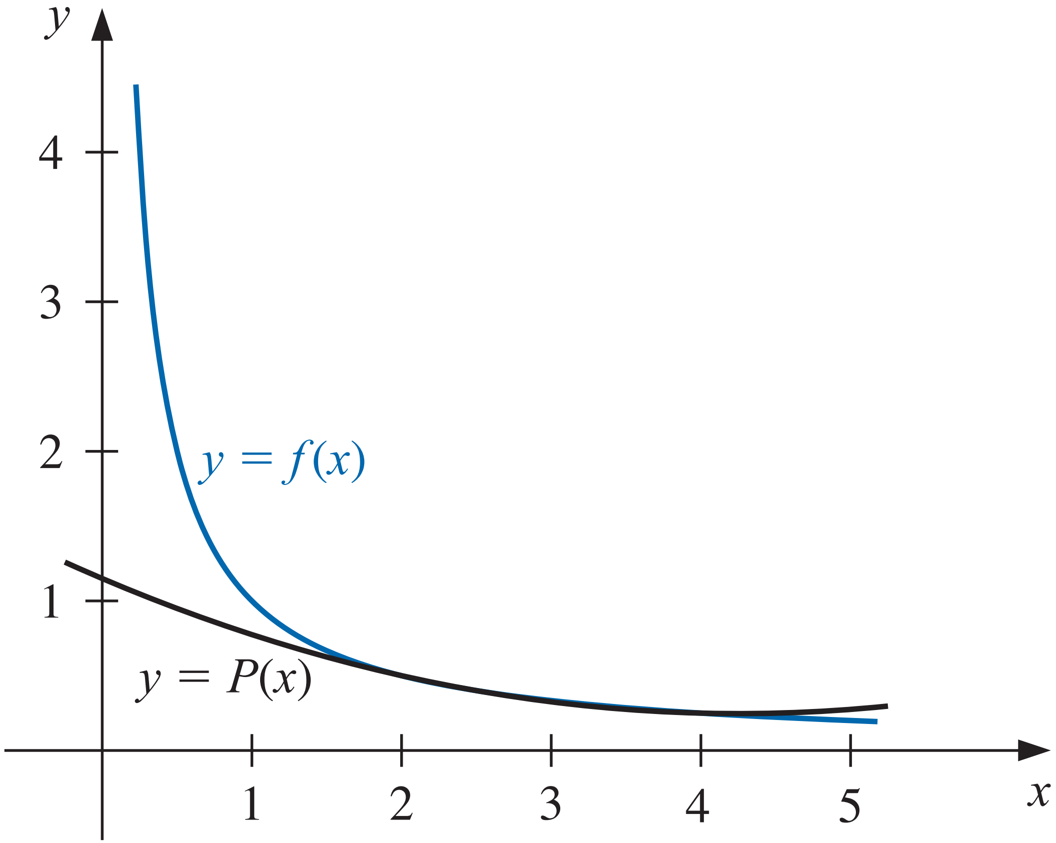
$$P(x) = \sum_{k=0}^{2} f(x_k) L_{2,k}(x) = \cdots = \frac{x^2}{22} - \frac{35x}{88} + \frac{49}{44}$$

Note that $P(3) = \frac{3^2}{22} - \frac{35\times 3}{88} + \frac{49}{44} \approx 0.32955$, close to $f(3) = \frac{1}{3}$.

Example

Example (Lagrange interpolating polynomial)

Let $f(x) = \frac{1}{x}$, $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$. Find the 2nd Lagrange interpolating polynomial $P(x)$ of $f(x)$ and compute $P(3)$.



Lagrange interpolating polynomial

Theorem (Error of Lagrange interpolating polynomial)

Suppose $f(x) \in C^{n+1}[a,b]$. *Then for every* $x \in [a,b]$, $\exists\, \xi(x)$ *between* $x_0, \dots, x_n$, *s.t.*

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)\dots(x - x_n)$$

Error of Lagrange interpolating polynomial

Proof.

For any given $x \in [a, b]$ different from $x_0, \dots, x_n$, define $g(t)$ as

$$g(t) = f(t) - P(t) - \underbrace{(f(x) - P(x))\frac{(t - x_0)\dots(t - x_n)}{(x - x_0)\dots(x - x_n)}}_{\text{polynomial of } t, \text{ degree } n+1}$$

Note that $f(t) = P(t)$ and $(t - x_0)\dots(t - x_n) = 0$ for $t = x_k$ and $k = 0, \dots, n$. So $g(t) = 0$ for $t = x, x_0, \dots, x_n$ (total $n + 2$ points). By generalized Rolle's Thm, $\exists\, \xi(x)$ between $x_0, \dots, x_n$ s.t.

$$0 = g^{(n+1)}(\xi(x)) = f^{(n+1)}(\xi(x)) - \frac{(n+1)! \cdot (f(x) - P(x))}{(x - x_0)\dots(x - x_n)}$$

since $P(t)$ is a poly of t with degree n and $(t - x_0)\cdots(t - x_n)$ is a monic poly of t with degree $n + 1$. □

Example

Example (Estimate error of Lagrange interpolating polynomial)

Let $f(x) = \frac{1}{x}$, $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$. Estimate the maximal error of the 2nd Lagrange interpolating polynomial $P(x)$ given above on $[2,4]$.

Example

Solution. Let $P(x)$ be the Lagrange interpolating polynomial, then

$$f(x) - P(x) = \frac{f^{(3)}(\xi(x))}{3!}(x-2)(x-2.75)(x-4)$$

We know $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = -\frac{3!}{x^4}$, so

$$\left|\frac{f^{(3)}(\xi(x))}{3!}\right| = \left|-\frac{1}{(\xi(x))^4}\right| \le \frac{1}{2^4} \qquad (\because \xi(x) \in [2,4])$$

Further, denote $h(x) := (x-2)(x-2.75)(x-4)$, find critical points and then the max/min values of $h(x)$ on $[2,4]$ to claim $|h(x)| \le \frac{9}{16}$ for all $x \in [2,4]$. Hence

$$|f(x) - P(x)| = \left|\frac{f^{(3)}(\xi(x))}{3!}h(x)\right| \le \frac{1}{2^4}\frac{9}{16} \approx 0.00586.$$

Example

Example (Estimate error of Lagrange interpolating polynomial)

Suppose we use uniform partition of $[0, 1]$ and linear Lagrange interpolating polynomial on each segment to approximate $f(x) = e^x$. How small the step size h should be to guarantee the error $< 10^{-6}$ everywhere?

Example

Solution. With step size h, we have $x_j = jh$ for $j = 0, 1, \dots$.

Then we use linear Lagrange polynomial to approximate e^x on each $[x_j, x_{j+1}]$. The error is

$$\frac{1}{2} f^{(2)}(\xi(x))(x - x_j)(x - x_{j+1})$$

So $|\frac{f^{(2)}(\xi(x))}{2}| = |\frac{e^{\xi(x)}}{2}| \le \frac{e}{2}$ $(\because \xi(x) \in [0, 1])$.

Again take $h(x) = (x - x_j)(x - x_{j+1})$ which has max $\frac{h^2}{2}$. Then

$$\left| \frac{f^{(2)}(\xi(x))}{2}(x - x_j)(x - x_{j+1}) \right| \le \frac{e}{2}\frac{h^2}{4} \le 10^{-6}$$

So we need $h \le (8 \times 10^{-6} \times e^{-1})^{1/2} \approx 1.72 \times 10^{-3}$.

Recursive constructions of interpolating polynomials

Given points $x_0, \dots, x_n$ and function values $f(x_k)$ for $k = 0, \dots, n$.

There are several questions regarding the use Lagrange interpolating polynomial:

- Can we use a subset of points to construct Lagrange interpolating polynomials with lower degree?
- If yes, which interpolating points among $x_0, \dots, x_n$ to choose?
- If the result is not satisfactory, can we improve the constructed polynomial to get a polynomial of higher degree?

Example

Example (Which points to choose?)

Consider the interpolation of the function f with 5 points:

k	x_k	$f(x_k)$
0	1.0	0.7651977
1	1.3	0.6200860
2	1.6	0.4554022
3	1.9	0.2818186
4	2.2	0.1103623

If we use an interpolating polynomial of degree $n < 4$, then we need to decide which points to use.

For example, if $n = 2$, then we need to chose 3 points. Should we choose x_0, x_1, x_2 or x_1, x_2, x_3, or x_0, x_2, x_4?

Neville's method

We do not know which choice is better, since true $f(x)$ is unknown. But we can compute all and see the trend.

Question: can we use polynomials obtained earlier (with lower degree) to get the later ones (with higher degree)?

Definition (Partial interpolating polynomial)

Let f be a function with known values at $x_0, \dots, x_n$ and suppose $m_1, \dots, m_k$ are k integers among $0, 1, \dots, n$. Then the partial Lagrange interpolating polynomial that agrees with f at $x_{m_1}, \dots, x_{m_k}$ is denoted by $P_{m_1,\dots,m_k}(x)$.

Example

Example (Partial interpolating polynomial)

Let $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 6$ for $f(x) = e^x$. Find $P_{1,2,4}(x)$ and approximate the value $f(5)$.

Solution. We only use x_1, x_2, x_4 to get $P_{1,2,4}(x)$:

$$\begin{aligned}
P_{1,2,4}(x) &= \frac{(x - x_2)(x - x_4)}{(x_1 - x_2)(x_1 - x_4)} f(x_1) + \frac{(x - x_1)(x - x_4)}{(x_2 - x_1)(x_2 - x_4)} f(x_2) + \frac{(x - x_1)(x - x_2)}{(x_4 - x_1)(x_4 - x_2)} f(x_4) \\
&= \frac{(x - 3)(x - 6)}{(2 - 3)(2 - 6)} e^2 + \frac{(x - 2)(x - 6)}{(3 - 2)(3 - 6)} e^3 + \frac{(x - 2)(x - 3)}{(6 - 2)(6 - 3)} e^6
\end{aligned}$$

$$P_{1,2,4}(5) = -\frac{1}{2}e^2 + e^3 + \frac{1}{2}e^6 \approx 218.105$$

Recursive construction of interpolating polynomials

Now we show how to recursively construct Lagrange interpolating polynomials:

Theorem (Recursive construction of interpolating polynomials)

Let f be defined at $x_0, \dots, x_k$, and x_i and x_j are two distinct points among them. Then

$$P_{0,1,\dots,k}(x) = \frac{(x - x_j) P_{0,\dots,\hat{j},\dots,k}(x) - (x - x_i) P_{0,\dots,\hat{i},\dots,k}(x)}{x_i - x_j}$$

Recursive construction of interpolating polynomials

Proof.

Denote the RHS by $P(x)$.

Both $P_{0,\dots,\hat{j},\dots,k}(x)$ and $P_{0,\dots,\hat{i},\dots,k}(x)$ are polynomials of degree $k-1$, we know $P(x)$ is a polynomial of degree $\le k$.

Verify that $P(x_s) = f(x_s)$ for $s = 0, 1, \dots, k$. So $P(x) = P_{0,\dots,k}(x)$. $\square$

Neville's method

Suppose there are 5 points $x_0, \dots, x_4$, and $P_i := f(x_i)$ for all i, then we can construct the following table:

$$
\begin{array}{lllll}
x_0 & P_0 & & & \\
x_1 & P_1 & P_{0,1} = \frac{(x-x_0)P_1-(x-x_1)P_0}{x_1-x_0} & & \\
x_2 & P_2 & P_{1,2} = \frac{(x-x_1)P_2-(x-x_2)P_1}{x_2-x_1} & P_{0,1,2} = \frac{(x-x_0)P_{1,2}-(x-x_2)P_{0,1}}{x_2-x_0} & \\
x_3 & P_3 & P_{2,3} = \frac{(x-x_2)P_3-(x-x_3)P_2}{x_3-x_2} & P_{1,2,3} = \frac{(x-x_1)P_{2,3}-(x-x_3)P_{1,2}}{x_3-x_1} & \ddots \\
x_4 & P_4 & P_{3,4} = \frac{(x-x_3)P_4-(x-x_4)P_3}{x_4-x_3} & P_{2,3,4} = \frac{(x-x_2)P_{3,4}-(x-x_4)P_{2,3}}{x_4-x_2} & \dots
\end{array}
$$

Neville's method

We introduce a new notation $Q_{ij} = P_{i-j,i-j+1,\dots,i}$ (i is the ending index and $j+1$ is the length), then the previous table is just

$$\begin{array}{llllll}
x_0 & Q_{0,0} & & & & \\
x_1 & Q_{1,0} & Q_{1,1} & & & \\
x_2 & Q_{2,0} & Q_{2,1} & Q_{2,2} & & \\
x_3 & Q_{3,0} & Q_{3,1} & Q_{3,2} & Q_{3,3} & \\
x_4 & Q_{4,0} & Q_{4,1} & Q_{4,2} & Q_{4,3} & Q_{4,4}
\end{array}$$

For example $Q_{3,3} = P_{0,1,2,3}$, $Q_{4,3} = P_{1,2,3,4}$, etc.

Example (Neville's method)

Consider the interpolation of the function f with 5 points:

k	x_k	$f(x_k)$
0	1.0	0.7651977
1	1.3	0.6200860
2	1.6	0.4554022
3	1.9	0.2818186
4	2.2	0.1103623

In addition, interpolate $f(1.5)$ and compare to the true value[2].

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[2] The data in this table were retrieved from a Bessel function with true value $f(1.5) = 0.5118277$.

Neville's iterated interpolation

Neville's iterated interpolation method:

- **Input.** $x_0, \dots, x_n$ and values $Q_{i,0} = f(x_i)$ for all i.
- For each $i = 1, \dots, n$: compute $Q_{i,j} = \frac{(x-x_{i-j})Q_{i,j-1}-(x-x_i)Q_{i-1,j-1}}{x_i-x_{i-j}}$ for $j = 1, \dots, i$.
- **Output.** Table Q with $P(x) = Q_{n,n}$.

Properties of Neville's method:

1. Add new interpolating nodes easily.
2. Can stop if $|Q_{i,i} - Q_{i-1,i-1}| < \epsilon_{\text{tol}}$.

Divided difference

We can also get the polynomials, not just the interpolating values.

Consider the polynomial $P_n(x)$ of degree n defined by

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

To make it the Lagrangian interpolating polynomial of f at $x_0, \dots, x_n$, we need to find a_i s.t. $P_n(x_i) = f(x_i)$ for all x_i.

It is easy to check that:

$$\begin{aligned} P_n(x_0) &= a_0 = f(x_0) && \Longrightarrow \quad a_0 = f(x_0) \\ P_n(x_1) &= a_0 + a_1(x_1 - x_0) = f(x_1) && \Longrightarrow \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &\vdots \end{aligned}$$

Divided difference

We define the following notations of **divided difference**:

$$
\begin{aligned}
f[x_i] &= f(x_i) \\
f[x_i, x_{i+1}] &= \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \\
f[x_i, x_{i+1}, x_{i+2}] &= \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i} \\
&\vdots
\end{aligned}
$$

Once the $(k-1)$th divided differences are determined, we can get the kth divided difference as

$$f[x_0, \dots, x_k] = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

until we get $f[x_0, \dots, x_n]$. Then set $a_k = f[x_0, \dots, x_k]$ for all k:

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k](x - x_0) \dots (x - x_k)$$

Divided difference

We can construct a table of divided difference as follows:

$$\begin{array}{llllll}
x_0 & f[x_0] & & & & \\
x_1 & f[x_1] & f[x_0, x_1] & & & \\
x_2 & f[x_2] & f[x_1, x_2] & f[x_0, x_1, x_2] & & \\
x_3 & f[x_3] & f[x_2, x_3] & f[x_1, x_2, x_3] & f[x_0, x_1, x_2, x_3] & \\
x_4 & f[x_4] & f[x_3, x_4] & f[x_2, x_3, x_4] & f[x_1, x_2, x_3, x_4] & f[x_0, x_1, x_2, x_3, x_4]
\end{array}$$

Divided difference

We can introduce a new notation $F_{i,j} = f[x_{i-j}, \ldots, x_i]$, then the table can be written as

$$\begin{array}{cccccc}
x_0 & F_{0,0} & & & & \\
x_1 & F_{1,0} & F_{1,1} & & & \\
x_2 & F_{2,0} & F_{2,1} & F_{2,2} & & \\
x_3 & F_{3,0} & F_{3,1} & F_{3,2} & F_{3,3} & \\
x_4 & F_{4,0} & F_{4,1} & F_{4,2} & F_{4,3} & F_{4,4}
\end{array}$$

Newton's divided difference formula

Newton's divided difference

- **Input.** $x_0, \dots, x_n$ and values $F_{i,0} = f(x_i)$ for all i.
- For each $i = 1, \dots, n$: set $F_{i,j} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$ for $j = 1, \dots, i$.
- **Output.** $F_{i,i}$ for $i = 0, \dots, n$, and set

$$P_n(x) = F_{0,0} + \sum_{i=1}^{n} F_{i,i}(x - x_0) \dots (x - x_{i-1})$$

Special case

In the special case where $x_{i+1} - x_i = h$ for all i, then $x_i = x_0 + ih$. Now if we want to know the value of f at $x_s = x_0 + sh$ (s can be non-integer), then

$$\begin{aligned}
P_n(x_s) &= f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k](x_s - x_0) \dots (x_s - x_{k-1}) \\
&= f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k](sh)((s-1)h) \dots ((s-k+1)h) \\
&= f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k] h^k \frac{s(s-1)\dots(s-k+1)}{k!} k! \\
&= f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k] h^k k! \binom{s}{k}
\end{aligned}$$

Special case

If we adopt the Aitkin's Δ^2 to simplify notations:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}(f(x_1) - f(x_0)) = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}(\frac{1}{h}\Delta f(x_1) - \frac{1}{h}\Delta f(x_0)) = \frac{1}{2h^2}\Delta^2 f(x_0)$$

$$\vdots$$

$$f[x_0, \dots, x_k] = \cdots = \frac{1}{k!h^k}\Delta^k f(x_0)$$

Newton's divided difference becomes:

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$$

Backward difference

We can also use the backward differences:

$$\nabla p_n := p_n - p_{n-1} \quad \text{and} \quad \nabla^k p_n = \nabla(\nabla^{k-1} p_n)$$ [3]

Suppose the points are in reverse order: $x_n, x_{n-1}, \dots, x_0$, then

$$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + \cdots + f[x_n, \dots, x_0](x - x_n)\dots(x - x_1).$$

If $x_s = x_n + sh$ (s is negative non-integer), then we can derive:

$$P_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

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[3] For example, $\nabla^2 p_n = (p_n - p_{n-1}) - (p_{n-1} - p_{n-2}) = p_n - 2p_{n-1} + p_{n-2}$.

Hermite interpolation

Suppose we also have derivatives $f^{(k)}(x_i)$ at points x_i for $k = 0, \dots, m_i$, we can find the polynomial $P(x)$ s.t.

$$P^{(k)}(x_i) = f^{(k)}(x_i), \qquad \forall\, i, k$$

The total number of conditions (values) we have is

$$\hat{n} := \sum_{i=0}^{n} (m_i + 1) = (n+1) + \sum_{i=0}^{n} m_i$$

So we can find a polynomial P of degree $\hat{n}$.

Such a polynomial is called an *osculating polynomial*.

Hermite polynomial

We're mostly interested in the case with $m_i = 1, \forall\, i$. That is, we have $f(x_i)$ and $f'(x_i)$ at each x_i.

We want to construct a polynomial $P(x)$ of degree $2n+1$, s.t. $P(x_i) = f(x_i)$ and $P'(x_i) = f(x_i), \forall\, i$.

Let $L_{n,j}(x)$ be the Lagrange polynomial of degree n such that

$$L_{n,j}(x_i) = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases}$$

We define two polynomials (both of degree $2n+1$):

$$\begin{aligned} H_{n,j}(x) &= (1 - 2(x - x_j)L'_{n,j}(x_j))L^2_{n,j}(x) \\ \hat{H}_{n,j}(x) &= (x - x_j)L^2_{n,j}(x) \end{aligned}$$

Hermite polynomial

Theorem (Construction of Hermite polynomial)

If $f \in C^1[a,b]$ *and* $x_0, \dots, x_n \in [a,b]$ *are distinct, then the polynomial of least degree that satisfies* $P(x_i) = f(x_i)$ *and* $P'(x_i) = f'(x_i)$ *is*

$$H_{2n+1}(x) := \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

which has degree $\leq 2n+1$*.*

Hermite polynomial

Proof.

It's clear the degree $\leq n+1$. Also,

$$H_{n,j}(x_i) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \quad \text{and} \quad \hat{H}_{n,j}(x_i) = 0, \forall i$$

So $H_{2n+1}(x_i) = f(x_i)\ \forall\, i$. Also

$$H'_{n,j}(x) = -2L'_{n,j}(x_j)L^2_{n,j}(x) + (2-4(x-x_j)L'_{n,j}(x_j))L_{n,j}(x)L'_{n,j}(x)$$
$$\hat{H}'_{n,j}(x) = L^2_{n,j}(x) + 2(x-x_j)L_{n,j}(x)L'_{n,j}(x)$$

Therefore

$$H'_{n,j}(x_i) = 0 \quad \forall\, i, \quad \text{and} \quad \hat{H}'_{n,j}(x) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

Hence $H'_{2n+1}(x) = f'(x_i)$, $\forall\, i$. □

Hermite polynomials

We can also construct Hermite polynomials using divided difference.

Suppose we have $x_0, x_1, \dots, x_n$ and $f(x_i), f'(x_i)$ are given. Define $z_{2i} = z_{2i+1} = x_i$ for $i = 0, \dots, n$

For example, $z_0 = z_1 = x_0$, $z_2 = z_3 = x_1$, etc.

Now we have $z_0, z_1, \dots, z_{2n+1}$, total of $2(n+1)$ points. So

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](z - z_0) \dots (z - z_k)$$

and use $f'(x_i)$ as $f[z_{2i}, z_{2i+1}]$ for all $i = 0, \dots, n$.

Hermite polynomial

Then we construct the table as follows,

$z_0 = x_0$	$f[z_0] = f(x_0)$				
$z_1 = x_0$	$f[z_1] = f(x_0)$	$f[z_0, z_1] = f'(x_0)$			
$z_2 = x_1$	$f[z_2] = f(x_1)$	$f[z_1, z_2] = \frac{f[z_2]-f[z_1]}{z_2-z_1}$	$f[z_0, z_1, z_2]$		
$z_3 = x_1$	$f[z_3] = f(x_1)$	$f[z_2, z_3] = f'(x_1)$	$f[z_1, z_2, z_3]$	$f[z_0, z_1, z_2, z_3]$	
$z_4 = x_2$	$f[z_4] = f(x_2)$	$f[z_3, z_4] = \frac{f[z_4]-f[z_3]}{z_4-z_3}$	$f[z_2, z_3, z_4]$	$f[z_1, z_2, z_3, z_4]$	$f[z_0, z_1, z_2, z_3, z_4]$
$z_5 = x_3$	$f[z_5] = f(x_3)$	$f[z_4, z_5] = f'(x_2)$	$f[z_3, z_4, z_5]$	$f[z_2, z_3, z_4, z_5]$	$f[z_1, z_2, z_3, z_4, z_5]$
$\vdots$					

Hermite interpolation

Hermite interpolation polynomial

- **Input.** Distinct $x_0, \dots, x_n$, $f(x_i), f'(x_i)\ \forall\, i$.
- For $i = 0, \dots, n$, do (# Assign values $Q_{\cdot,0}, Q_{\cdot,1}$)
    1. Set $z_{2i} = z_{2i+1} = x_i$, $Q_{2i,0} = Q_{2i+1,0} = f(x_i)$, $Q_{2i+1,1} = f'(x_i)$.
    2. If $i \neq 0$, then set $Q_{2i,1} = \frac{Q_{2i,0} - Q_{2i-1,0}}{z_{2i} - z_{2i-1}}$.
- For $i = 2, \dots, 2n+1$ and $j = 2, \dots, i$, set

$$Q_{i,j} = \frac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$$

- **Output.** Hermite polynomial coeff. $Q_{0,0}, \dots, Q_{2n+1,2n+1}$, s.t.

$$\begin{aligned} H(x) = & Q_{0,0} + Q_{1,1}(x - x_0) + Q_{2,2}(x - x_0)^2 + \cdots \\ & + Q_{2n+1,2n+1}(x - x_0)^2 \dots (x - x_n)^2 \end{aligned}$$