Linearly independent functions

Definition

The set of functions $\{\phi_1, \dots, \phi_n\}$ is called **linearly independent** on $[a, b]$ if

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0, \quad \text{for all } x \in [a, b]$$

implies that $c_1 = c_2 = \cdots = c_n = 0$.

Otherwise the set of functions is called **linearly dependent**.

Linearly independent functions

Example

Suppose $\phi_j(x)$ is a polynomial of degree j for $j = 0, 1, \ldots, n$, then $\{\phi_0, \ldots, \phi_n\}$ is linearly independent on any interval $[a, b]$.

Proof.

Suppose there exist $c_0, \ldots, c_n$ such that

$$c_0\phi_0(x) + \cdots + c_n\phi_n(x) = 0$$

for all $x \in [a, b]$. If $c_n \neq 0$, then this is a polynomial of degree n and can have at most n roots, contradiction. Hence $c_n = 0$. Repeat this to show that $c_0 = \cdots = c_n = 0$. □

Linearly independent functions

Example

Suppose $\phi_0(x) = 2, \phi_1(x) = x - 3, \phi_2(x) = x^2 + 2x + 7$, and $Q(x) = a_0 + a_1 x + a_2 x^2$. Show that there exist constants c_0, c_1, c_2 such that $Q(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x)$.

Solution. Substitute ϕ_j into $Q(x)$, and solve for c_0, c_1, c_2.

Linearly independent functions

We denote $\Pi_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$, i.e., Π_n is the set of polynomials of degree $\leq n$.

Theorem

Suppose $\{\phi_0, \ldots, \phi_n\}$ is a collection of linearly independent polynomials in Π_n, then any polynomial in Π_n can be written uniquely as a linear combination of $\phi_0(x), \ldots, \phi_n(x)$.

$\{\phi_0, \ldots, \phi_n\}$ is called a **basis** of Π_n.

Orthogonal functions

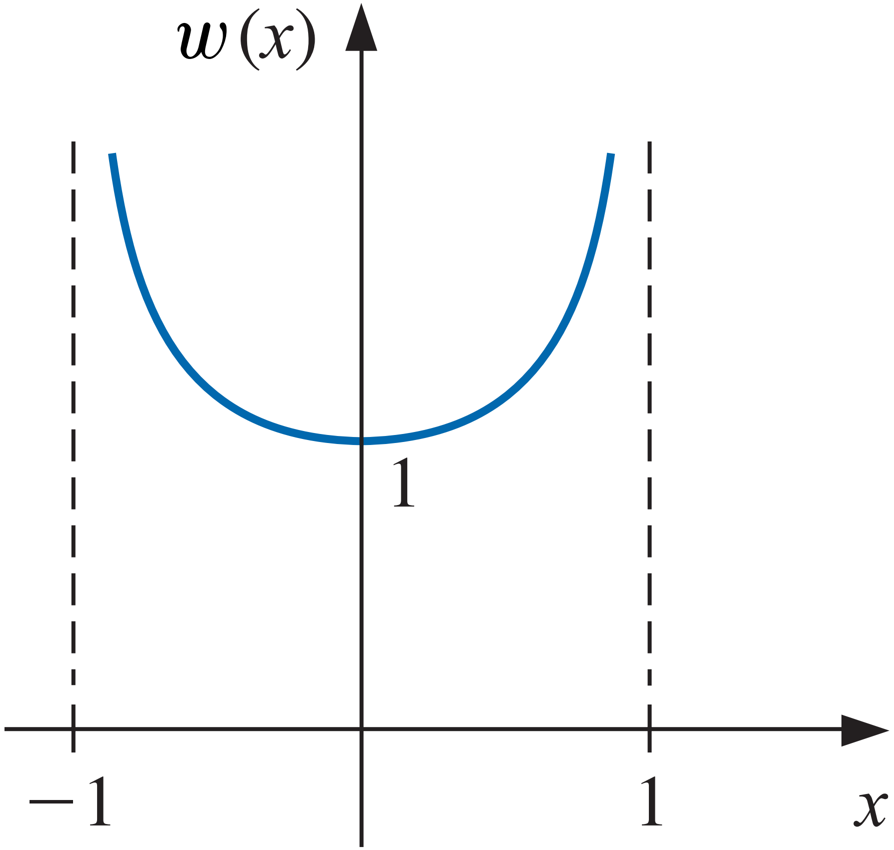
Definition

An integrable function w is called a **weight function** on the interval I if $w(x) \geq 0$, for all $x \in I$, but $w(x) \not\equiv 0$ on any subinterval of I.

Orthogonal functions

Example

Define a weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on interval $(-1, 1)$.



Orthogonal functions

Suppose $\{\phi_0, \dots, \phi_n\}$ is a set of linearly independent functions in $C[a, b]$ and w is a weight function on $[a, b]$. Given $f(x) \in C[a, b]$, we seek a linear combination

$$\sum_{k=0}^{n} a_k \phi_k(x)$$

to minimize the least squares error:

$$E(a) = \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x) \right]^2 \mathrm{d}x$$

where $a = (a_0, \dots, a_n)$.

Orthogonal functions

As before, we need to solve a^* from $\nabla E(a) = 0$:

$$\frac{\partial E}{\partial a_j} = \int_a^b w(x) \left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x) \right] \phi_j(x) \, \mathrm{d}x = 0$$

for all $j = 0, \dots, n$. Then we obtain the normal equation

$$\sum_{k=0}^{n} \left(\int_a^b w(x) \phi_k(x) \phi_j(x) \, \mathrm{d}x \right) a_k = \int_a^b w(x) f(x) \phi_j(x) \, \mathrm{d}x$$

which is a linear system of $n+1$ equations about $a = (a_0, \dots, a_n)^\top$.

Orthogonal functions

If we chose the basis $\{\phi_0, \dots, \phi_n\}$ such that

$$\int_a^b w(x)\phi_k(x)\phi_j(x)\,\mathrm{d}x = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_j, & \text{when } j = k \end{cases}$$

for some $\alpha_j > 0$, then the LHS of the normal equation simplifies to $\alpha_j a_j$. Hence we obtain closed form solution a_j:

$$a_j = \frac{1}{\alpha_j}\int_a^b w(x)f(x)\phi_j(x)\,\mathrm{d}x$$

for $j = 0, \dots, n$.

Orthogonal functions

Definition

A set $\{\phi_0, \dots, \phi_n\}$ is called **orthogonal** on the interval $[a, b]$ with respect to weight function w if

$$\int_a^b w(x)\phi_k(x)\phi_j(x)\,\mathrm{d}x = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_j, & \text{when } j = k \end{cases}$$

for some $\alpha_j > 0$ for all $j = 0, \dots, n$.

If in addition $\alpha_j = 1$ for all $j = 0, \dots, n$, then the set is called **orthonormal** with respect to w.

The definition above applies to general functions, but for now we focus on orthogonal/orthonormal polynomials only.

Gram-Schmidt process

Theorem

A set of orthogonal polynomials $\{\phi_0, \dots, \phi_n\}$ *on* $[a, b]$ *with respect to weight function* w *can be constructed in the recursive way*

- *First define*

$$\phi_0(x) = 1, \quad \phi_1(x) = x - \frac{\int_a^b x w(x)\, \mathrm{d}x}{\int_a^b w(x)\, \mathrm{d}x}$$

- *Then for every* $k \geq 2$, *define*

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k \phi_{k-2}(x)$$

where

$$B_k = \frac{\int_a^b x w(x)[\phi_{k-1}(x)]^2\, \mathrm{d}x}{\int_a^b w(x)[\phi_{k-1}(x)]^2\, \mathrm{d}x}, \quad C_k = \frac{\int_a^b x w(x)\phi_{k-1}(x)\phi_{k-2}(x)\, \mathrm{d}x}{\int_a^b w(x)[\phi_{k-2}(x)]^2\, \mathrm{d}x}$$

Orthogonal polynomials

Corollary

Let $\{\phi_0, \dots, \phi_n\}$ *be constructed by the Gram-Schmidt process in the theorem above, then for any polynomial* $Q_k(x)$ *of degree* $k < n$, *there is*

$$\int_a^b w(x)\phi_n(x)Q_k(x)\,\mathrm{d}x = 0$$

Proof.

$Q_k(x)$ can be written as a linear combination of $\phi_0(x), \dots, \phi_k(x)$, which are all orthogonal to ϕ_n with respect to w. □

Legendre polynomials

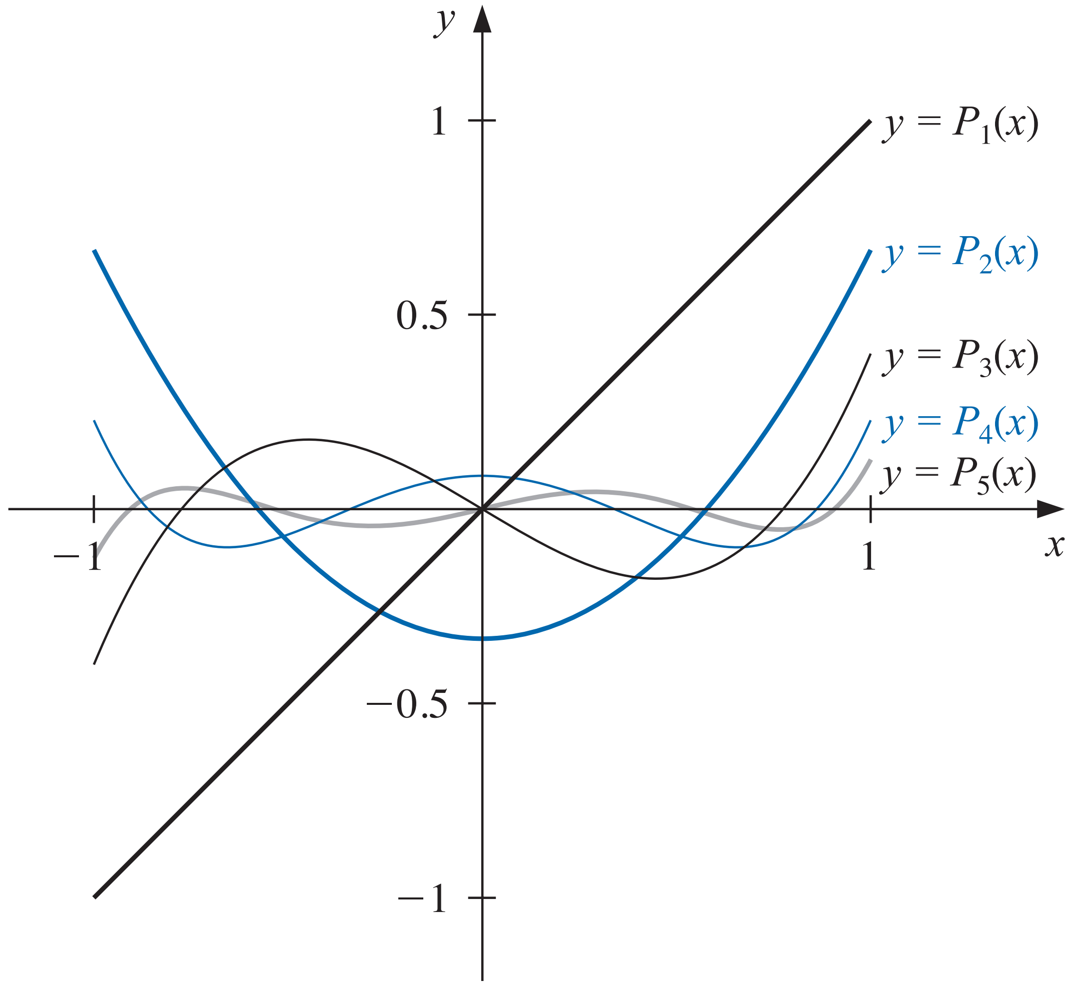
Using weight function $w(x) \equiv 1$ on $[-1, 1]$, we can construct **Legendre polynomials** using the recursive process above to get

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - \frac{1}{3} \\
P_3(x) &= x^3 - \frac{3}{5}x \\
P_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35} \\
P_5(x) &= x^5 - \frac{10}{9}x^3 + \frac{5}{21}x \\
&\vdots
\end{aligned}
$$

Use the Gram-Schmidt process to construct them by yourself.

Legendre polynomials

The first few Legendre polynomials:



Chebyshev polynomials

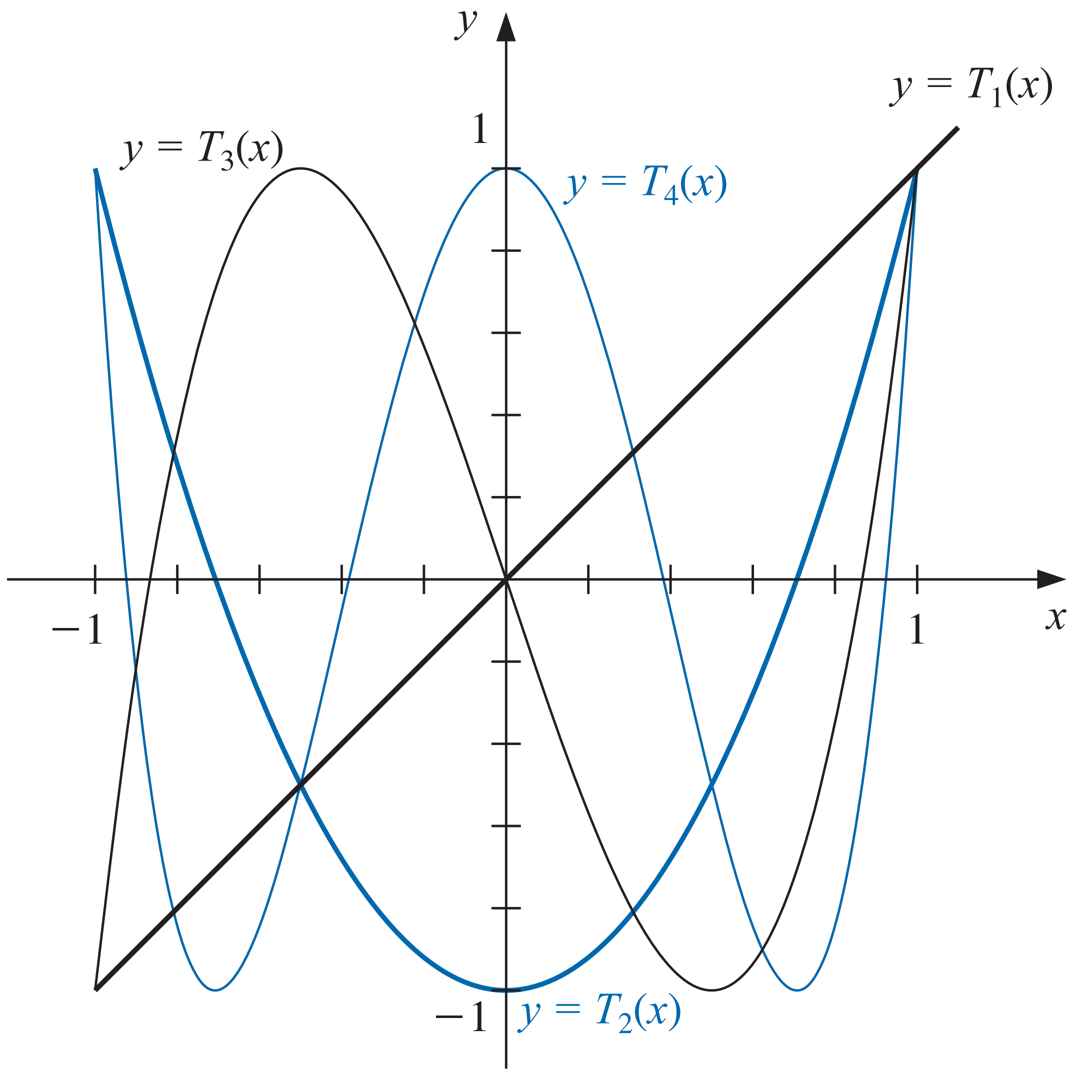
Using weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $(-1, 1)$, we can construct **Chebyshev polynomials** using the recursive process above to get

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
&\vdots
\end{aligned}
$$

It can be shown that $T_n(x) = \cos(n \arccos x)$ for $n = 0, 1, \dots$

Chebyshev polynomials

The first few Chebyshev polynomials:



Chebyshev polynomials

The Chebyshev polynomials $T_n(x)$ of degree $n \geq 1$ has n simple zeros in $[-1,1]$ (from right to left) at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad \text{for each } k = 1,2,\dots,n$$

Moreover, T_n has maximum/minimum (from right to left) at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right) \;\; \textit{where } T_n(\bar{x}'_k) = (-1)^k \text{ for each } k = 0,1,2,\dots,n$$

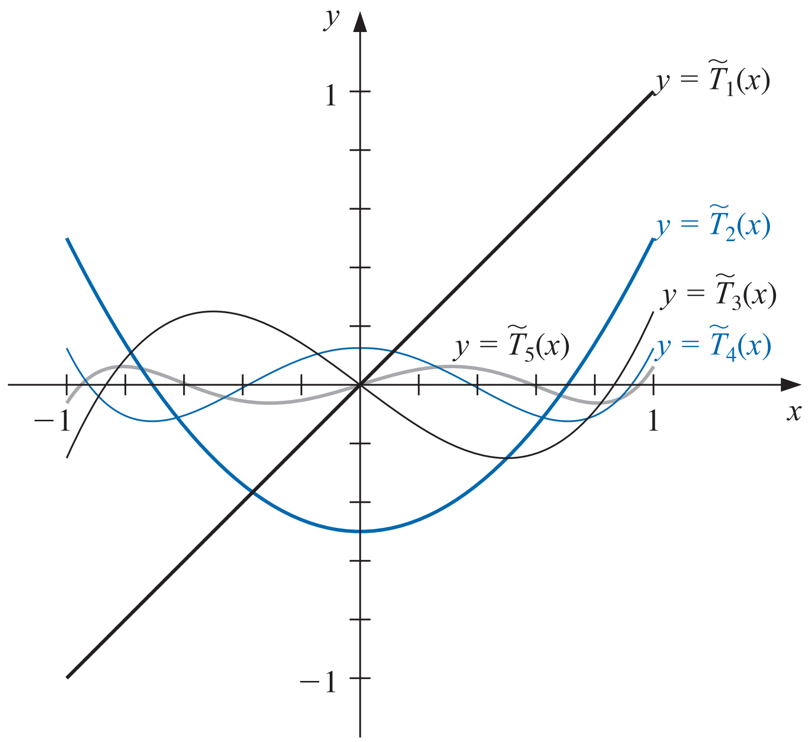
Therefore $T_n(x)$ has n distinct roots and $n+1$ extreme points on $[-1,1]$. These $2n+1$ points, from right to left, are max, zero, min, zero, max ...

Monic Chebyshev polynomials

The monic Chebyshev polynomials $\tilde{T}_n(x)$ are given by $\tilde{T}_0 = 1$ and

$$\tilde{T}_n = \frac{1}{2^{n-1}} T_n(x)$$

for $n \geq 1$.



Monic Chebyshev polynomials

The monic Chebyshev polynomials are

$$\begin{aligned}
\tilde{T}_0(x) &= 1 \\
\tilde{T}_1(x) &= x \\
\tilde{T}_2(x) &= x^2 - \frac{1}{2} \\
\tilde{T}_3(x) &= x^3 - \frac{3}{4}x \\
\tilde{T}_4(x) &= x^4 - x^2 + \frac{1}{8} \\
&\vdots
\end{aligned}$$

Monic Chebyshev polynomials

The monic Chebyshev polynomials $\tilde{T}_n(x)$ of degree $n \geq 1$ has n simple zeros in $[-1, 1]$ at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad \text{for each } k = 1, 2, \ldots, n$$

Moreover, T_n has maximum/minimum at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right) \text{ where } T_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}}, \text{ for each } k = 0, 1, \ldots, n$$

Therefore $\tilde{T}_n(x)$ also has n distinct roots and $n+1$ extreme points on $[-1, 1]$.

Monic Chebyshev polynomials

Denote $\tilde{\Pi}_n$ be the set of monic polynomials of degree n.

Theorem

For any $P_n \in \tilde{\Pi}_n$, there is

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)| \leq \max_{x\in[-1,1]} |P_n(x)|$$

The "$=$" holds only if $P_n \equiv \tilde{T}_n$.

Monic Chebyshev polynomials

Proof.

Assume not, then $\exists P_n(x) \in \tilde{\Pi}_n$, s.t. $\max_{x\in[-1,1]} |P_n(x)| < \frac{1}{2^{n-1}}$.

Let $Q(x) := \tilde{T}_n(x) - P_n(x)$. Since $\tilde{T}_n, P_n \in \tilde{\Pi}_n$, we know $Q(x)$ is a ploynomial of degree at most $n-1$. At the $n+1$ extreme points $\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right)$ for $k = 0, 1, \dots, n$, there are

$$Q(\bar{x}'_k) = \tilde{T}_n(\bar{x}'_k) - P_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}'_k)$$

Hence $Q(\bar{x}'_k) > 0$ when k is even and < 0 when k odd. By intermediate value theorem, Q has at least n distinct roots, contradiction to $\deg(Q) \leq n-1$. $\square$

Minimizing Lagrange interpolation error

Let $x_0, \ldots, x_n$ be $n+1$ distinct points on $[-1,1]$ and $f(x) \in C^{n+1}[-1,1]$, recall that the Lagrange interpolating polynomial $P(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$ satisfies

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

for some $\xi(x) \in (-1,1)$ at every $x \in [-1,1]$.

We can control the size of $(x-x_0)(x-x_1)\cdots(x-x_n)$ since it belongs to $\tilde{\Pi}_{n+1}$: set $(x-x_0)(x-x_1)\cdots(x-x_n) = \tilde{T}_{n+1}(x)$. That is, set $x_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, the kth root of $\tilde{T}_{n+1}(x)$ for $k = 1, \ldots, n+1$. This results in the minimal $\max_{x\in[-1,1]} |(x-x_0)(x-x_1)\cdots(x-x_n)| = \frac{1}{2^n}$.

Minimizing Lagrange interpolation error

Corollary

Let $P(x)$ be the Lagrange interpolating polynomial with $n+1$ points chosen as the roots of $\tilde{T}_{n+1}(x)$, there is

$$\max_{x\in[-1,1]} |f(x) - P(x)| \le \frac{1}{2^n(n+1)!} \max_{x\in[-1,1]} |f^{(n+1)}(x)|$$

Minimizing Lagrange interpolation error

If the interval of apporximation is on $[a, b]$ instead of $[-1, 1]$, we can apply change of variable

$$\tilde{x} = \frac{1}{2}[(b-a)x + (a+b)]$$

Hence, we can convert the roots $\bar{x}_k$ on $[-1, 1]$ to $\tilde{x}_k$ on $[a, b]$,

Minimizing Lagrange interpolation error

Example

Let $f(x) = xe^x$ on $[0, 1.5]$. Find the Lagrange interpolating polynomial using

1. the 4 equally spaced points $0, 0.5, 1, 1.5$.
2. the 4 points transformed from roots of $\tilde{T}_4$.

Minimizing Lagrange interpolation error

Solution. For each of the four points $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5$, we obtain $L_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}$ for $i = 0, 1, 2, 3$:

$$\begin{aligned}
L_0(x) &= -1.3333x^3 + 4.0000x^2 - 3.6667x + 1, \\
L_1(x) &= 4.0000x^3 - 10.000x^2 + 6.0000x, \\
L_2(x) &= -4.0000x^3 + 8.0000x^2 - 3.0000x, \\
L_3(x) &= 1.3333x^3 - 2.000x^2 + 0.66667x
\end{aligned}$$

so the Lagrange interpolating polynomial is

$$P_3(x) = \sum_{i=0}^{3} f(x_i)L_i(x) = 1.3875x^3 + 0.057570x^2 + 1.2730x.$$

Minimizing Lagrange interpolation error

Solution. (cont.) The four roots of $\tilde{T}_4(x)$ on $[-1, 1]$ are $\bar{x}_k = \cos(\frac{2k-1}{8}\pi)$ for $k = 1, 2, 3, 4$. Shifting the points using $\tilde{x} = \frac{1}{2}(1.5x + 1.5)$, we obtain four points

$$\tilde{x}_0 = 1.44291, \tilde{x}_1 = 1.03701, \tilde{x}_2 = 0.46299, \tilde{x}_3 = 0.05709$$

with the same procedure as above to get $\tilde{L}_0, \dots, \tilde{L}_3$ using these 4 points, and then the Lagrange interpolating polynomial:

$$\tilde{P}_3(x) = 1.3811x^3 + 0.044652x^2 + 1.3031x - 0.014352.$$

Minimizing Lagrange interpolation error

Now compare the approximation accuracy of the two polynomials

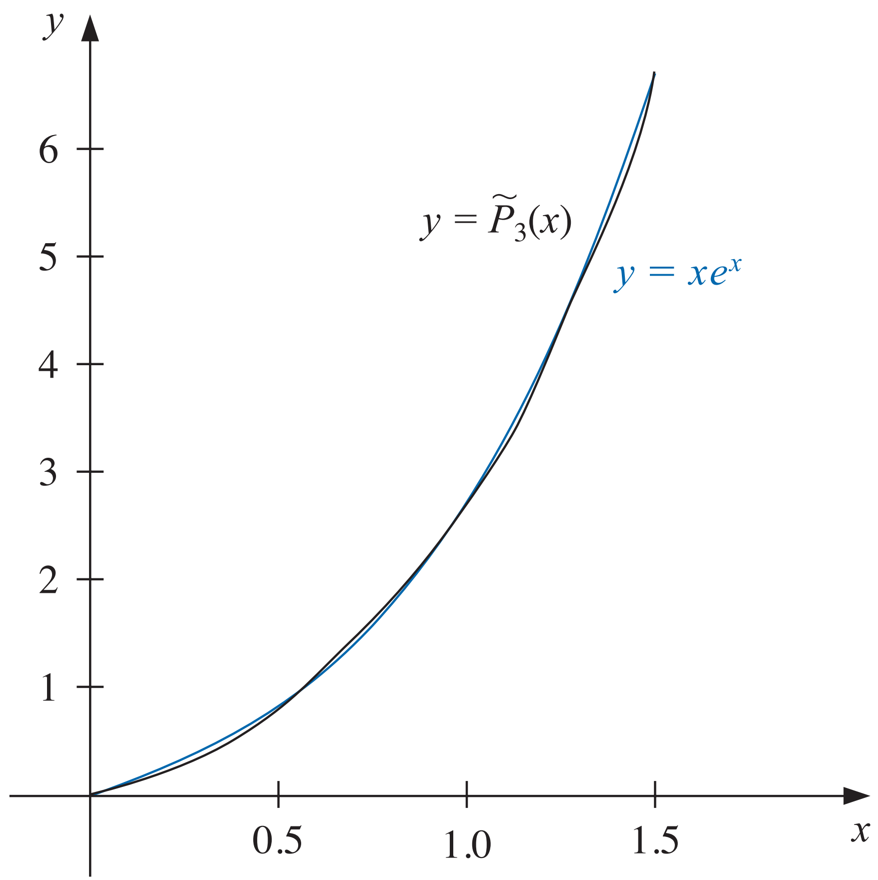
$$P_3(x) = 1.3875x^3 + 0.057570x^2 + 1.2730x$$
$$\tilde{P}_3(x) = 1.3811x^3 + 0.044652x^2 + 1.3031x - 0.014352$$

x	$f(x) = xe^x$	$P_3(x)$	$\lvert xe^x - P_3(x)\rvert$	$\tilde{P}_3(x)$	$\lvert xe^x - \tilde{P}_3(x)\rvert$
0.15	0.1743	0.1969	0.0226	0.1868	0.0125
0.25	0.3210	0.3435	0.0225	0.3358	0.0148
0.35	0.4967	0.5121	0.0154	0.5064	0.0097
0.65	1.245	1.233	0.012	1.231	0.014
0.75	1.588	1.572	0.016	1.571	0.017
0.85	1.989	1.976	0.013	1.974	0.015
1.15	3.632	3.650	0.018	3.644	0.012
1.25	4.363	4.391	0.028	4.382	0.019
1.35	5.208	5.237	0.029	5.224	0.016

Minimizing Lagrange interpolation error

The approximation using $\tilde{P}_3(x)$

$$\tilde{P}_3(x) = 1.3811x^3 + 0.044652x^2 + 1.3031x - 0.014352$$



Reducing the degree of approximating polynomials

As Chebyshev polynomials are efficient in approximating functions, we may use approximating polynomials of smaller degree for a given error tolerance.

For example, let $Q_n(x) = a_0 + \cdots + a_n x^n$ be a polynomial of degree n on $[-1, 1]$. Can we find a polynomial of degree $n - 1$ to approximate Q_n?

Reducing the degree of approximating polynomials

So our goal is to find $P_{n-1}(x) \in \Pi_{n-1}$ such that

$$\max_{x\in[-1,1]} |Q_n(x) - P_{n-1}(x)|$$

is minimized. Note that $\frac{1}{a_n}(Q_n(x) - P_{n-1}(x)) \in \tilde{\Pi}_n$, we know the best choice is $\frac{1}{a_n}(Q_n(x) - P_{n-1}(x)) = \tilde{T}_n(x)$, i.e., $P_{n-1} = Q_n - a_n\tilde{T}_n$. In this case, we have approximation error

$$\max_{x\in[-1,1]} |Q_n(x) - P_{n-1}(x)| = \max_{x\in[-1,1]} |a_n\tilde{T}_n| = \frac{|a_n|}{2^{n-1}}$$

Reducing the degree of approximating polynomials

Example

Recall that $Q_4(x)$ be the 4th Maclaurin polynomial of $f(x) = e^x$ about 0 on $[-1, 1]$. That is

$$Q_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

which has $a_4 = \frac{1}{24}$ and truncation error

$$|R_4(x)| = |\frac{f^{(5)}(\xi(x))x^5}{5!}| = |\frac{e^{\xi(x)}x^5}{5!}| \le \frac{e}{5!} \approx 0.023$$

for $x \in (-1, 1)$. Given error tolerance 0.05, find the polynomial of small degree to approximate $f(x)$.

Reducing the degree of approximating polynomials

Solution. Let's first try Π_3. Note that $\tilde{T}_4(x) = x^4 - x^2 + \frac{1}{8}$, so we can set

$$\begin{aligned} P_3(x) &= Q_4(x) - a_4 \tilde{T}_4(x) \\ &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right) - \frac{1}{24}\left(x^4 - x^2 + \frac{1}{8}\right) \\ &= \frac{191}{192} + x + \frac{13}{24}x^2 + \frac{1}{6}x^3 \in \Pi_3 \end{aligned}$$

Therefore, the approximating error is bounded by

$$\begin{aligned} |f(x) - P_3(x)| &\le |f(x) - Q_4(x)| + |Q_4(x) - P_3(x)| \\ &\le 0.023 + \frac{|a_4|}{2^3} = 0.023 + \frac{1}{192} \le 0.0283. \end{aligned}$$

Reducing the degree of approximating polynomials

Solution. (cont.) We can further try Π_2. Then we need to approximate P_3 (note $a_3 = \frac{1}{6}$) above by the following $P_2 \in \Pi_2$:

$$\begin{aligned} P_2(x) &= P_3(x) - a_3 \tilde{T}_3(x) \\ &= \frac{191}{192} + x + \frac{13}{24}x^2 + \frac{1}{6}x^3 - \frac{1}{6}\left(x^3 - \frac{3}{4}x\right) \\ &= \frac{191}{192} + \frac{9}{8}x + \frac{13}{24}x^2 \in \Pi_2 \end{aligned}$$

Therefore, the approximating error is bounded by

$$\begin{aligned} |f(x) - P_2(x)| &\le |f(x) - Q_4(x)| + |Q_4(x) - P_3(x)| + |P_3(x) - P_2(x)| \\ &\le 0.0283 + \frac{|a_3|}{2^2} = 0.0283 + \frac{1}{24} = 0.0703. \end{aligned}$$

Unfortunately this is larger than 0.05.

Reducing the degree of approximating polynomials

Although the error bound is larger than 0.05, the actual error is much smaller:

x	e^x	$P_4(x)$	$P_3(x)$	$P_2(x)$	$\|e^x - P_2(x)\|$
-0.75	0.47237	0.47412	0.47917	0.45573	0.01664
-0.25	0.77880	0.77881	0.77604	0.74740	0.03140
0.00	1.00000	1.00000	0.99479	0.99479	0.00521
0.25	1.28403	1.28402	1.28125	1.30990	0.02587
0.75	2.11700	2.11475	2.11979	2.14323	0.02623

Pros and cons of polynomial approxiamtion

Advantages:

- Polynomials can approximate continuous function to arbitrary accuracy;
- Polynomials are easy to evaluate;
- Derivatives and integrals are easy to compute.

Disadvantages:

- Significant oscillations;
- Large max absolute error in approximating;
- Not accurate when approximating discontinuous functions.