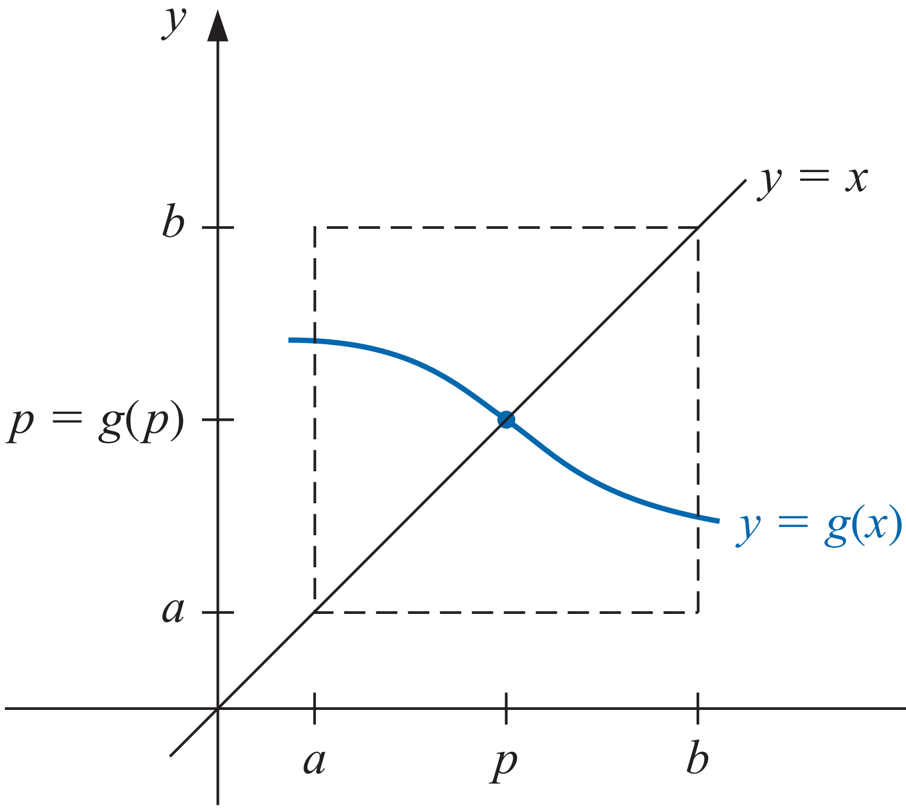
Fixed point iteration

Definition

Let $g : \mathbb{R} \to \mathbb{R}$, then p is a **fixed point** of g if $g(p) = p$.



Fixed point

Example (Fixed point and root)

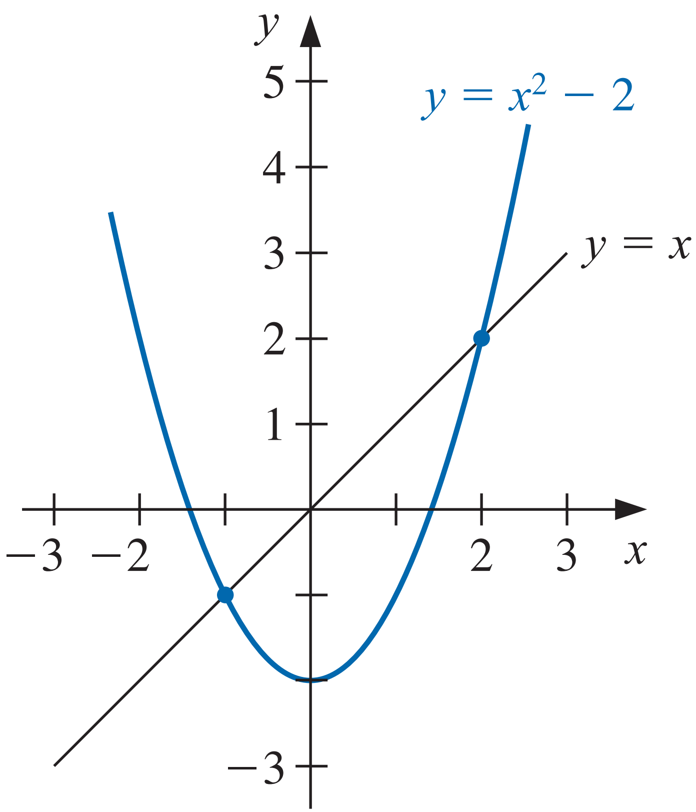
Suppose $\alpha \neq 0$. Show that p is a root of $f(x)$ iff p is a fixed point of $g(x) := x - \alpha f(x)$

Example

Example (Fixed point)

Find the fixed point(s) of $g(x) = x^2 - 2$.

Solution. p is a fixed point of g if $p = g(p) = p^2 - 2$. Solve for p to get $p = 2, -1$.



Fixed point theorem

Theorem (Fixed point theorem)

1. *If* $g \in C[a,b]$ *and* $a \leq g(x) \leq b$ *for all* $x \in [a,b]$*, then* g *has at least one fixed point in* $[a,b]$*.*
2. *If, in addition,* g' *exists in* $[a,b]$*, and* $\exists k < 1$ *such that* $|g'(x)| \leq k < 1$ *for all* x*, then* g *has a unique fixed point in* $[a,b]$*.*

Fixed point theorem

Proof.

1. If $g(a) = a$ or $g(b) = b$, then done. Otherwise, $g(a) > a$ and $g(b) < b$. Define $f(x) = x - g(x)$, then $f(a) = a - g(a) < 0$, and $f(b) = b - g(b) > 0$. By IVT and f is continuous, $\exists p \in (a, b)$ s.t. $f(p) = 0$, i.e., $p - g(p) = 0$.
2. If $\exists p, q \in [a, b]$ are two distinct fixed points of g, then $\exists \xi \in (p, q)$ s.t.

$$1 = \frac{p-q}{p-q} = \left|\frac{g(p)-g(q)}{p-q}\right| = |g'(\xi)| \leq k < 1$$

by MVT. Contradiction.

□

Example

Example (Application of Fixed Point Theorem)

$g(x) = \frac{x^2-1}{3}$ has a unique fixed point in $[-1, 1]$.

Proof.

First we need show $g(x) \in [-1, 1]$, $\forall x \in [-1, 1]$. Find the max and min values of g as $-\frac{1}{3}$ and 0 (Hint: find critical points of g first). So $g(x) \in [-\frac{1}{3}, 0] \subset [-1, 1]$.

Also $|g'(x) = |\frac{2x}{3}| \leq \frac{2}{3} < 1$, $\forall x \in [-1, 1]$, so g has unique fixed point in $[-1, 1]$ by FPT. □

Remark: We can solve for this fixed point: $p = g(p) = \frac{p^2-1}{3} \implies p = \frac{3-\sqrt{13}}{2}$.

Example

Example (Fixed Point Theorem – Failed Case 1)

$g(x) = \frac{x^2-1}{3}$ has a unique fixed point in $[3, 4]$. But we can't use FPT to show this.

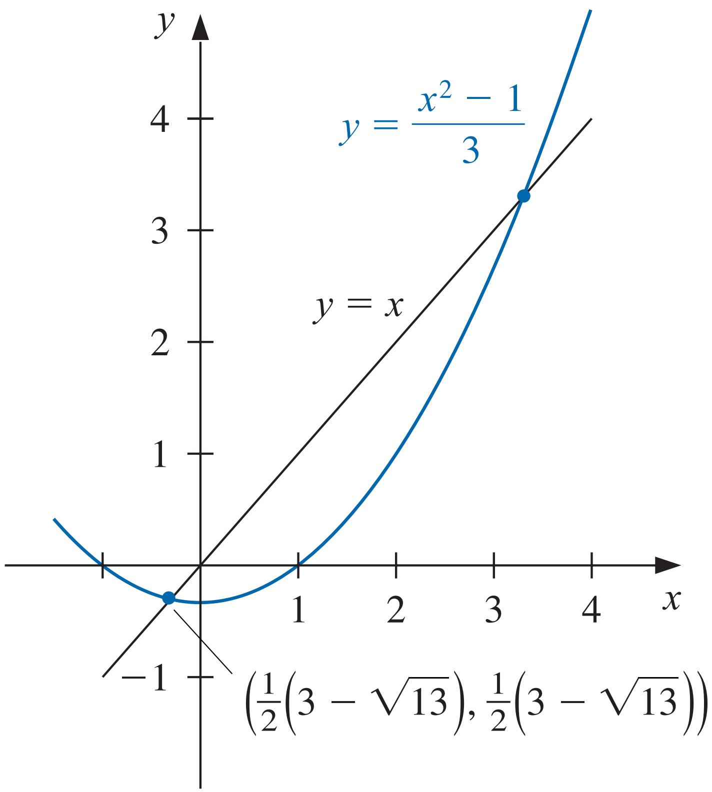
Remark: Note that there is a unique fixed point in $[3, 4]$ ($p = \frac{3+\sqrt{13}}{2}$), but $g(4) = 5 \notin [3, 4]$, and $g'(4) = 8/3 > 1$ so we cannot apply FPT here.

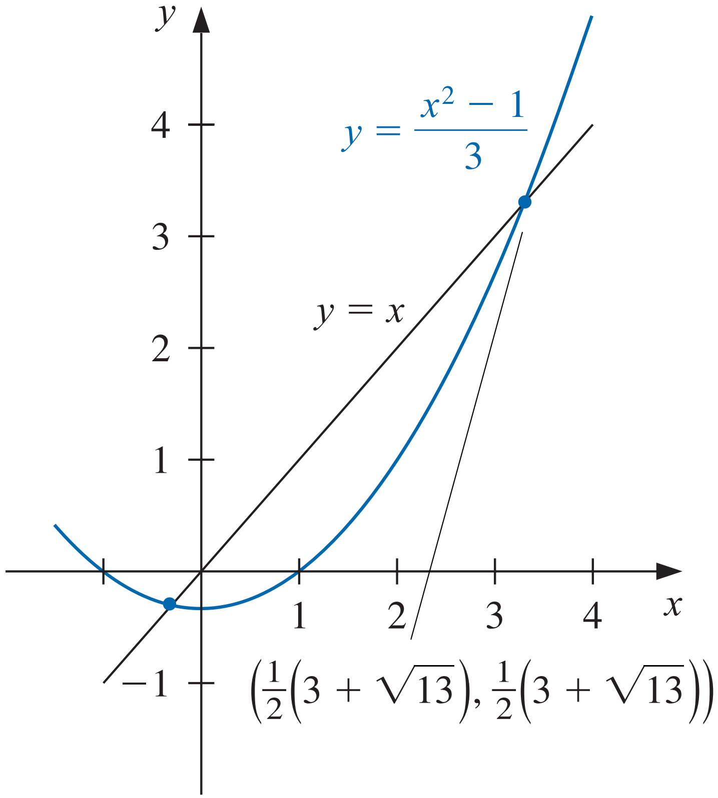
From this example, we know FPT provides a *sufficient but not necessary* condition.

Example

Example (Fixed Point Theorem – Failed Case 1)

$g(x) = \frac{x^2-1}{3}$ has a unique fixed point in $[3, 4]$. But we can't use FPT to show this.





Example

Example (Fixed Point Theorem – Failed Case 2)

We can use FPT to show that $g(x) = 3^{-x}$ must have FP on $[0, 1]$, but we can't use FPT to show if it's unique (even though the FP on $[0, 1]$ is unique in this example).

Solution. $g'(x) = (3^{-x})' = -3^{-x} \ln 3 < 0$, therefore $g(x)$ is strictly decreasing on $[0, 1]$. Also $g(0) = 3^0 = 1$ and $g(1) = 3^{-1}$, so $g(x) \in [0, 1]$, $\forall x \in [0, 1]$. So a FP exists by FPT.

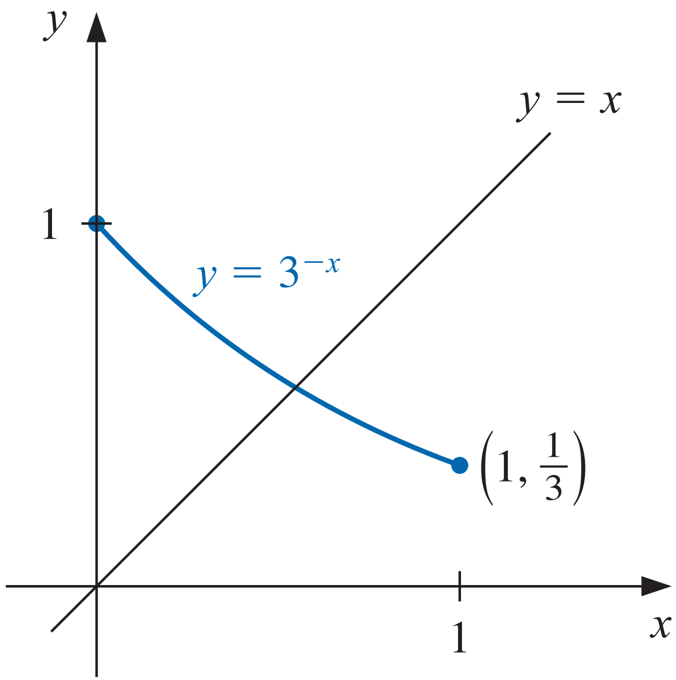
However, $g'(0) = -\ln 3 \approx -1.098$, so we do not have $|g'(x)| < 1$ over $[0, 1]$. Hence FPT does not apply.

Nevertheless, the FP must be unique since g strictly decreases and intercepts with $y = x$ line only once.

Example

Example (Fixed Point Theorem – Failed Case 2)

We can use FPT to show that $g(x) = 3^{-x}$ must have FP on $[0, 1]$, but we can't use FPT to show if it's unique (even though the FP on $[0, 1]$ is unique in this example).



Fixed point iteration

We now introduce a method to find a fixed point of a *continuous* function g.

Fixed point iteration:
Start with an initial guess p_0, recursively define a sequence p_n by

$$p_{n+1} = g(p_n)$$

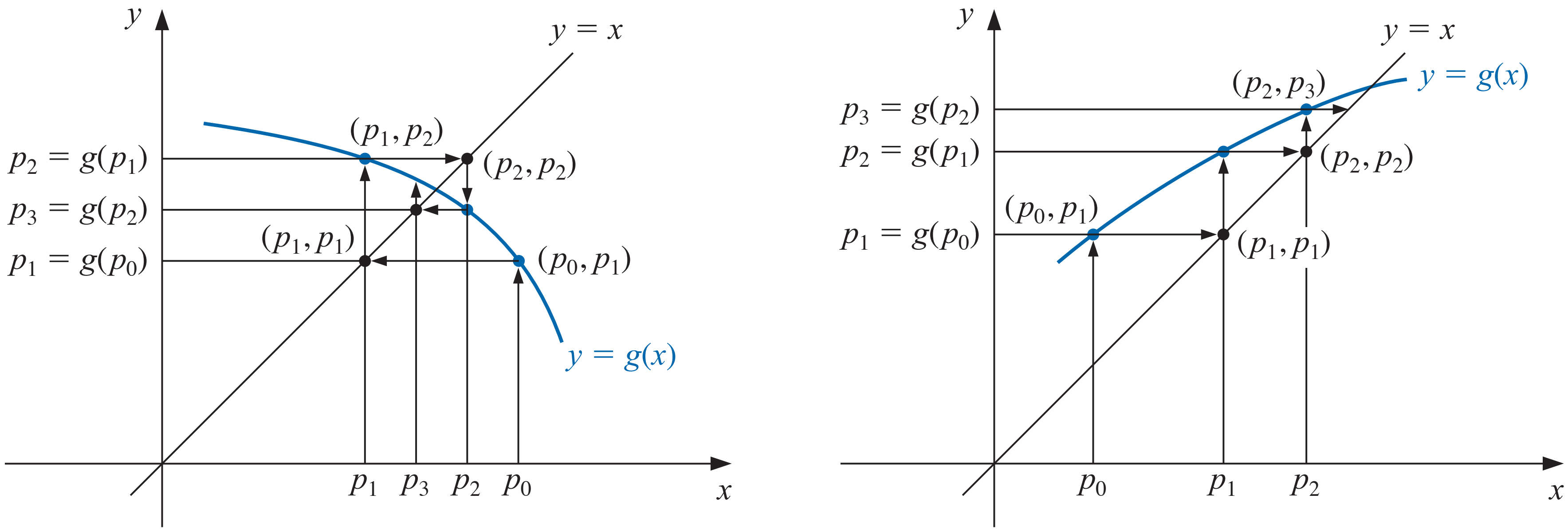
If $p_n \to p$, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(\lim_{n\to\infty} p_{n-1}) = g(p)$$

i.e., the limit of p_n is a fixed point of g.

Fixed point iteration

Example trajectories of fixed point iteration:



Fixed point iteration

Fixed Point Iteration Algorithm:

- **Input:** initial p_0, tolerence ϵ_{tol}, max iteration N_{max}. Set iteration counter $N = 1$.
- While $N \leq N_{\text{max}}$, do:
  1. Set $p = g(p_0)$ (update p_N to p_{N+1})
  2. If $|p - p_0| < \epsilon_{\text{tol}}$, then STOP
  3. Set $N \leftarrow N + 1$
  4. Set $p_0 = p$ (prepare p_N for the next iteration)
- **Output:** If $N \geq N_{\text{max}}$, print("Max iteration reached."). Return p.

FPI for root-finding

We can also use FPI to find the root of a function f:

1. Determine a function g, such that $p = g(p)$ iff $f(p) = 0$.[1]
2. Apply FPI to g and find FP p.

[1] We can use $\Longrightarrow$ but we may miss some roots of f.

Example

Example (FPI algorithm for root-finding)

Find a root of $f(x) = x^3 + 4x^2 - 10$ using FPI.

Solution. First notice that

$$
\begin{aligned}
x^3 + 4x^2 - 10 = 0 \quad &\Longleftrightarrow \quad 4x^2 = 10 - x^3 \\
&\Longleftrightarrow \quad x^2 = \frac{10 - x^3}{4} \\
&\Longleftrightarrow \quad x = \pm\sqrt{\frac{10 - x^3}{4}} \\
&\Longleftrightarrow \quad x^2 = \frac{10 - 4x^2}{x} \\
&\Longleftrightarrow \quad \dots
\end{aligned}
$$

Example

Example (FPI algorithm for root-finding)

Find a root of $f(x) = x^3 + 4x^2 - 10$ using FPI.

Solution. So we can define several g:

$$
\begin{aligned}
g_1(x) &= x - (x^3 + 4x^2 - 10) \\
g_2(x) &= \sqrt{\frac{10}{x} - 4x} \\
g_3(x) &= \frac{10 - x^3}{4} \\
g_4(x) &= \sqrt{\frac{10}{4 + x}} \\
g_5(x) &= x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}
\end{aligned}
$$

Which g to choose? – All these g have the the same FP p. But g_3, g_4, g_5 converge (g_5 fastest) while g_1, g_2 do not.

Convergence of FPI algorithm

Theorem (Convergence of FPI Algorithm)

Suppose $g \in C[a,b]$ s.t. $g(x) \in [a,b]$, $\forall x \in [a,b]$. If $\exists k \in (0,1)$ s.t. $|g'(x)| \le k$, $\forall x \in (a,b)$, then $\{p_n\}$ generated by FPI algorithm converges to the unique FP of $g(x)$ on $[a,b]$.

Proof.

$g(x) \in [a,b]$ and $|g'(x)| \le k < 1$, $\forall x \in [a,b] \Longrightarrow \exists!$ FP p on $[a,b]$ by FPT. Moreover, $\exists \xi(p_{n-1})$ between p and p_{n-1} s.t.

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi(p_{n-1}))||p_{n-1} - p| \le k|p_{n-1} - p|$$

Apply this inductively, we get

$$|p_n - p| \le k|p_{n-1} - p| \le k^2|p_{n-2} - p| \le \cdots \le k^n|p_0 - p| \to 0$$

since $k^n \to 0$ as $n \to \infty$. □

Convergence rate of FPI algorithm

Corollary (Convergence rate of FPI Algorithm)

With the same conditions as above, we have for all $n \geq 1$

- $|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$
- $|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$

Proof.

1. $|p_0 - p| \leq \max\{p_0 - a, b - p_0\}$. Then apply the proof above.
2. Apply the proof above to get $|p_{n+1} - p_n| \leq k^n|p_1 - p_0|$. Then

$$|p_m - p_n| \leq |p_1 - p_0| \sum_{i=0}^{m-n-1} k^{n+i} = \frac{1 - k^{m-n}}{1 - k} k^n |p_1 - p_0|$$

Let $m \to \infty$ to get the estimate.

$\square$

Example

Example (FPI algorithm for root-finding)

Find a root of $f(x) = x^3 + 4x^2 - 10$ using FPI algorithm.

Solution. Recall the functions g we defined:

$$
\begin{aligned}
g_1(x) &= x - (x^3 + 4x^2 - 10) \\
g_2(x) &= \sqrt{\frac{10}{x} - 4x} \\
g_3(x) &= \frac{10 - x^3}{4} \\
g_4(x) &= \sqrt{\frac{10}{4 + x}} \\
g_5(x) &= x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}
\end{aligned}
$$

Apply the theorem above, check $|g'(x)|$, and explain why FPI algorithm converges with g_3, g_4, g_5.

Fixed point iteration for root-finding

To find a good FPI algorithm for root-finding $f(p) = 0$, find a function g s.t.

- $g(p) = p \Longrightarrow f(p) = 0$
- g is continuous, differentiable
- $|g'(x)| \le k \in (0, 1)$, $\forall x$ with k as small as possible