Numerical Analysis

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Section 1

Review of Calculus

Limit and continuity

Definition

An **Euclidean ball** $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

Definition

The **limit** of $f(x)$ as x approaches x_0 is L if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in B_\delta(x_0)$.

Definition

f is **continuous** at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$. f is continuous in X if f is continuous at every $x \in X$.

Limit and continuity

Definition

A sequence $\{x_n : n \in \mathbb{N}\}$ has **limit** x if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that $|x_n - x| < \epsilon$ for all $n \geq N$.

Theorem

The following two statements are equivalent:

- *f is continuous at x.*
- *If $x_n \to x$, then $f(x_n) \to f(x)$.*

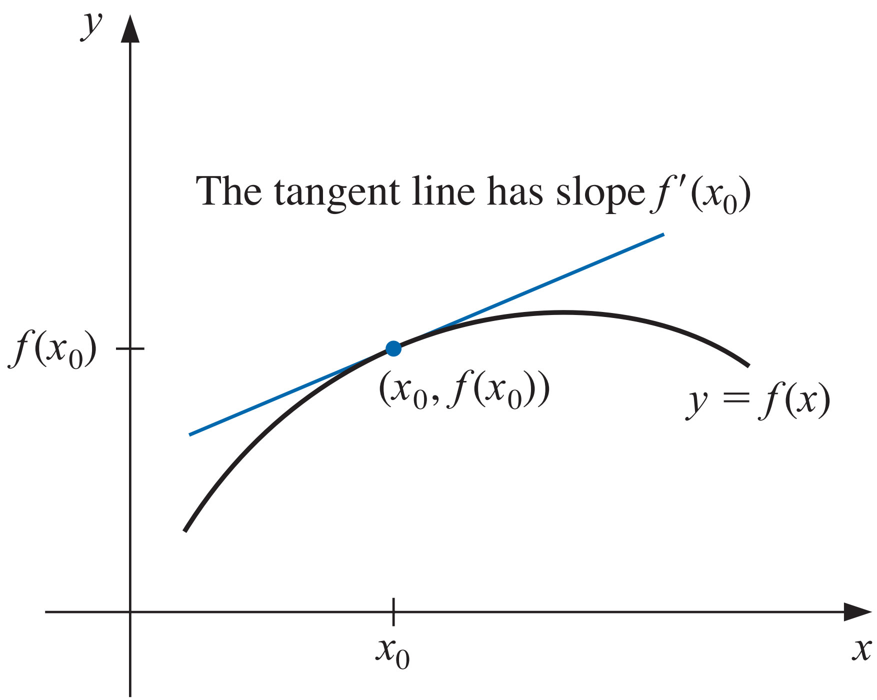
Differentiability

Definition

f is **differentiable** at x_0 if the following limit exists:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The value of this limit is called the **derivative** of f at x_0.



Differentiability

Theorem

f is differentiable at $x \Longrightarrow f$ is continuous at x.

Theorem (Rolle's Theorem)

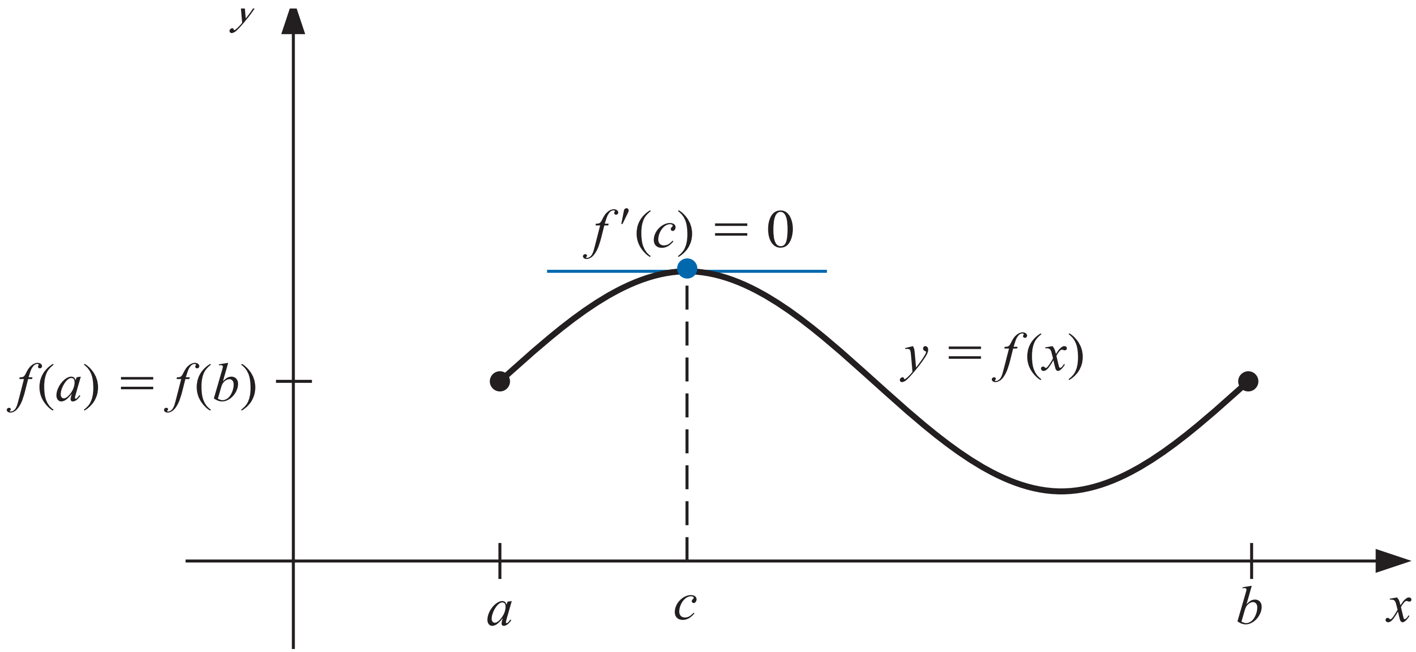
Suppose $f \in C[a, b]$, f is differentiable in (a, b) and $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof.

Hint: $f \in C[a, b]$ implies that f attains max or min in $[a, b]$ by the extreme value theorem (see soon). □

Rolle's theorem

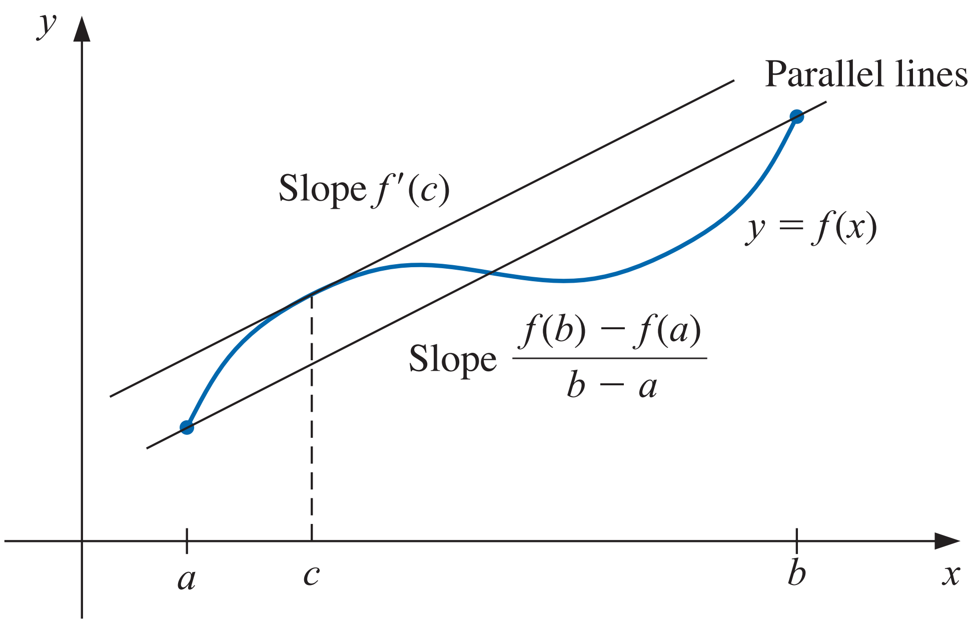
Illustration of the Rolle's theorem:



Mean Value Theorem

Theorem (Mean Value Theorem)

If $f \in C[a,b]$ and f is differentiable on (a,b), then $\exists c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Proof.

Define $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then $g(a) = g(b) = 0$. Apply Rolle's theorem to g. □

Extreme Value Theorem

Theorem (Extreme Value Theorem)

If $f \in C[a, b]$, then $\exists c_1, c_2 \in [a, b]$ such that

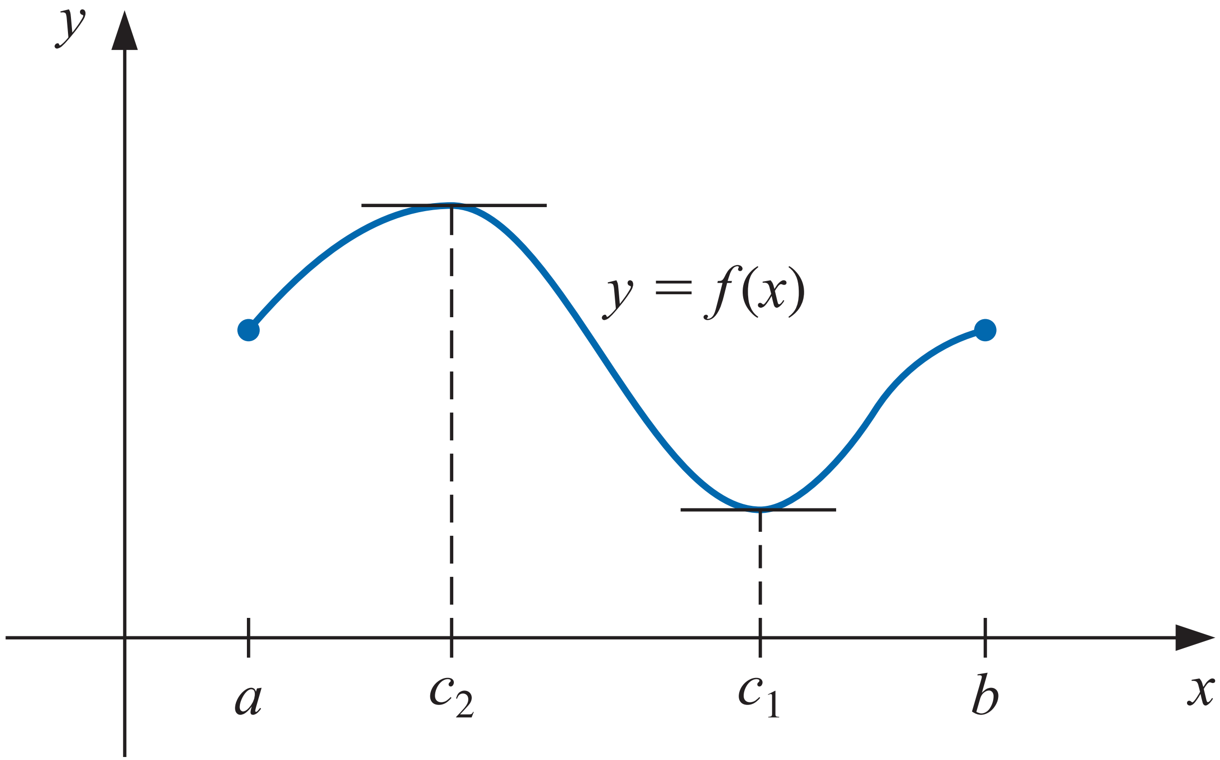
$$f(c_1) \le f(x) \le f(c_2)$$

for all $x \in [a, b]$. In addition, if f is differentiable in (a, b), then c_1 and c_2 occur either at a, b, or where $f' = 0$.

Proof.

Suppose $f(x_k) \to \sup_{a \le x \le b} f(x)$, then $\exists$ subseq $x_{k_j} \to c_1 \in [a, b]$ such that $f(x_{k_j}) \to f(c_1)$ ($\because$ f continuous). Hence we have $f(c_1) = \max_{a \le x \le b} f(x)$. □

Extreme value theorem



Generalized Rolle's theorem

Theorem (Generalized Rolle's Theorem)

Suppose $f \in [a, b]$ and is n times differentiable. Let $\{x_0, \dots, x_n\}$ be a partition of $[a, b]$, i.e., $a = x_0 < x_1 < \cdots < x_n = b$, such that $f(x_i) = 0$ for all $i = 1, \dots, n$, then $\exists c \in (a, b)$ such that $f^{(n)}(c) = 0$.

Proof.

By Rolle's theorem, $\exists y_1, \dots, y_n$ s.t. $x_0 < y_1 < x_1 < \cdots < y_n < x_n$ and $f'(y_i) = 0$ for $i = 1, \dots, n$. Keep applying Rolle's theorem for another $n - 1$ times to show that $\exists c \in (a, b)$ s.t. $f^{(n)}(c) = 0$. □

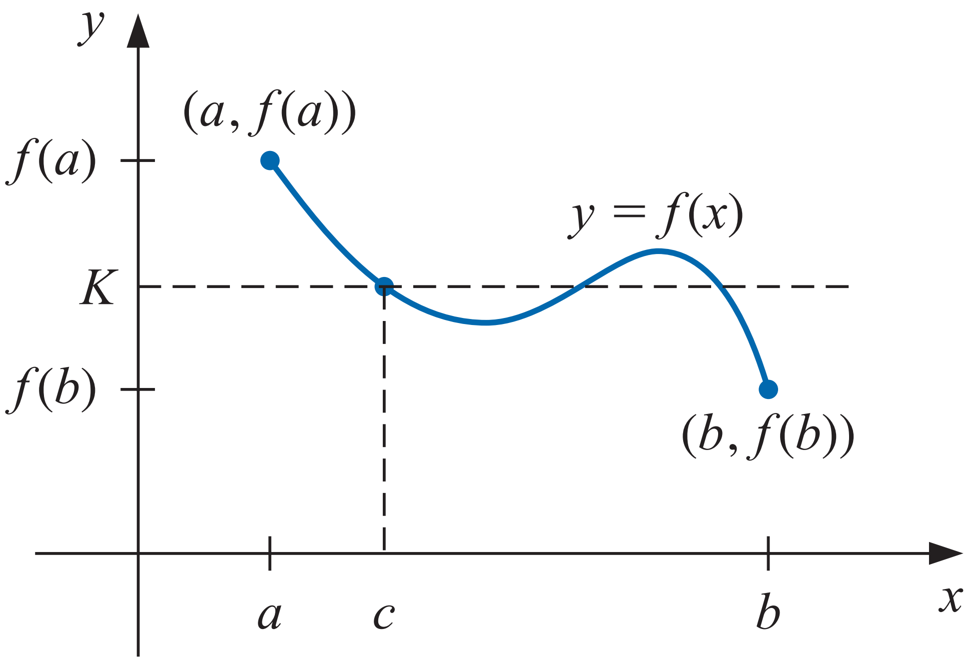
Intermediate value theorem

Theorem (Intermediate Value Theorem)

If $f \in C[a, b]$ and k is a number between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ such that $f(c) = k$.

Proof.

By continuity of f on $[a, b]$. □



Example

Example (Application of IVT)

Show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in $[0, 1]$.

Solution. Set $f(x) = x^5 - 2x^3 + 3x^2 - 1$. Then we need to show that $\exists c \in [0, 1]$ such that $f(c) = 0$. Since $f(0) = -1$ and $f(1) = 1$, we know such c exists by IVT.

Integration

Definition (Riemann integral)

The **Riemann integral** of f on $[a, b]$ is the limit

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{\max_i \Delta x_i \to 0} \sum_{i=1}^n f(z_i)\Delta x_i$$

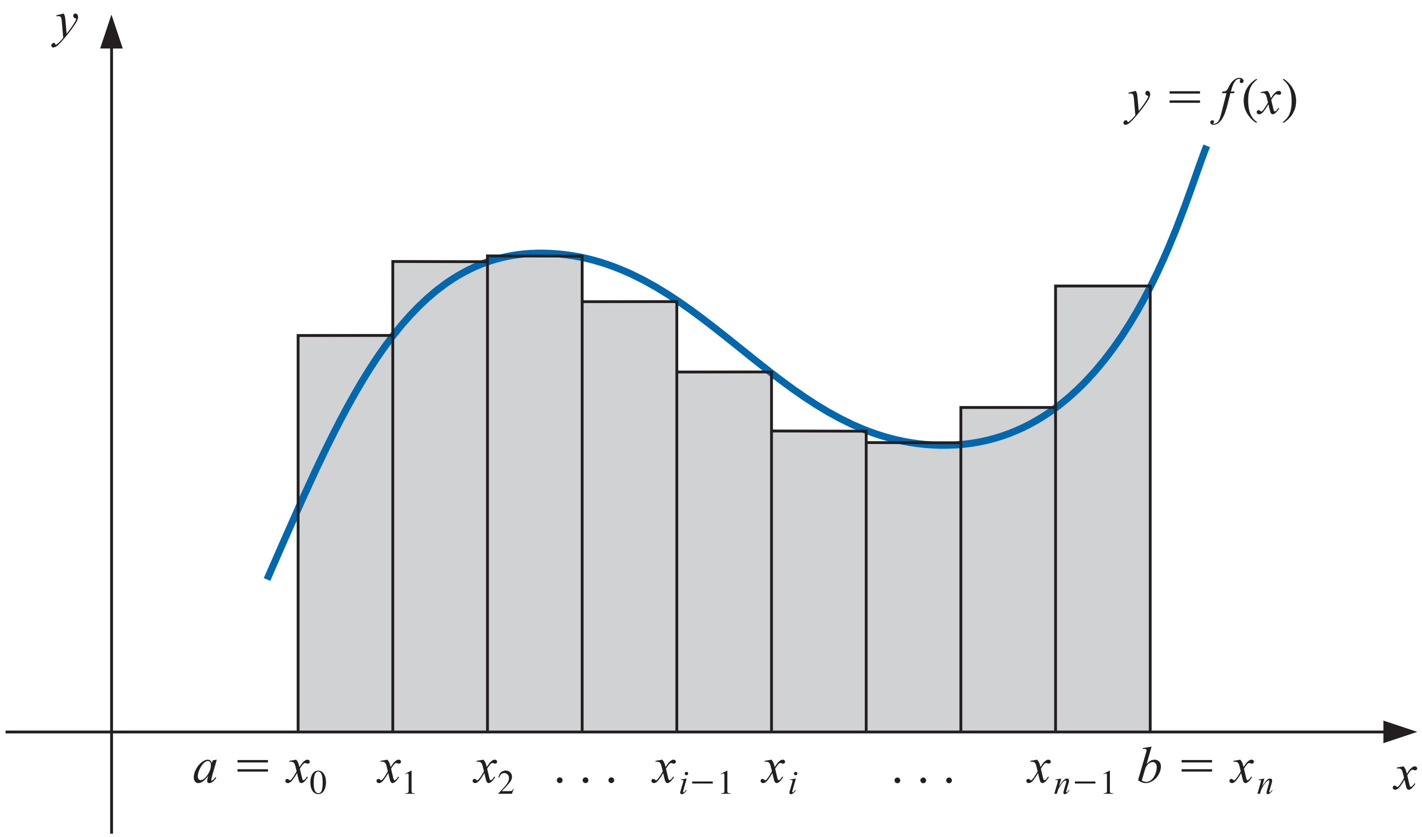
where $\{x_0, \dots, x_n\}$ is a paritition of $[a, b]$, $\Delta x_i := x_i - x_{i-1}$ and z_i is arbitrary in $[x_{i-1}, x_i]$.

If $f \in C[a, b]$, this simply means

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n f(x_i)$$

where $\{x_0, \dots, x_n\}$ is an equal partition of $[a, b]$ into n segments, $\Delta x_i = \frac{b-a}{n}$, $\forall i$.

Riemann integral



Mean value theorem for integrals

Theorem (Mean Value Theorem for Integrals)

Suppose $f \in C[a, b]$, and g is Riemann integrable over $[a, b]$ and does not change sign, then $\exists c \in (a, b)$ s.t.

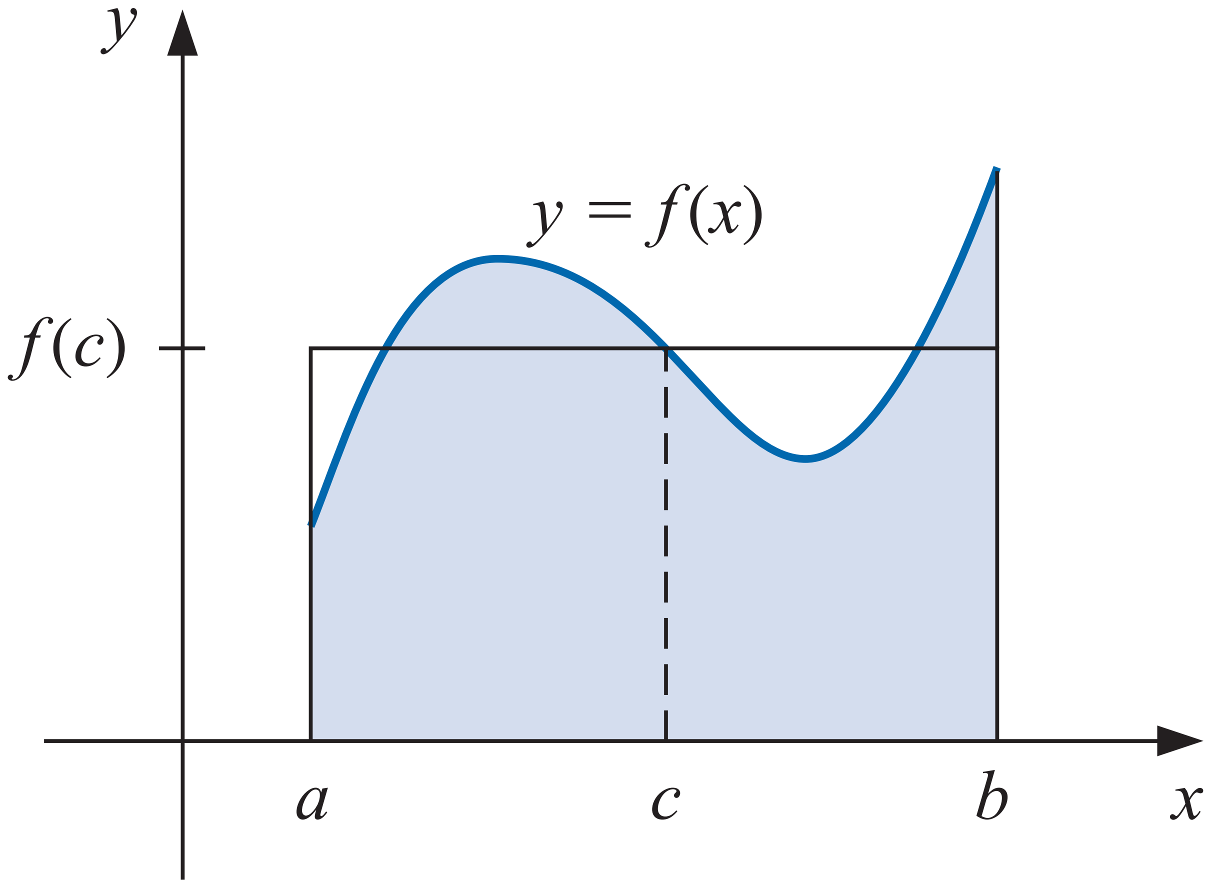
$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x$$

If $g(x) \equiv 1$, then $\exists c \in [a, b]$, s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$

Proof.

Hint: WLOG $g \geq 0$, then $m\int_a^b g(x)\,\mathrm{d}x \leq \int_a^b f(x)g(x)\,\mathrm{d}x \leq M\int_a^b g(x)\,\mathrm{d}x$ where m, M are min,max of f. So $m \leq r := \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \leq M$. By IVT $\exists c \in [a, b]$ s.t. $f(c) = r$. □

Mean value theorem for integrals



Taylor series and polynomials

Theorem (Taylor's theorem)

Suppose $f \in C^n[a,b]$, $f^{(n+1)}$ exists in (a,b), $x_0 \in [a,b]$. Then for every $x \in (a,b)$, there exists a number $\xi(x)$ such that $f(x) = P_n(x) + R_n(x)$, where $P_n(x)$ is a polynomial of degree n:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

and $R_n(x)$ is the remainder term:

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi(x))(x - x_0)^{n+1}.$$

Proof.

For any fixed x_0, x, define $r := \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}$ and $F(t) := f(t) - P_n(t) - r \cdot (t - x_0)^{n+1}$. Prove that $F(x_0) = F'(x_0) = \cdots = F^{(n)}(x_0) = 0$ and $F(x) = 0$. Then apply Rolle's theorem repeatedly to show $\exists \xi(x) \in (x_0, x)$ s.t. $F^{(n+1)}(\xi(x)) = 0$, i.e., $r = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}$. $\square$

Example

Example (Taylor polynomial)

Let $f(x) = \cos x$ and $x_0 = 0$. Find $P_3(x)$, the Taylor polynomial of degree 3 (i.e., the polynomial by expanding f at x_0 to the 3rd order).

Solution. $f(x_0) = \cos(0) = 1$, $f'(x_0) = -\sin(0) = 0$, $f''(x_0) = -\cos(0) = 1$, $f'''(x_0) = \sin(0)$. So

$$P_3(x) = \sum_{k=0}^{3} f^{(k)}(x_0)(x - x_0)^k = 1 - \frac{1}{2}x^2$$

Taylor series and polynomials

Approximating $f(x) = \cos x$ by Taylor's polynomial $P_2(x)$: