Analysis – Exercise Problems and Solutions

Real and Complex Numbers

1. If r is a nonzero rational number and x is irrational, prove that $r+x$ and rx are irrational.
2. Convert $0.456123123123\cdots$ into the form of m/n where m and n are co-prime integers. Why does the presence of a repeating block in the decimal form imply that the number is rational?
3. Give an example of two irrational numbers whose product is rational. Is it true that for every irrational number x there exists another irrational number y such that xy is rational?
4. Let p denote a prime number. Prove that $\sqrt{p}$ is irrational.
5. Prove that there is no rational number whose square is 12.
6. Prove that for every $n \in \mathbb{N}$, $\sqrt{n+1}+\sqrt{n-1}$ is irrational.
7. Prove that the least upper bound of a set in an ordered set is unique.
8. Prove that a nonempty finite set in an ordered set contains its supremum and infimum.
9. Let $A \subset \mathbb{R}$ and A nonempty, and denote $-A = \{-x \mid x \in A\}$. Prove that $\inf A = -\sup(-A)$.
10. Find the supremum and infimum of set $\{x \in \mathbb{R} \mid 3x^2+3 < 10x\}$. Do they belong to this set?
11. Find the supremum and infimum of set $\{1/n \mid n \in \mathbb{N}\}$. Do they belong to this set?
12. Let $A, B \subset \mathbb{R}$ and $C = \{x+y \mid x \in A, y \in B\}$. How are the numbers $\inf A$, $\inf B$, and $\inf C$ related? How are the numbers $\sup A$, $\sup B$, and $\sup C$ related?
13. Let $A, B \subset \mathbb{R}$ and $C = A \cap B$, how are the numbers $\inf A$, $\inf B$, and $\inf C$ related? If $C = A \cup B$, how are the numbers $\sup A$, $\sup B$, and $\sup C$ related?
14. Let $A \subset \mathbb{R}$, and $A^2 = \{x^2 \mid x \in A\}$. Is there any relation between $\sup A$ and $\sup(A^2)$?
15. Let $a, b \in \mathbb{R}$ and $b-a > 1$. Prove that there exists at least one integer $c \in \mathbb{N}$ such that
16. Prove that there exists a real number $x > 0$ such that $x^3 = 5$.
17. Under what condition is $\sup A$ not a limit point of the set A?
18. Prove that no order can be defined in the set of complex numbers to turn it into an ordered field.
19. Suppose $z = (a,b), w = (c,d)$, where $a,b,c,d \in \mathbb{R}$. Define $z < w$ if $a < c$ and also if $a = c$ but $b < d$. Prove that this turns the set of complex numbers into an ordered set (this is called the dictionary order or lexicographic order). Does this set have the least-upper-bound property?
20. Let $z_i \in \mathbb{C}$ for $i = 1, 2, \cdots, n$. Prove that $|\sum_i z_i| \le \sum_i |z_i|$.
21. Prove that $||z|-|w|| \le |z|+|w|$ for all $z, w \in \mathbb{C}$.
22. Suppose that $z \in \mathbb{C}$ and $|z| = 1$. Compute $|1+z|^2 + |1-z|^2$.
23. Prove that $|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2(|\mathbf{x}|^2 + |\mathbf{y}|^2)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
24. If $n \ge 2$ and $\mathbf{x} \in \mathbb{R}^n$, prove that there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \ne \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true for $n = 1$?
25. Suppose that $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Find $\mathbf{c} \in \mathbb{R}^n$ and $r > 0$ such that $|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$ if and only if $|\mathbf{x}-\mathbf{c}| = r$.

Numerical Sequences

1. Give the definition in formula of the sequence $\{2, 1, 4, 1, 6, 1, 8, 1, \cdots\}$.

2. Find $x \in \mathbb{R}$ and an $N \in \mathbb{N}$ such that $|x_n - x| \leq 10^{-3}$ for all $n \leq N$:

   (a) $x_n = \frac{2}{\sqrt{n+1}}$.

   (b) $x_n = 1 - \frac{1}{n^3}$.

   (c) $x_n = 2 + 2^{-n}$.

3. Prove convergence or divergence of the sequences defined as follows.

   (a) $x_n = \frac{2n^2+5n-6}{n^3}$.

   (b) $x_n = \frac{3n^5}{6n+11}$.

   (c) $x_n = \frac{n\sqrt{n+2}+1}{n^2+4}$.

   (d) $x_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n})$.

4. Suppose that $x_n \in \mathbb{Z}$. Under what conditions does this sequence converge?

5. Show that the sequences $\{x_n\}$ and $\{y_n\}$ where $y_n = x_{n+100}$ for all $n \in \mathbb{N}$ are either both convergent or both divergent.

6. Let $x_1 = 1$ and $x_{n+1} = \sqrt{x_n + 1}$. List the first few terms of this sequence. Prove that the sequence converges to $(1 + \sqrt{5})/2$.

7. Determine which of the followings about numerical sequences are true and justify your answer.

   (a) If $\{x_n\}$ is unbounded, then either $\lim_{n\to\infty} x_n = \infty$ or $\lim_{n\to\infty} x_n = -\infty$.

   (b) If $\{x_n\}$ is unbounded, then $\lim_{n\to\infty} |x_n| = \infty$.

   (c) If $\{x_n\}$ and $\{y_n\}$ are both bounded, then so is $\{x_n + y_n\}$.

   (d) If $\{x_n\}$ and $\{y_n\}$ are both unbounded, then so is $\{x_n + y_n\}$.

   (e) If $\{x_n\}$ and $\{y_n\}$ are both bounded, then so is $\{x_n y_n\}$.

   (f) If $\{x_n\}$ and $\{y_n\}$ are both unbounded, then so is $\{x_n y_n\}$.

8. Determine which of the followings about numerical sequences are true and justify your answer.

   (a) If $\{x_n\}$ and $\{y_n\}$ are both divergent, then so is $\{x_n + y_n\}$.

   (b) If $\{x_n\}$ and $\{y_n\}$ are both divergent, then so is $\{x_n y_n\}$.

   (c) If $\{x_n\}$ and $\{x_n + y_n\}$ are both convergent, then so is $\{y_n\}$.

   (d) If $\{x_n\}$ and $\{x_n y_n\}$ are both convergent, then so is $\{y_n\}$.

   (e) If $\{x_n\}$ is convergent, then so is $\{x_n^2\}$.

   (f) If $\{x_n\}$ is convergent, then so is $\{1/x_n\}$.

   (g) If $\{x_n^2\}$ is convergent, then so is $\{x_n\}$.

9. Suppose that a sequence $\{x_n\}$ satisfies $|x_{n+1} - x_n| < 2^{-n}$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ is Cauchy. Is this result true under the condition $|x_{n+1} - x_n| < \frac{1}{n}$?

10. Let $x_1 = 1$ and $x_{n+1} = (x_n + 1)/3$ for all $n \in \mathbb{N}$. Find the first five terms in this sequence. Use induction to show that $x_n > 1/2$ for all $n \in \mathbb{N}$. Prove that this sequence is non-increasing, convergent, and find the limit.

11. Let $x_1 = 1$ and $x_{n+1} = (1 - \frac{1}{4n^2})x_n$ for $n \in \mathbb{N}$. Determine if the sequence converges.

12. Which statements are true?

    (a) a sequence is convergent if and only if all its subsequences are convergent.

(b) a sequence is bounded if and only if all its subsequences are bounded.

(c) a sequence is monotone if and only if all its subsequences are monotone.

(d) a sequence is divergent if and only if all its subsequences are divergent.

13. The sequence $\{x_n\}$ has the property that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that $|x_{n+1} - x_n| < \epsilon$ for all $n \geq N$. Is this sequence necessarily a Cauchy sequence?

14. Prove that the convergence of $\{x_n\}$ implies the convergence of $\{|x_n|\}$. Is the converse true?

15. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

16. Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + \sqrt{x_n}}$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges and $x_n < 2$ for all $n \in \mathbb{N}$.

17. Find the upper and lower limits of the sequence $\{x_n\}$ defined by $x_0 = 0$, $x_{2m} = x_{2m-1}/2$, and $x_{2m+1} = \frac{1}{2} + x_{2m}$ for all $m \in \mathbb{N}$.

18. For any two real numerical sequences $\{x_n\}, \{y_n\}$, prove that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

19. Prove that the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} \sqrt{a_n}/n$ if $a_n \geq 0$.

20. If $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}$ is monotonic and bounded, prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

21. Suppose that $\{x_n\}$ and $\{y_n\}$ are Cauchy. Prove that $\{|x_n - y_n|\}$ is Cauchy.

Basic Topology

1. Find all the interior points, isolated points, accumation points and boundary points for the following sets:

   (a) $\mathbb{N}, \mathbb{Q}, \mathbb{R}$.

   (b) (a, b), $(a, b]$ and $[a, b]$ as intervals in $\mathbb{R}$.

   (c) $\mathbb{R} \setminus \mathbb{N}$.

   (d) $\mathbb{R} \setminus \mathbb{Q}$.

2. Give an example of:

   (a) A set with no accumulation points.

   (b) A set with infinitely many accumulation points, none of which belong to the set.

   (c) A set that contains some, but not all, of its accumulation points.

3. Give an example of a nonempty set with the following properties or explain why no such set can exist:

   (a) a set with no accumulation points and no isolated points.

   (b) a set with no interior points and no isolated points.

   (c) a set with no boundary points and no isolated points.

4. Is every interior point of a set A an accumulation point? Is every accumulation point of a set A an interior point?

5. Let x be an interior point of set A and suppose $\{x_n\}$ is a sequence of points, not necessarily in A, but converging to x. Show that there exists an integer N such that $x_n \in A$ for all $n \geq N$.

6. Prove that a set F is closed if and only if F contains all its boundary points.

7. Find the interior and boundary of each of the sets $\{1/\sqrt{n} : n \in \mathbb{N}\}$ and $\{x \in \mathbb{Q} : 0 < x^2 < 2\}$.

8. Is the set of irrational real numbers countable? Justify your claim.

9. Construct a bounded set of real numbers with exactly three limit points.

10. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E} := E \cup E'$ have the same limit points. Do E and E' always have the same limit points?

11. Let $A_1, A_2, \cdots$ be subsets of a metric space. (a) If $B_n = \cup_{i=1}^{n} A_i$, prove that $\overline{B_n} = \cup_{i=1}^{n} \overline{A_i}$, for $n = 1, 2, \cdots$. (a) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\cup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$. Show, by an example, that this inclusion can be proper.

12. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

13. Let X be an infinity set. For $p, q \in X$, define $d(p, q) = 1$ if $p \neq q$ and 0 otherwise. Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

14. For $x, y \in \mathbb{R}$, define $d_1(x, y) = (x - y)^2$, $d_2(x, y) = \sqrt{|x - y|}$, $d_3(x, y) = |x^2 - y^2|$, $d_4(x, y) = |x - 2y|$, and $d_5(x, y) = \frac{|x-y|}{1+|x-y|}$. Determine, for each of these, whether it is a metric or not.

15. Prove directly from the definition that the set $E := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

16. Construct a compact set of real numbers whose limit points form a countable set.

17. Give an example of an open cover of the segment $(0, 1)$ which has no finite sub-cover.

18. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space with $d(x, y) = |x - y|$. Let E be the set of all $x \in \mathbb{Q}$ such that $2 < x^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Functions and Continuity

1. Suppose f is a real function defined on $\mathbb{R}$ which satisfies

$$\lim_{h \to 0} (f(x + h) - f(x - h)) = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

2. If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subset \overline{f(E)}$ for every set $E \subset X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

3. Let f be a continuous real function on a metric space X. Let $Z(f) := \{x \in X : f(x) = 0\}$. Prove that $Z(f)$ is closed in X.

4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense set in X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

5. If f is defined on E, the *graph* of f is the set $\{(x, f(x)) : x \in E\}$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane. Suppose E is compact, prove that f is continuous on E if and only if its graph is compact.

6. Let $E \subset X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain is E such that $g(p) = f(p)$ for all $p \in E$. Define f and g on $\mathbb{R}^2$ by: $f(0, 0) = g(0, 0) = 0$, $f(x, y) = xy^2/(x^2 + y^4)$, $g(x, y) = xy^2/(x^2 + y^6)$ if $(x, y) \neq (0, 0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ is continous.

7. Let f be a real uniformly continuous functions on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

8. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathrm{diam}(f(E)) < \epsilon$ for all $E \subset X$ with $\mathrm{diam}(E) < \delta$, where $\mathrm{diam}(E) := \sup\{d(x,y) : x, y \in E\}$ is the diameter of a set E.

9. Suppose f is a uniformly continuous mapping of a metric space (X, d) into a metric space (Y, ρ), and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

10. Prove that a uniformly continuous function of a uniformly continuously function is uniformly continuous.

11. Let $I = [0, 1]$ be the closed unit interval in $\mathbb{R}$. Suppose f is a continuous mapping of I onto I. Prove that $f(x) = x$ for at least one $x \in I$.

12. Call a mapping of X into Y *open* if $f(V)$ is open in Y whenever V is open in X. Prove that every continuous open mapping of $\mathbb{R}$ to $\mathbb{R}$ is monotonic.

13. Let $\lfloor x \rfloor$ denote the integer that $x - 1 < \lfloor x \rfloor \leq x$, and let $(x) = x - \lfloor x \rfloor$ denote the fractional part of x. What discontinuities do the function $\lfloor x \rfloor$ and (x) have?

14. Every rational number x can be written as m/n for integers m, n such that $n > 0$ and m, n having no common divisors. When $x = 0$, take $n = 1$. Consider the function f defined on $\mathbb{R}$ by $f(x) = 1/n$ if $x = m/n$ is rational, and $f(x) = 0$ if x is irrational. Prove that f is continuous at every irrational point, and that f has simple discontinuity at every rational point.

15. Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$. Prove that f is uniformly continuous if h is uniformly continuous. Prove also that f is continuous if h is continuous.

Differentiation of real functions

1. Suppose g is a real function on $\mathbb{R}$ with bounded derivatives, i.e., $\exists M > 0$ such that $|g'(x)| \leq M$ for all $x \in \mathbb{R}$. Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough.

2. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

3. Suppose that f is defined in a neighborhood of x, and $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

   In addition, show by an example that this limit may exist even if $f''(x)$ does not.

4. Suppose $a \in \mathbb{R}$ and f is twice differentiable in (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|$, and $|f''(x)|$ in (a, ∞), respectively. Prove that $M_1^2 \leq 4M_0M_2$.

Riemann integrals

1. Suppose f is a bounded real function on $[a, b]$, and f^2 is Riemann integrable on $[a, b]$. Does it follow that f is Riemann integrable on $[a, b]$? Does the answer change if we assume that f^3 is Riemann integrable?

2. For fixed $a \in \mathbb{R}$, suppose f is integrable on $[a, b]$ for all $b > a$. Define

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x)\,\mathrm{d}x$$

   if this limit exists and is finite. In this case we say that the integral on the left converges. Assume that $f(x) \geq 0$ and f is non-increasing on $[1, \infty)$. Prove that $\int_1^\infty f(x)\,\mathrm{d}x$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

3. Let p and q be positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Prove the following statements:
(a) If $u \geq 0$ and $v \geq 0$, then
$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$
and the equality holds if and only if $u^p = v^q$. [Hint: apply the convex function "$-\log$" on both sides.]
(b) If f and g are Riemann integrable, $f, g \geq 0$, and $\int_a^b f^p \,\mathrm{d}x = \int_a^b g^q \,\mathrm{d}x = 1$, then $\int_a^b fg \,\mathrm{d}x \leq 1$.
(c) If f and g are Riemann integrable, then
$$\left| \int_a^b fg \,\mathrm{d}x \right| \leq \left(\int_a^b |f|^p \,\mathrm{d}x \right)^{1/p} \left(\int_a^b |g|^q \,\mathrm{d}x \right)^{1/q}.$$

4. For Riemann integrable function $u : [a, b] \to \mathbb{R}$, define
$$\|u\|_2 := \left(\int_a^b |u|^2 \,\mathrm{d}x \right)^{1/2}.$$
Suppose $f, g, h : [a, b] \to \mathbb{R}$ are integrable, prove the triangle inequality
$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$
[Hint: take square on both sides and apply the inequality from previous homework].

5. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable. Prove that for any $\epsilon > 0$, there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that $\|f - g\|_2 < \epsilon$. [Hint: Find a suitable partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and define
$$g(t) = \frac{x_{k+1} - t}{x_{k+1} - x_k} f(x_k) + \frac{t - x_k}{x_{k+1} - x_k} f(x_{k+1}),$$
for $t \in [x_k, x_{k+1}]$.]

Sequence of Functions

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a subset E of a metric space, prove that $\{f_n + g_n\}$ converges uniformly on E. If in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

3. Prove a comparison test for uniform convergence of series: if f_n and g_n are functions and $0 \leq f_n \leq g_n$, and the series $\sum_n g_n$ conveges uniformly then so also does the series $\sum_n f_n$.

4. If $f_n \to f$ uniformly on a domain E and if f_n, f never vanish on E (i.e. $f_n(x) \neq 0$ and $f(x) \neq 0$ for all $x \in E$ and $n \in \mathbb{N}$) then does it folow that functions $1/f_n$ converge uniformly to $1/f$ on E?

5. A function is called "piecewise linear" if it is (i) continuous and (ii) its graph consists of finitely many linear segments. Prove that a continuous function on an interval $[a, b]$ is the uniform limit of a sequence of piecewise linear functions.

6. Let $f_n(x) = \frac{x}{1+nx^2}$ for all $x \in \mathbb{R}$ and $n = 1, 2, \ldots$. Show that $\{f_n\}$ converges uniformly to a function f and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x \neq 0$ but false if $x = 0$.

7. Consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n = 1, 2, \cdots$ defined by $f_n(x) = \frac{x}{n} \sin\left(\frac{x}{n}\right)$. (a) Give the set of all points in $\mathbb{R}$ where $\{f_n\}$ converges pointwisely. (b) Does $\{f_n\}$ converge uniformly on $\mathbb{R}$? Justify your claim.

8. If $I(x) = 0$ if $x \leq 0$ and $I(x) = 1$ if $x > 0$, and if $\{x_n\}$ is a distinct sequence in $[a, b]$, and $\sum_n |c_n|$ converges, prove that the series $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ for $a \leq x \leq b$ converges uniformly, and that f is continuous for every $x \neq x_n$.

9. Let $\{f_n\}$ be a sequence of functions where $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Prove that $\{f_n\}$ is uniformly bounded but does not contain uniformly convergent subsequence.

10. Prove that every function in an equicontinuous family of functions is uniformly continuous.

11. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

12. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put $F_n(x) = \int_a^x f_n(t)dt$ for $x \in [a, b]$. Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

13. Let $f : [a, b] \to \mathbb{R}$ be a bounded and Riemann integrable function. Prove that there are polynomials P_n such that $\lim_{n\to\infty} \int_a^b |f - P_n|^2 \,\mathrm{d}x = 0$.

14. If f is continuous on $[0, 1]$ and if $\int_0^1 f(x)x^n \,\mathrm{d}x = 0$ for all $n = 0, 1, 2, \ldots$. Prove that $f(x) = 0$ on $[0, 1]$. [Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x) \,\mathrm{d}x = 0$.]

15. Let K be a compact metric space, and S be a subset of $C(K)$, the set of continuous real-valued functions on K. Equip $C(K)$ with the norm $\|f\| := \sup_{x\in K} |f(x)|$ for every $f \in C(K)$ and define the metric $d(f, g) = \|f - g\|$ for $f, g \in C(K)$. Prove that S is compact with respect to this metric if and only if S is uniformly closed, pointwise bounded, and equicontinuous.

16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

17. Suppose f is a real continuous function on $\mathbb{R}$, and $f_n(t) = f(nt)$ for $n = 1, 2, \ldots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f and justify it.

18. Suppose X is a metric space. Let S be a subset of $C(X)$, the set of continuous real-valued functions on X. If S is equicontinuous and bounded, and define $g : X \to \mathbb{R}$ such that for every $x \in X$ there is $g(x) = \sup_{f\in S} f(x)$. Prove that $g \in C(X)$.

Functions of Several Variables

1. Let $\mathbf{y} \in \mathbb{R}^n$ and define $A : \mathbb{R}^n \to \mathbb{R}$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove that $A \in L(\mathbb{R}^n, \mathbb{R})$ and $\|A\| = |y|$.

2. Give an example of two 2×2 matrices such that the operator norm of the product is less than the product of the operator norms.

3. Suppose $\mathbf{f}$ is a differentiable mapping of $\mathbb{R}$ to $\mathbb{R}^3$ such that $|\mathbf{f}(t)| = 1$ for every $t \in \mathbb{R}$. Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$.

4. Show that both partial derivatives of the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if}(x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

exist at $(0, 0)$ but the function is not differentiable there.