International Journal of Bifurcation and Chaos, Vol. 17, No. 10 (2007) 3387–3395

WHEN SYMMETRIZATION GUARANTEES SYNCHRONIZATION IN DIRECTED NETWORKS

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Received October 18, 2005; Revised March 6, 2006

We review and illustrate our recent results on globally stable synchronization in directed oscillator networks. We consider asymmetrically connected networks with node balance, the property that the sum of the coupling coefficients of all edges directed to a node equals the sum of the coupling coefficients of all the edges directed outward from the node. We show that for such directed but node balanced networks, it is sufficient to symmetrize all connections by replacing a unidirectional coupling with a bidirectional coupling of half the coupling strength. The synchronization condition for the symmetrized network then guarantees synchronization in the original directed network. By considering an example of coupled driven pendula, we show how to prove global stability of synchronization in a concrete unidirectional network. We also discuss the relation between local and global synchronization.

Keywords: Synchronization; stability; directed networks; node balance.

1. Introduction

Since the pioneering works by Fujisaka and Yamada [1983], Afraimovich *et al.* [1986], and Pecora and Carroll [1990], synchronization of chaotic dynamical systems has attracted a rapidly growing interest in physics, mathematics, biology and engineering. Similarly, synchronization of dynamical systems with periodic behavior continues to be of great interest (for a review see [Pikovsky *et al.*, 2001; Strogatz, 2003]).

Until recently, most studies of synchronization were concerned with a small number of coupled systems, but the interest has now shifted toward the analysis of large oscillator networks. An important question in this study is how a synchronous behavior is influenced by the network topology and network size. Especially, complete synchronization in undirected and directed networks of linearly coupled limit-cycle and chaotic oscillators has received much attention (see [Wu & Chua, 1996;

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Pecora & Carroll, 1998; Pecora, 1998; Pogromsky & Nijmeijer, 2001; Barahona & Pecora, 2002; Wang & Chen, 2002; Nishikawa *et al.*, 2003; Chavez *et al.*, 2005; Motter *et al.*, 2005; Wu, 2005] for a sampling of this large field). These studies show that both local and global stabilities of complete synchronization depend on the eigenvalues of the Laplacian connectivity matrix.

In a recent paper [Belykh *et al.*, 2004a], we proved that the synchronization condition can also be derived from *graph theoretical quantities* (the connection graph stability method). The main step of the method is to establish a bound on the total length of all paths passing through an edge on the connection graph. This approach was originally developed for undirected graphs and applied to global synchronization in complex networks [Belykh *et al.*, 2004b; Belykh *et al.*, 2005]. More recently, we showed how the method could be applied to directed networks with node balance [Belykh *et al.*, 2006a].

The purpose of the present paper is to review and illustrate the results obtained in [Belykh *et al.*, 2006a] by considering a few examples of directed networks with node balance. We also compare local and global network synchronization and show that even though the synchronization criteria in both cases are similar, there is no direct relation between them.

2. System Considered

We consider a network of n interacting nonlinear l-dimensional dynamical systems. We suppose that the individual dynamical systems are all identical, even though our results could be generalized to slightly differing systems. The composed dynamical system is described by the $n \times l$ ordinary differential equations

$$\dot{x}_i = F(x_i) + \sum_{k=1}^{n} c_{ik}(t) P x_k, \quad i = 1, \dots, n, \tag{1}$$

where $x_i \in R^l$, the dynamical law of the individual system is expressed by $F : R^l \to R^l$ and P is a projection operator that selects the components of x_i that are involved in the interaction between the individual dynamical systems. They can range from a single component to all l components. For clarity, we shall consider a vector version of the coupling when the first s components are involved, and $P = \mathrm{diag}(p_1, p_2, \dots, p_l)$, where $p_h = 1$, $h = 1, 2, \dots, s$ and $p_h = 0$ for $h = s+1, \dots, l$. We will also consider the case where the projection matrix P is not diagonal (cf. Example 1).

The interaction is assumed to be of diffusive nature (on an arbitrary coupling graph). The coupling matrix C is assumed to have non-negative off-diagonal elements and zero row-sums:

$$c_{ik} \geq 0 \quad \text{for } i \neq k \quad \text{and} \quad \sum_{k=1}^{n} c_{ik} = 0 \quad \text{for } i = 1, \dots, n. \tag{2}$$

We have also indicated in the system (1) the possibility that the interaction depends on time. In this case, the constraints (2) have to hold for all times. Note that we *do not* require the symmetry of the coupling matrix as in our previous papers [Belykh *et al.*, 2004b, 2005], in general, the matrix C will be asymmetric. We associate with the coupling matrix *the connection graph*. To each individual system corresponds a node and to each nonzero off-diagonal element c_{ik}, there corresponds an edge directed from node k to node i. Thus, the connection graph is directed. We suppose that the connection graph is connected.

3. Synchronization Considered

There are many different notions of synchronization [Pikovsky *et al.*, 2001]. We concentrate on the strongest possible form, namely global complete synchronization.

Definition 1. A solution of the system (1) synchronizes *completely*, if

$$\|x_i(t) - x_j(t)\| \to_{t\to\infty} 0 \quad \text{for } i, j = 1, \dots, n. \tag{3}$$

The system (1) synchronizes *globally* and *completely*, if the condition (3) holds for any solution, and it synchronizes *locally* and *completely*, if the *complete synchronization manifold* $\{\underline{x} \,|\, x_i = x_j \text{ for all } i, j = 1, \dots, n\}$ is asymptotically stable, i.e. if any solution of the system (1) that starts sufficiently close to the complete synchronization manifold synchronizes completely.

Note that so far we have imposed no restriction on the individual (uncoupled) dynamical systems $\dot{x}_i = F(x_i)$, $i = \overline{1, n}$. They may have different forms of instabilities/multi-stabilities. In particular, they may have (i) various stable and unstable equilibrium points; (ii) one or more limit cycles; (iii) chaotic behavior. Global complete synchronization then means that for any initial conditions (i) all individual systems converge to the same equilibrium point; (ii) all individual systems converge to the same limit cycle, and on this limit cycle they have the same phase; (iii) all individual systems converge to the same chaotic trajectory.

4. Constraints on the Individual Systems

As in [Belykh *et al.*, 2004a], we impose two constraints on the dynamics of the (uncoupled) individual dynamical systems.

Hypothesis 1. All solutions of the individual system $\dot{x}_i = F(x_i)$ reach in finite time a compact set $B_1 \in R^l$.

It is not difficult to prove that this implies that all the solutions of the composed system (1) reach also in finite time a compact set $B_n \in R^{nl}$. Thus, all interesting dynamical phenomena take place in B_n. Hypothesis 1 is satisfied by most systems of interest.

The second constraint basically requires that the individual dynamical systems can be stabilized by adding a diagonal term for each state component that is involved in the interaction. We state this in a more technical and slightly stronger form.

Hypothesis 2. There exist a parameter $a > 0$ and a matrix

$$H = \mathrm{diag}(h_1, \ldots, h_s, \tilde{H}), \quad \text{where } h_i = 1$$
$$\text{for } i = 1, \ldots, s \ \text{ and } \tilde{H} \ \text{ is positive definite}$$

such that the quadratic form defined by H is a Lyapunov function for all the auxiliary linear systems (varying $x \in B_1$)

$$\dot{\xi} = \frac{\partial F}{\partial x}(x)\xi - aP\xi \tag{4}$$

simultaneously. Equivalently, all matrices

$$H\left(\frac{\partial F}{\partial x}(x) - aP\right) + \left(\frac{\partial F}{\partial x}(x) - aP\right)^T H \tag{5}$$

must be negative definite.

It must be emphasized that we do not alter the original system (1). System (4) is only introduced for proving synchronization in the entire network later on. The stabilizing term will be compensated in the complete equations for the network.

5. Synchrony Between Two Unidirectionally Coupled Oscillators

Note that Hypothesis 2 is closely related to the requirement that the network (1) be composed of *two* unidirectionally coupled systems which *globally* synchronizes when the coupling c_{12} exceeds the critical value a. This is not always true in general [Pecora, 1998; Belykh *et al.*, 2000; Pogromsky & Nijmeijer, 2001]. The standard example is x-coupled Rössler oscillators where global synchronization is *never* stable due to the existence of limit sets lying outside the synchronization manifold [Belykh *et al.*, 2000]. Such systems undergo so-called short-wave bifurcations and only local synchronization can occur in an interval of the coupling parameter. Hence, Hypothesis 2 has to be proven for the chosen type of oscillators and the projection matrix P. Then, having calculated the critical value a sufficient for global synchronization, we can tackle the synchronization problem in larger coupling configurations of that oscillator.

It is worth noticing that once global synchronization is settled at the critical value of coupling, a further increase in coupling strength (even up to an infinite value) *cannot* desynchronize the regime of global synchrony (the proof follows from the Lyapunov function, see Example 1). This is in contrast to local synchronization, which can lose its stability with increasing coupling. We discuss this point in more detail in Sec. 9.

Many networks of linearly coupled limit-cycle and chaotic oscillator exhibit global synchronization such that Hypothesis 2 is satisfied. It was proved for coupled Lorenz systems [Belykh *et al.*, 2004a], Chua circuits, Hindmarsh–Rose neuron models [Belykh *et al.*, 2005], etc. As an illustrative example, we show below how to calculate $c_{12}^* = a$, sufficient for globally stable synchronization of two unidirectionally coupled nonautonomous pendula.

Example 1. Two coupled driven pendula. As an individual dynamical system, we chose a periodically forced single degree of freedom oscillator, the classical nonlinear pendulum that is both damped and driven. The equations of motion are

$$\ddot{\varphi} + d\dot{\varphi} + \sin\varphi = \gamma + A\sin\omega t, \tag{6}$$

where the dimensionless variable $\varphi = \varphi(\bmod 2\pi)$ is the angle of the pendulum, d characterizes damping, γ is a constant torque, and A is the amplitude of the driving force, while ω is the frequency of that force. Depending on the values of these parameters, the system can exhibit dynamics ranging from periodic to chaotic.

The model equations for a system of two *unidirectionally* coupled pendula are given by

$$\begin{aligned}&\ddot{\varphi}_1 + d\dot{\varphi}_1 + \sin\varphi_1\\&\quad = \gamma + A\sin\omega t + c_{12}(\varphi_2 - \varphi_1)\\&\ddot{\varphi}_2 + d\dot{\varphi}_2 + \sin\varphi_1 = \gamma + A\sin\omega t\end{aligned} \tag{7}$$

or, in equivalent first order form,

$$\begin{aligned}&\dot{\varphi}_1 = y_1\\&\dot{y}_1 = -dy_1 - \sin\varphi_1 + \gamma + A\sin\omega t\\&\qquad + c_{12}(\varphi_2 - \varphi_1)\\&\dot{\varphi}_2 = y_2\\&\dot{y}_2 = -dy_2 - \sin\varphi_2 + \gamma + A\sin\omega t\end{aligned} \tag{8}$$

In terms of the system (1), Eqs. (8) are associated with the variables $x_i = \{\varphi_i, y_i\}$, $i = 1, 2$ and matrices $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$.

Introducing the notation for the differences $\Phi = \varphi_1 - \varphi_2$, $Y = y_1 - y_2$, we obtain the stability system

$$\dot{\Phi} = Y, \quad \dot{Y} = -dY - S\Phi - c_{12}\Phi. \tag{9}$$

Here, $S \equiv S(\varphi_1, \varphi_2) = \cos\varphi^*$, where $\varphi^* \in [\varphi_1, \varphi_2]$ comes from the Lagrange mean-value theorem. Obviously, $|S| \leq 1$. Note that the terms representing the torque and driving force, γ and $A\sin\omega t$, are not explicitly present in the stability system (9), but they drive the coefficient S via the system (8).

Global stability of the equilibrium $O = \{\Phi = 0, Y = 0\}$ of the system (9) amounts to global stability of complete synchronization in the system (8).

To prove global stability, we consider a Lyapunov function candidate

$$V = \frac{c_{12}}{2}\Phi^2 + \frac{Y^2}{2} + \beta\Phi Y, \tag{10}$$

where β is a positive auxiliary parameter to be defined. For $c_{12} > \beta^2$ and $\Phi \neq 0$, $Y \neq 0$, the function V is positive. Its derivative with respect to the system (9) is

$$\dot{V} = -\left[\beta(S + c_{12})\Phi^2 + (S + d\beta)\Phi Y + (d - \beta)Y^2\right] \tag{11}$$

Applying Silvester criterion for the negative definiteness of the quadratic form $\dot{V}$, we obtain the conditions:

$$c_{12} > S \quad \text{for any} \quad \varphi^*, \quad d > \beta,$$
$$S + c_{12} > \frac{(S + d\beta)^2}{4\beta(d - \beta)}. \tag{12}$$

Taking into account that $|S| \leq 1$ for any φ^*, we can rewrite the conditions (12)

$$c_{12} > \frac{(1 + d\beta)^2}{4\beta(d - \beta)} + 1, \quad d > \beta, \tag{13}$$

Choosing $\beta = d/4$, we transform the condition (13) as follows

$$c_{12} > \frac{4 + 5d^2 + d^4/4}{3d^2}. \tag{14}$$

The above mentioned condition $c_{12} > \beta^2 = d^2/16$ is also satisfied if the inequality (14) holds. Therefore, under the condition (14), $\dot{V} < 0$ and the trivial equilibrium O of the system (9) is globally asymptotically stable. Consequently, the coupling strength $c_{12}^* = (4 + 5d^2 + d^4/4)/3d^2$ is an upper bound for the minimal value that guarantees global stability of synchronization in the system (8). The value c_{12}^* is also the required constant a, sufficient to stabilize the auxiliary system (4)–(9).

Note that it follows from the proof that the bound (14) is not restricted to the coupled system with the periodic force $A\sin\omega t$, but is directly applicable to coupled pendula or Josephson junctions driven by any common bounded aperiodic signal or noise.

6. Lyapunov Function for the Network

Since we are interested in complete synchronization, we introduce the difference variables $X_{ij} = x_j - x_i$ for any i and j. We will prove synchronization in the network (1) by establishing that the function

$$V = \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}^T \cdot H \cdot X_{ij} \tag{15}$$

is a Lyapunov function for the system (1). For this purpose, we have to show that its derivative along any solution is negative

$$\begin{aligned}\dot{V} = {} & \frac{1}{4}\sum_{i,j=1}^{n} X_{ij}^T H \Bigg[F(x_j) - F(x_i)\\& + \sum_{k=1}^{n}(c_{jk}PX_{jk} - c_{ik}PX_{ik})\Bigg]\\& + \frac{1}{4}\sum_{i,j=1}^{n}\Bigg[F(x_j) - F(x_i)\\& + \sum_{k=1}^{n}(c_{jk}PX_{jk} - c_{ik}PX_{ik})\Bigg]^T HX_{ij}.\end{aligned} \tag{16}$$

Rewriting $F(x_j) - F(x_i) = [\int_0^1 d\beta(\partial F/\partial x)(\beta x_j + (1 - \beta)x_i)]X_{ij}$ and adding and subtracting a term

$aX_{ij}^T HPX_{ij}$ from both sums, we have, according to Hypothesis 2

$$\begin{aligned}\dot{V} \le \frac{1}{4}\sum_{i,j,k}^{n} X_{ij}^T H(c_{jk}PX_{jk} - c_{ik}PX_{ik}) \\ + \frac{1}{4}\sum_{i,j,k}^{n}(c_{jk}PX_{jk} - c_{ik}PX_{ik})^T HX_{ij} \\ + \frac{1}{2}\sum_{i,j=1}^{n} aX_{ij}^T HPX_{ij}. \end{aligned} \tag{17}$$

After some algebraic manipulations [Belykh *et al.*, 2006a] one obtains

$$\begin{aligned}\dot{V} \le -\sum_{\nu=1}^{s}\sum_{j=1,i>j}^{n} h\left(n\frac{c_{ij}+c_{ji}}{2} - a\right){X_{ij}^{\nu}}^2 \\ + \sum_{\nu=1}^{s}\sum_{j=1,i>i}^{n} h\frac{C_i + C_j}{2}{X_{ij}^{\nu}}^2, \end{aligned} \tag{18}$$

where $C_j = \sum_{i=1}^{n} c_{ij}$ are the column sums of the connection matrix C.

7. Node Balance and Main Theorem

We have to show that the time derivative of V along any solution is negative. If the connection matrix C was symmetric, as in our previous papers [Belykh *et al.*, 2004a, 2005], the column sums of C, being equal to the row sums, would vanish and we only have to worry about the negativity of the first term on the RHS of Eq. (18). In this paper, we discuss the case when the row sums are equal to the column sums without the symmetry of C. This means $C_j = \sum_{i=1}^{n} c_{ij} = \sum_{j=1}^{n} c_{ji} = 0$ for $j = 1, \ldots, n$ and implies

$$\sum_{i=1,i\neq j}^{n} c_{ij} = \sum_{i=1,i\neq j}^{n} c_{ji} = 0 \quad \text{for } j = 1, \ldots, n. \tag{19}$$

Definition 2. A directed graph with a connection coefficient associated with each edge has the property of *node balance* if, for each node, the sum of all the coefficients of the edges directed towards the node is *equal* to the sum of the coefficients of all edges directed away from the node. In terms of the connection matrix, this condition amounts to Eq. (19).

Under the hypothesis of node balance, the sufficient condition for global complete synchronization derived from the inequality (18) is the positivity of the quadratic form

$$\sum_{\nu=1}^{s}\sum_{j=1,i>i}^{n} h\left(n\frac{c_{ij}+c_{ji}}{2} - a\right){X_{ij}^{\nu}}^2.$$

Note that now only the symmetrized version $(C + C^T)/2$ of the connection matrix C plays a role.

In our previous work [Belykh *et al.*, 2004a], we have linked the positivity of this quadratic form both to the second largest eigenvalue of the connection matrix C and to a purely graph-theoretical quantity. Similar to [Belykh *et al.*, 2006a], these results can now be directly applied.

Theorem 1. *Suppose the network described by Eq. (1) satisfies the following conditions*

(i) *The individual dynamical systems satisfy Hypotheses 1 and 2.*
(ii) *The connection matrix has nonnegative off-diagonal elements and zero row sums.*
(iii) *The network graph has node balance.*

Then the network synchronizes globally and completely if either the inequalities (a) *or* (b) *below are satisfied.*

(a) $|\lambda_2| > a$, *where* λ_2 *is the second largest eigenvalue of the symmetrized matrix* $(C + C^T)/2$.
(b) *If we number the edges of the connection graph by* $k = 1, \ldots, m$, *then*

$$c_{sym,k} > \frac{a}{n}\, b_k(n, m) \quad \textit{for } k = 1, \ldots, m, \tag{20}$$

where $c_{sym,k} = (c_{ij} + c_{ji})/2$ *if the edge* k *connects nodes* i *and* j. *The quantities* b_k *are computed for the symmetrized undirected graph as follows* [*Belykh et al.*, *2004*]: *a path* P_{ij} *between nodes* i *and* j *is chosen for each pair of nodes* i *and* j. *Then* b_k *is the sum of all lengths of the chosen paths that go through edge* k: $b_k = \sum_{i,j=1,\, k\in P_{ij}}^{n} |P_{ij}|$, *where* $|.|$ *denotes the length of a path.*

We emphasize again that thanks to node balance, only the symmetrized connection matrix plays a role in the theorem. The symmetrized connection matrix is the connection matrix of the undirected graph obtained from the original directed graph by the substitution of Fig. 1, edge by edge.

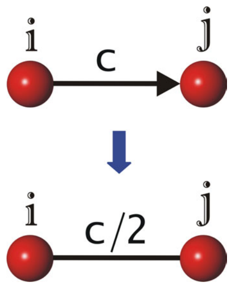

Fig. 1. Symmetrization operation. Each directed edge is replaced by an undirected edge of half the coupling strength.

8. Examples of Networks with Node Balance

8.1. *Directed lattice on a torus*

If all connection coefficients are equal, then node balance amounts to equal in- and out- degree of each node. Consider an example of such a network, the two-dimensional lattice on a torus with a uniformly directed coupling and coupling coefficients c [Fig. 2]. The symmetrized undirected network with coupling $c/2$ is depicted in Fig. 2. The torus is composed of $n_{\rm ml}$ and $n_{\rm pr}$ oscillators in the meridian and parallel directions, respectively, $n_{\rm pr} > n_{\rm ml}$.

Synchronization in this symmetrized network can be proved by calculating b_k (cf. Theorem 1). Due to the lattice structure of the network, we can simplify calculations by considering separately synchronization of oscillators in the meridian and parallel directions. It can be shown that this amounts to obtaining the synchronization conditions in rings of the sizes $n_{\rm pr}$ and $n_{\rm ml}$. Considered together, these two conditions will then give the synchronization criterion for the entire network. More precisely, the maximum from $b_k(n_{\rm pr})$ and $b_k(n_{\rm ml})$ will guarantee synchronization in the torus.

Synchronization in a ring of n coupled oscillators was already studied within the connection graph method [Belykh *et al.*, 2004a]. It was shown that the graph quantity can be calculated as follows $b_k = \{(n^3/24 - n/24)$ for odd n; $(n^3/24 + n/12)$ for even $n\}$. Therefore, since $n_{\rm pr} > n_{\rm ml}$, the sufficient synchronization condition for the symmetrized torus network becomes

$$\frac{c}{2} > \begin{cases} a\left(\dfrac{n_{\rm pr}^2}{24} - \dfrac{1}{24}\right) & \text{for odd } n_{\rm pr} \\ a\left(\dfrac{n_{\rm pr}^2}{24} + \dfrac{1}{12}\right) & \text{for even } n_{\rm pr}. \end{cases} \tag{21}$$

According to Theorem 1, this sufficient condition also guarantees synchronization in the original directed network Fig. 2.

We have also calculated the eigenvalues numerically for specific torus network examples (for different $n_{\rm pr}$ and $n_{\rm ml}$). For all these examples, the real parts of the eigenvalues of the directed and symmetrized undirected networks are the same. For example, for $n_{\rm pr} = 10$ and $n_{\rm ml} = 7$, the second largest eigenvalues of the directed and undirected networks are $-0.1910 \pm 0.5878i$ and -0.1910, respectively. It shows that our graph-based analysis correctly predicts the real relation between the synchronization properties of the two networks.

It must be emphasized that the real parts of the eigenvalues of directed but node balanced networks and their symmetrized analogs do not coincide in general. Typically, the symmetrized network is slightly more difficult to synchronize than the directed coupled one. This also supports our analytical results that the symmetrization applied to networks with node balance provides the sufficient condition that *guarantees* synchronization.

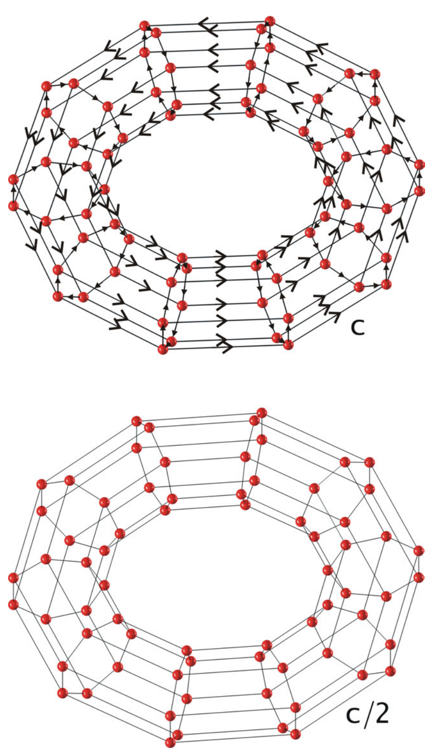

Fig. 2. Directed torus network and the corresponding symmetrized network with half the coupling strength per link. Arrows indicate direction of coupling along an edge; edges without arrows are coupled bidirectionally. The width of the links may be thought of as the coupling strength.

8.2. *How to create a node balanced network*

Regularly structured networks of interest often have an equal in- and out- degree. Examples include rings of K-nearest neighbor coupled nodes.

To construct node balanced networks with a given network graph that does not necessarily have, at each node, an equal in- and out- degree, it is convenient to interpret the connection coefficients as currents. Indeed, the node-balance condition is nothing else than Kirchhoff's current law (KCL). According to classical circuit theory [Chua *et al.*, 1987], a vector of edge currents satisfying KCL can always be represented as a linear combination of loop currents. To give an example, clearly, the network of Fig. 3(left) satisfies node-balance. In Fig. 3 (right) it is shown how the connection coefficients can be obtained by superimposition of "loop currents", where the loops are the windows of this planar graph.

9. Global Versus Local Synchronization

The theorem for local synchronization [Pecora & Carroll, 1998; Pecora, 1998] says that the nonzero eigenvalues of the connection matrix have to lie in the stability region of the master stability function of the individual dynamical system. In coupled systems exhibiting only long-wave length bifurcations [Pecora, 1998], this leads to a bound of the type $-\mathrm{Re}\lambda_2 > \alpha$ where $\alpha > 0$ is the largest Lyapunov exponent of the individual system and where λ_2 is the second largest eigenvalue of the original, possibly asymmetric, connection matrix. Our criterion, when expressed with the eigenvalue, leads to an inequality of the type $-\lambda_2 > a$ for some constant $a > 0$, where a is the maximum of the positive diagonal elements needed to stabilize the individual system. Here, λ_2 is the (necessarily real) second largest eigenvalue of *the symmetrized connection matrix*. What is the relation between the two criteria? Even though the criteria are very similar, there is no direct relation between them. The following remarks make a case in point.

From the phenomenological point of view, clearly global complete synchronization is a stronger condition than local synchronization. In fact, for global synchronization, not only the behavior of the system in the vicinity of the synchronization manifold matters, but also the behavior far from it. The fact that one cannot infer global from local synchronization can be seen by the following example. Suppose that the individual systems are one-dimensional, with a function F as represented in Fig. 4.

For this example, the qualitative dynamical behavior of the uncoupled individual system is easily understood. All solutions, except those starting exactly in an unstable equilibrium point (crosses) converge towards a stable equilibrium point (circles). The largest Lyapunov exponent is given by the maximum α of the slopes of the two unstable equilibrium points. Local synchronization takes place, if $-\mathrm{Re}\,\lambda_2 > \alpha$.

On the other hand, a necessary condition for global synchronization is that there is no equilibrium point of the network outside of the synchronization manifold. Now, when the coupling coefficients are zero, all 5^n combinations of equilibrium points of the individual systems are equilibrium points of the network. Gradually increase all coupling coefficients, keeping identical values c. Then, for any directed or undirected graph, λ_2 is proportional to c. Suppose that we do not have the contrived case where $-\mathrm{Re}\,\lambda_2 = 0$. At sufficiently small values of c, at least some of the equilibrium points are still lying outside of the synchronization

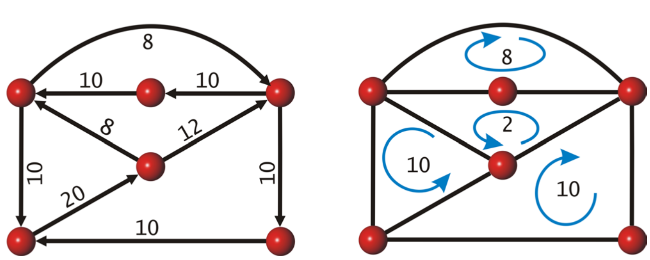

Fig. 3. (Left) Example of a node balanced network. Numbers correspond to coupling strengths c_{ij}. Two nodes do not have an equal in- and out- degree. (Right) Construction of the node balanced networks by means of Kirchhoff's current law. The coupling strength of each edge is defined by a linear combination of loop currents.

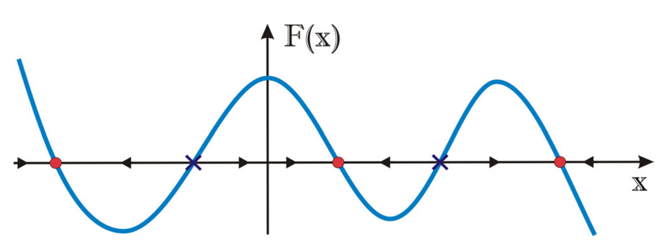

Fig. 4. Graph of the function F, defining the dynamics of the one-dimensional individual dynamical system. Stable and unstable equilibria are depicted by circles and crosses, respectively.

manifold. Now, change the function F in a very small neighborhood of the unstable equilibrium points of the isolated system such that the slope at these points becomes very small, until $-\mathrm{Re}\,\lambda_2 > \alpha$ is satisfied. Outside of these neighborhoods, nothing is changed. It is not difficult to see that this can be done in such a way that the equilibrium outside of the synchronization manifolds are not touched. Then, for this value of coupling and for the modified function F we have local, but not global complete synchronization.

From the point of view of the mathematical criteria for local and global synchronization, let us remark that α and a are constants linked to somewhat different dynamical properties of the individual dynamical systems. In addition, even though in some special cases the $-\mathrm{Re}\,\lambda_2$ of the asymmetrically coupled system and $-\lambda_2$ of the symmetrized system coincide, this is in general not the case.

Finally, we note that in the presence of *short-wave length* bifurcations [Pecora, 1998], when global synchronization is impossible, the comparison between local and global synchronization conditions makes no sense.

10. Conclusions

We have introduced the notion of node balance for directed graphs with a positive coupling coefficient associated to each edge of the graph. It means that the sum of the coupling coefficients of the edge directed towards a vertex equals the sum of the coefficients of all edges directed away from the same vertex. If we interpret the coupling coefficients as currents, this is equivalent to Kirchhoff's current law.

We showed that for networks with node balance, it is sufficient to symmetrize all connections by replacing a unidirectional coupling of strength c by a bidirectional coupling of strength $c/2$. The bounds for global synchronization for this directed network then hold also for the original directed network. If the node balance is not satisfied, the directed network may have very different synchronization behavior from the symmetrized network. The extension of the connection graph method to this most general case of weighted networks is given in [Belykh *et al.*, 2006b].

The explicit bounds on the coupling coefficients that guarantee global complete synchronization can be expressed either in terms of the second largest eigenvalue of the symmetrized coupling matrix or by a purely graph theoretical criterion. We remark that local synchronization depends also on the eigenvalue with the second largest real part, in this case, however, of the asymmetrical coupling matrix. Nevertheless, we showed with an example that global and local synchronization are no more related in general than that the former implies the later.

Acknowledgments

This work was supported in part by the SNSF (grant No. 21-6526801), the EU Commission (FP6-NEST project No. 517133), and the RFFI-NWO (grant No. 047.011.2004.004).

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