Dynamical networks with on-off stochastic connections: beyond fast switching

Russell Jeter
Department of Mathematics
and Statistics
Georgia State University
Atlanta, Georgia 30303

Igor Belykh
Department of Mathematics & Statistics
and Neuroscience Institute
Georgia State University
Atlanta, Georgia 30303
Email: ibelykh@gsu.edu

***Abstract*—We consider networks whose topology changes in time according to a stochastic rule. While the literature gives insight into the effects of fast stochastic connections, little is known about the effects of slower switching on the evolution of a network. We review recent analytical results on convergence properties of fast switching dynamical networks, including bounds on the probability of converging towards an attractor of a multistable network. We also discuss the advantages of slower switching over fast switching, and consider an example in which slow switching provides opportunities for network synchronization while fast switching does not. It is shown that there is an optimal window in which the switching frequency causes an unstable system to stabilize.**

I. Introduction

Networks of dynamical systems are common models for various types of systems across many disciplines, including physics, engineering, chemistry, biology, and even the social sciences. The best known examples (and most studied) include Internet routers, genetic networks, ecological networks, neuronal networks, and communication/social networks. A great deal of attention has been focused on examining the interconnectedness of the dynamical properties of the individual nodes and those of the network topology. Specifically, researchers have studied the interplay between these network characteristics as they apply to causing synchronization within the network, as synchronization is a key property among both biological and technological networks (as it is observed in the field and mathematical models). Given the already complex nature of the issue, most studies have looked at networks whose connections are fixed in time, or are governed by a strict dynamical rule. Only very recently researchers have considered networks with topology that evolves in time based on a deterministic or stochastic rule [1]–[15]. This is currently a hot research topic due to its potential in a variety of emerging applications.

In many engineering and biological networks, the individual nodes that compose the network interact only sporadically via short on-off interactions. Packet switched networks such as the Internet are an important example. To model realistic networks with intermittent connections, a class of dynamical networks with fast on-off connections, called "blinking" networks, was introduced in [1], [2]. These networks are composed of oscillatory dynamical systems with connections that switch on and off randomly; the switching time is fast, with respect to the characteristic time of the individual node dynamics. It was proved in [1] that global synchronization occurs almost surely in a blinking network, provided that coupling strengths are strong enough and the switching time of blinking connections is fast.

This work focuses on the mathematical analysis and modeling of dynamical networks whose coupling or internal parameters stochastically switch on and off, on a time scale that is not necessarily fast. We review the recent results [1]–[5] on blinking networks and study dynamical properties of a switching network as a function of the switching frequency, within and beyond the fast-switching limit.

II. On-Off Dynamical Networks: fast switching and beyond

We consider the asymptotic dynamics of the general continuous-time dynamical network with identically distributed independent random switching variables. The general equation for the model network of n nodes has the following form:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}_i(\mathbf{x}_i) + \sum_{j=1}^{n} \varepsilon_{ij} g_{ij}(t) \mathbf{E} \mathbf{x}_j, \qquad (1)$$

where the dynamics of the individual nodes are governed by $\mathbf{F}_i(\mathbf{x}_i)$, and $\mathbf{F}$ is a set of differential equations that act on the state vector, $\mathbf{x}_i$. g_{ij} is the component of $\mathbf{G}$, the $n \times n$ zero row-sum, connectivity matrix (Laplacian), that represents the possible node-to-node connections (switches). $\mathbf{E}$ projects the coupling onto the appropriate variable and ε_{ij} is the coupling strength between the i-th and j-th nodes. Building on the theory established in [3], [4], we replace the non-zero entries of $\mathbf{G}$ in the following manner, the i-jth entry (where $i \neq j$) of $\mathbf{G}$, g_{ij}, is replaced with $\mathbf{s}_{ij}$. We shall call this new matrix $\tilde{\mathbf{G}}$. $\mathbf{s}_{ij}$ is a binary function that is either $\{1\}$ with probability p or $\{0\}$ with probability $1-p$, and is then chosen again (with the same probabilities, p and $1-p$ respectively) after some time τ. To maintain the symmetry of $\tilde{\mathbf{G}}$, we set $\mathbf{s}_{ij} = \mathbf{s}_{ji}$, and lastly, we point out that the diagonal elements of $\tilde{\mathbf{G}}$ are still chosen to make $\tilde{\mathbf{G}}$ a zero row-sum matrix. In simple terms, we replace the edges in the graph represented by $\mathbf{G}$ with edges that blink on and off randomly at discrete intervals of length τ. The resulting graph at any time t is given by an Erdös-Rényi graph of n vertices. The existence of an edge from vertex j to vertex $i \neq j$ is determined randomly and independently of other edges (switches) with a given probability p. Thus, every switch in the network is operated independently, according to a similar probability law, and each switch opens and closes in different time intervals independently. It is important to note that we are considering a stochastic schematic such that there is the possibility of a connection between any two nodes, which is not always the case. One can decide which nodes they want allow to be connected, and assign g_{ij} accordingly. Figure 1 gives an example of a "blinking" graph.

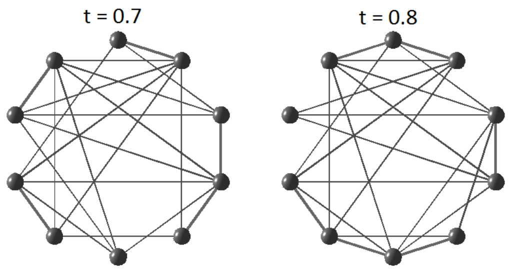

Fig. 1. Graph at two different time steps that fall in different discrete intervals (recall the intervals are each of length τ). In this graph, edges could be between any two edges, and the probability of there being an edge between two nodes is $p = 0.5$, and the coupling strength along each edge is ε. In this example, $\tau = 0.1$, which means that the graph reconfigures itself after 0.1 time has passed.

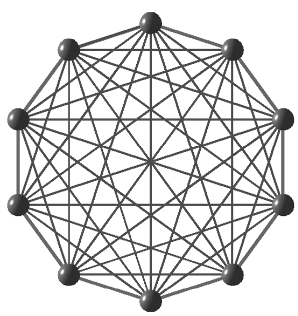

Fig. 2. The graph of an averaged system, where there is an edge between every node (10 in this example) in which the coupling along each edge is just $p\varepsilon$.

If switching is fast compared to the oscillator's intrinsic time scale, it is natural to expect the switching system to follow the averaged system, which is obtained from taking the expectation of all of the stochastic variables ($\mathbf{s}_{ij}$). This amounts to replacing the non-zero entries of $\mathbf{G}_{ij}$ ($i \neq j$) with p and assigning the diagonal elements that preserve the zero row-sum property of $\mathbf{G}$. We denote the averaged system Φ. Conceptually, this equates to connections between nodes that are always present, but that are weaker than 'on' connections (as a connection in Φ has coupling $p\varepsilon$ and a 'on' connection in $\mathbf{F}$ has coupling ε) (see Fig. 2).

The relation between the dynamics of the stochastically blinking network and its averaged analog is a non-trivial problem and a substantial contribution to its solution has been made in the previous work [1]–[6], [9]. While averaging is a classical technique in the study of nonlinear oscillators, averaging for blinking systems needs some special mathematical techniques for obtaining rigorous convergence proofs. Such techniques have been developed for synchronization of blinking networks of chaotic dynamical systems [1] and for the convergence of the blinking network to an attractor [2]–[5].

It was proven [1], [6], in different contexts, that switching networks (1) of coupled identical oscillators can synchronize even if the network is insufficiently coupled to support synchronization at every instant of time. In particular, it was rigorously proven in [1] that, for almost all switching sequences, the threshold for complete synchronization in the blinking network is the same as the threshold in the averaged model, where the remaining links are constant, with value $p\varepsilon$. In other words, the set of on-off shortcut switching sequences that fail to force total synchronization has probability zero. For this property to be true, the switching time τ must be much smaller than the characteristic synchronization time $T_{\rm syn}$ of the network. This allows the use of averaging. The explicit bound for the switching time τ that satisfies this requirement is $C_1 \exp\left\{-\frac{C_2 T_{\rm syn}}{N\varepsilon}\cdot\frac{1}{\tau}\right\} < 1$ where C_1 and C_2 are functions of the static network topology and the dynamics of the individual node (their explicit dependences along with $T_{\rm syn}$ are given in [1]). The switching period τ appears only in the dominator of the exponent. Therefore, the left-hand side of the inequality decreases rapidly when τ decreases and the inequality can always be satisfied for small enough τ.

The proof for global stability of complete synchronization in network (1) involves the construction of a Lyapunov function for the difference (transverse) oscillators' variables that decreases along solutions of the blinking system. Actually, because of the stochastic nature of the switching, this is not always true. The Lyapunov function may increase temporarily, but the general tendency is to decrease. Switching is a stochastic process, therefore, the convergence properties also have a probabilistic flavor. This can be expressed by showing that after a certain time the Lyapunov function decreases with high probability [1], [4] as long as the switching frequency is sufficiently high.

In [3]–[5], rigorous theory for the behavior of stochastically switching networks that blink rapidly was developed. There are four distinct classes of switching dynamical networks. Two properties differentiate them: single or multiple attractors of the averaged system and their invariance or non-invariance under the dynamics of the switching system. In the case of invariance, one proves that the trajectories of the switching system converge to the attractor of the averaged system with high probability. In the non-invariant single attractor case, the trajectories rapidly reach a ghost attractor and remain close most of the time with high probability. In the non-invariant multiple attractor case where the averaged network is multistable and one of its attractors is not shared by the switching network, the trajectory may escape to another ghost attractor with small probability [3], [4]. The developed theory allows deriving explicit bounds that connect the probability of converging towards the ghost attractor, the switching frequency, and the chosen initial conditions [3]–[5]. This theory and its application to specific networks will be reviewed in the talk.

There are circumstances for which not converging to the averaged system is favorable, and the present theory is not able to make definitive claims about the behavior of the stochastic

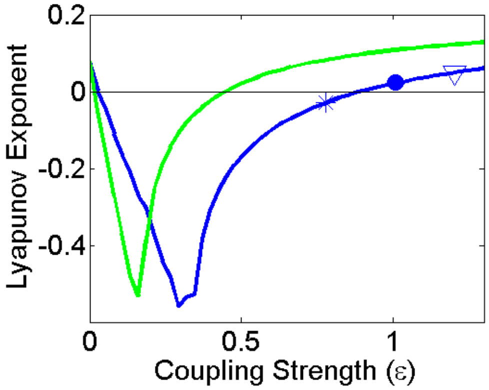

Fig. 3. Transversal Lyapunov Exponent for the stability of synchronization in the 10-node network of Rössler systems, both averaged (light) and static (dark). Note that increasing the coupling strength, beyond a critical coupling strength ε^* on the right side of the stability well, makes synchronization unstable [16].

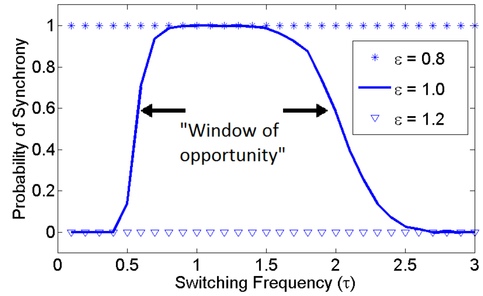

Fig. 4. The effects of varying τ for three different coupling strengths, $\varepsilon = 0.8$, $\varepsilon = 1$, and $\varepsilon = 1.2$. These are coupling strengths for which the averaged system is stable, barely unstable, and unstable respectively (cf. Fig. 3). Notice the bell-shaped curve corresponding to an optimal range of non-fast switching $0.6 < \tau < 2.2$ ("the window of opportunity"), where synchronization in the stochastic network becomes stable with high probability, whereas synchronization in the corresponding averaged system is unstable ($\varepsilon = 1$). Probability calculations are based on 1000 trials.

system *beyond fast switching*. This leads us to explore the effects of not-fast-switching (where τ is not necessarily small) on the dynamics of the network.

III. Beyond Fast Switching: An Example

We consider a network of ten coupled oscillators, whose individual dynamics are governed by the Rössler equations in the chaotic regime coupled through the x variables. The blinking system is given by (2).

$$\begin{aligned} \dot{x}_i &= -(y_i + z_i) + \sum_{j=1}^{n} \varepsilon_{ij} \mathbf{s}_{ij}(x_j - x_i) \\ \dot{y}_i &= x_i + a y_i \\ \dot{z}_i &= b + z_i(x_i - c) \end{aligned} \tag{2}$$

The static equivalent to the network we are considering (and therefore, the basic structure of the averaged system) resembles the complete graph of ten vertices, K_{10}, see Fig. 2. Synchronization in a network of x-coupled Rössler systems is known [16] to destablize after a critical coupling strength ε^*, which depends on the eigenvalues of the connectivity matrix, $\mathbf{G}$. For our uncoupled, not-averaged system, we turn to the Master Stability Function [16] to give us an idea for the value of ε^*. In the averaged system we set $p\varepsilon_1^* = \varepsilon^*$, and see that the critical coupling for the averaged system (ε_1^*) is just the critical coupling for the static, non-averaged system over the probability of having an edge in the stochastic system, $\varepsilon_1^* = \frac{\varepsilon^*}{p}$ (see Fig. 3).

Before delving into the exploration of "not-fast-switching," we must first discuss the invariant attracting set of our network (both stochastic and averaged). The invariant set is a three dimensional manifold, characterized by $x_i = x_j$, $y_i = y_j$, and $z_i = z_j$ for $i, j = 1, 2, \ldots, n$. We refer to this as the synchronization manifold, and the dynamics along the manifold are characterized by the the dynamics of the uncoupled system.

We find that the system behaves like the averaged system, even close to the critical value $\varepsilon^* \approx 0.9$, for $\tau \leq 0.01$. Notice that for all values of τ on the range $\tau \in [0.1, 3]$ we observe synchrony for $\varepsilon = 0.8$, which is in the synchrony window (the stability well) seen in Fig. 3. What is startling about this observation is that there is synchrony even for $\tau = 3$, which is somewhat unexpected, because the individual nodes spend a fair amount of time uncoupled. Looking further, we consider a wide range of values for τ in Fig. 4 and see that for a value of $\varepsilon > \varepsilon^*$, we can observe synchrony in the stochastic system, even when synchrony *is not present* in the averaged system. We find that this optimal window for τ is actually quite large, with synchrony appearing with a high probability ($P_{Synchrony} > 0.5$) around $\tau = 0.6$. Moreover, we see that the system synchronizes consistently until around $\tau = 2.2$ While for $\tau > 2.5$, we see that the system is once again unstable. If the system spends a large portion of its time with either coupling that is destabilizing, or entirely uncoupled. Both of these scenarios would result in a system that fails to synchronize. This gives us the impression that making the system stochastic, in some way, reforms the graph given in Fig. 3. We then observe that for $\varepsilon = 1.2$, which corresponds to the ∇-laden curve in Fig. 4, that there is synchrony for no values of τ in the range that we consider. While this tells us that for the network topology we are considering, there is a critical ε^* such that no switching frequencies can synchronize the system.

IV. Conclusions

An important research problem is to develop a rigorous theory for understanding and dynamical networks *beyond fast switching*. We have used an example to show that connections, that are only present with some probability p in a complex network, can stabilize synchronization even in a normally unstable regime. We have explored the possibilities when the time scale for the stochastic process does not approach 0, and showed that so call "slow switching" or "not-fast-switching" can be favorable, compared to fast switching, when one does not want to follow the dynamics of the averaged system. We have also shown that switching cannot be too slow, as this can make the system even more unpredictable. This gives the impression that there is some window for each system for which we have a sense of "controlled unpredictability." Moreover, this phenomenon also seemed to pop up in [5] when we were analyzing an entirely different system whose parameters blink stochastically (i.e, not a network with stochastic connections). We named this controlled unpredictability, "windows of opportunity," to further emphasize that there seem to consistently be favorable conditions in which the stochastic and deterministic parameters match up appropriately, and allow the system to behave favorably, against all odds.

While the numerical results alone give us plenty of insight as to the effects of stochastic coupling in dynamical networks (and how stochastic connections can actually be favorable to static ones), we are currently working on developing an analytical approach to this problem. Namely, examining (i) what creates this phenomenon of synchrony in systems that are unstable when the connections are static; (ii) the role the network topology in the averaged system in how it effects the possibilities present in the stochastic system. Are there topologies for which there is an optimal value τ^* for which the stochastic system synchronizes consistently for any value of ε, even when the averaged system has a finite window ($\varepsilon \in [\varepsilon_1, \varepsilon_2]$), outside of which there is no synchrony? What is the optimal switching frequency and rewiring strategy for a desired performance objective? What are critical feedback mechanisms linking the adaptation of complex network structure and dynamics of switching networks? In particular, can the network topology be designed to vary, according to a Markov stochastic process, such that the (multistable) network converges to the desired mode with high probability that can be explicitly given apriori? The work which gives explicit analytical insights into these complicated issues is in preparation.

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