Supplementary Material for "From Delay to Inertia and Triadic Interactions: A Reduction of Coupled Time-Delayed Oscillators"

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DERIVATION OF THE SECOND-ORDER REDUCED MODEL

We study a generalized KD network with heterogeneous natural frequencies, external forcing, and time-delayed pairwise coupling:

$$\frac{d\theta_j(t)}{dt} = \varpi + \eta_j(t) + \frac{\varkappa}{N}\sum_{k=1}^{N} F_{jk}\big(\theta_k(t-\tau) - \theta_j(t)\big), \tag{S.1}$$

where $\theta_j(t)$ is the phase of oscillator j ($j = 1, \ldots, N$), ϖ is a baseline (mean) natural frequency, $\eta_j(t)$ collects deviations from this baseline (including external forcing or detuning), $\varkappa$ is the coupling strength, and $F_{jk}(\cdot)$ are 2π-periodic pairwise coupling functions that may differ across oscillator pairs and represent, for example, random interactions. The parameter $\tau > 0$ is a uniform coupling delay.

Basic Assumptions for Reduction: Weak Heterogeneity and Small Coupling Strength

To derive a reduced description of the dynamics governed by Eq. (S.1), we consider the regime in which both the heterogeneity ($\eta_j(t)$ and the coupling strength $\varkappa$ are weak. We introduce a small parameter $\varepsilon \ll 1$ and scale these quantities as

$$\eta_j(t) = \varepsilon\omega_j + \varepsilon\zeta_j(t), \qquad \varkappa = \varepsilon\kappa.$$

Here, ω_j denotes the (scaled) deviations of the natural frequencies from the mean value ϖ, $\zeta_j(t)$ represents additional time-dependent perturbations, and κ is the unscaled coupling constant.

This setting corresponds to a self-oscillatory regime in which the dominant behavior is nearly uniform rotation at frequency ϖ. Indeed, in the limit $\varepsilon = 0$, all oscillators evolve independently with $\dot{\theta}_j = \varpi$, yielding identical uniform rotation. The introduction of ε thus enables a systematic perturbative treatment of weak heterogeneity and weak coupling around this uniform rotational state.

Multiple Time Scale Expansion

The assumption that the oscillators exhibit nearly uniform rotation at the dominant frequency ϖ, with slow modulations induced by weak disorder, external influences, and pairwise interactions, enables the use of a multiple-timescale expansion. This perturbative technique is well-suited to systems with dynamics that evolve on distinct temporal scales.

To systematically capture the slow evolution of the phases, we introduce a hierarchy of time scales $t_s = \varepsilon^s t$ for $s = 0, 1, 2, \ldots$. This separation allows us to distinguish the fast oscillatory motion from the slow modulations driven by heterogeneity and coupling. Before proceeding with the reduction, we make one technical remark that clarifies the structure of the analysis.

Because the variables $t_s = \varepsilon^s t$ are treated as independent, the total time derivative expands as a sum of partial derivatives:

$$\frac{d}{dt} = \sum_{s=0}^{\infty} \varepsilon^s \frac{\partial}{\partial t_s} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2} + \ldots. \tag{S.2}$$

We seek solutions of Eq. (S.1) in the form of an asymptotic expansion in powers of ε:

$$\theta_j(t) = \varpi t_0 + \phi_j(t_1, t_2, \ldots) + \sum_{p=1}^{\infty} \varepsilon^p \varphi_j^{(p)}(t_0, t_1, t_2, \ldots), \tag{S.3}$$

where the leading term ϖt_0 represents the primary fast rotation, while $\phi_j(t_1, t_2, \ldots)$ describes the slow collective phase drift responsible for the emergent network-level dynamics. The functions $\varphi_j^{(p)}$ represent higher-order corrections that include rapid oscillatory components and transient effects, which average out over long time scales.

Expansion of Delayed Phase Differences

To substitute the multi–scale expansion into the Kuramoto–Daido model, we first evaluate the delayed phase difference $\theta_k(t-\tau) - \theta_j(t)$. Using the asymptotic series (S.3), and assuming that the slow time variables $t_1, t_2, \ldots$ are only weakly affected by the finite delay τ at leading order (i.e., τ does not generate additional fast time scales), we obtain the expansion

$$\begin{aligned}\theta_k(t-\tau) - \theta_j(t) = -\varpi\tau + \phi_k(t_1, t_2, \ldots) - \phi_j(t_1, t_2, \ldots)\\ - \varepsilon\tau \frac{\partial \phi_k}{\partial t_1} + \varepsilon \varphi_k^{(1)}(t_0 - \tau, t_1, t_2, \ldots) - \varepsilon \varphi_j^{(1)}(t_0, t_1, t_2, \ldots) + o(\varepsilon).\end{aligned} \tag{S.4}$$

Here, the term $-\varpi\tau$ reflects the difference in the rapid rotation between times t and $t-\tau$. Since ϖ is not assumed to satisfy any resonance conditions with τ, this constant phase shift may take arbitrary values. The next terms, $\phi_k(t_1, t_2, \ldots) - \phi_j(t_1, t_2, \ldots)$, represent the slow phase difference between oscillators k and j. The contribution $-\varepsilon\tau\, \partial\phi_k/\partial t_1$ accounts for the slow drift of oscillator k over the delay interval τ when observed on the t_1 time scale. Although the variables ϕ_j evolve slowly compared with the fast rotation, their change over time τ must still be retained at order ε. The remaining terms in (S.4) contain the first-order fast corrections $\varphi_j^{(1)}$, evaluated at t_0 and $t_0 - \tau$, respectively. These describe rapid fluctuations around the slow phase manifold. The $o(\varepsilon)$ term indicates truncation at first order, appropriate for small ε.

Hierarchy of Equations by Order of Smallness

Substituting the expansion (S.4) into Eq. (S.1) and using the multi–scale representation of the time derivative, we obtain

$$\begin{aligned}
&\varepsilon\left(\frac{\partial \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_1} + \frac{\partial \varphi_j^{(1)}\left(t_0, t_1, t_2, \ldots\right)}{\partial t_0}\right) + \varepsilon^2\left(\frac{\partial \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_2} + \frac{\partial \varphi_j^{(1)}\left(t_0, t_1, t_2, \ldots\right)}{\partial t_1} + \frac{\partial \varphi_j^{(2)}\left(t_0, t_1, t_2, \ldots\right)}{\partial t_0}\right) = \\
&\varepsilon\omega_j + \varepsilon\zeta_j(t_0, t_1, t_2, \ldots) + \frac{\varepsilon\kappa}{N}\sum_{k=1}^{N} F_{jk}\left(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\right) \\
&- \frac{\varepsilon^2\tau\kappa}{N}\sum_{k=1}^{N} F_{jk}'\left(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\right)\frac{\partial \phi_k\left(t_1, t_2, \ldots\right)}{\partial t_1} \\
&+ \frac{\varepsilon^2\kappa}{N}\sum_{k=1}^{N} F_{jk}'\Big(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\Big)\Big(\varphi_k^{(1)}\left(t_0 - \tau, t_1, t_2, \ldots\right) - \varphi_j^{(1)}\left(t_0, t_1, t_2, \ldots\right)\Big) + o\left(\varepsilon^2\right),
\end{aligned}$$

where we have grouped terms according to powers of ε. The contributions on the left-hand side arise from the multi-scale expansion of the time derivative and the asymptotic series for $\theta_j(t)$, while the right-hand side results from the scaled heterogeneity, ω_j and $\zeta_j(t)$, the scaled coupling constant κ, and the Taylor expansion of the coupling functions F_{jk} around their arguments $\phi_k - \phi_j - \varpi\tau$.

This substitution produces a hierarchy of equations, each corresponding to a particular power of ε. The solvability condition at each order (specifically, the removal of secular terms that would otherwise lead to unbounded growth in the fast time variable t_0) determines the evolution equations for the slow phase variables ϕ_j. This procedure is analogous to the averaging method: secular terms encode resonant forcing on the slow manifold, and their elimination yields the correct slow dynamics governing long-term behavior.

In what follows, we analyze this hierarchy order by order, extracting the dynamical equations for the slow modulation of phases that ultimately produce the first- and second-order reduced models.

Order ε^1: Leading Slow Dynamics and First Fast Correction

For the Kuramoto–Daido model (S.1), the multiple–time–scale expansion yields, at order ε^1, an equation determining the first fast correction $\varphi_j^{(1)}(t_0, t_1, t_2, \ldots)$. Collecting all terms proportional to ε gives

$$\frac{\partial \varphi_j^{(1)}(t_0, t_1, t_2, \ldots)}{\partial t_0} = \zeta_j(t_0, t_1, t_2, \ldots) - \bar{\zeta}_j(t_1, t_2, \ldots), \quad \bar{\zeta}_j(t_1, t_2, \ldots) = \frac{\varpi}{2\pi} \int\limits_{t-\pi/\varpi}^{t+\pi/\varpi} \zeta_j(\varsigma)\, d\varsigma, \tag{S.5}$$

Here, $\bar{\zeta}_j$ is the average of the external forcing over one fast rotation period $2\pi/\varpi$, so the right–hand side represents only the oscillatory (zero-mean) part of ζ_j. Enforcing this equation ensures that no secular terms appear in $\varphi_j^{(1)}$, preventing unbounded growth in the fast time variable t_0.

The solvability condition at this order then yields the evolution equation for the slow phase $\phi_j(t_1, t_2, \ldots)$ on the t_1 time scale:

$$\frac{\partial \phi_j(t_1, t_2, \ldots)}{\partial t_1} = \omega_j + \bar{\zeta}_j(t_1, t_2, \ldots) + \frac{\kappa}{N} \sum_{k=1}^{N} F_{jk}\big(\phi_k(t_1, t_2, \ldots) - \phi_j(t_1, t_2, \ldots) - \varpi\tau\big). \tag{S.6}$$

This is the first-order reduced Kuramoto–Daido equation: the delay appears only as an effective phase shift $\varpi\tau$, and the natural frequencies are modified by the averaged external perturbations $\bar{\zeta}_j$.

Remark (First-order reduction in the original time variable). If one stops at order ε and rewrites the dynamics using the original time derivative d/dt from Eq. (S.2), then the reduced system takes the standard Kuramoto–Daido form

$$\frac{d\phi_j}{dt} = \bar{\eta}_j(t) + \frac{\varkappa}{N} \sum_{k=1}^{N} F_{jk}(\phi_k - \phi_j - \varpi\tau). \tag{S.7}$$

where $\bar{\eta}_j(t) = \varepsilon\omega_j + \varepsilon\bar{\zeta}_j(t)$. Thus, $\bar{\eta}_j(t)$ represents the combined effect of the small intrinsic frequency deviation of oscillator j from ϖ and the averaged external forcing acting on it. Terms of order $o(\varepsilon)$ are consistently omitted.

Order ε^2: Second Fast Correction and Higher-Order Slow Dynamics

At order ε^2, collecting all terms proportional to ε^2 yields

$$\frac{\partial \varphi_j^{(2)}(t_0, t_1, t_2, \ldots)}{\partial t_0} = -\frac{\partial \varphi_j^{(1)}(t_0, t_1, t_2, \ldots)}{\partial t_1} + \\ + \frac{\kappa}{N} \sum_{k=1}^{N} F'_{jk}\big(\phi_k(t_1, t_2, \ldots) - \phi_j(t_1, t_2, \ldots) - \varpi\tau\big)\Big(\varphi_k^{(1)}(t_0 - \tau, t_1, t_2, \ldots) - \varphi_j^{(1)}(t_0, t_1, t_2, \ldots)\Big). \tag{S.8}$$

which defines the second fast correction $\varphi_j^{(2)}$ while ensuring the absence of secular terms.

The first term on the right-hand side accounts for the slow modulation of the first-order fast correction $\varphi_j^{(1)}$. Eliminating secular terms at this order gives the next slow-time equation for ϕ_j:

$$\frac{\partial \phi_j(t_1, t_2, \ldots)}{\partial t_2} = -\frac{\tau\kappa}{N} \sum_{k=1}^{N} F'_{jk}\big(\phi_k(t_1, t_2, \ldots) - \phi_j(t_1, t_2, \ldots) - \varpi\tau\big) \frac{\partial \phi_k(t_1, t_2, \ldots)}{\partial t_1}, \tag{S.9}$$

which represents the second-order correction to the slow phase drift induced jointly by delay and coupling.

Explicit form obtained by substituting the first-order slow dynamics. Using Eq. (S.6) inside Eq. (S.9) yields

$$\frac{\partial \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_2} = -\frac{\tau\kappa}{N}\sum_{k=1}^{N}\left(\omega_k + \bar{\zeta}_k\left(t_1, t_2, \ldots\right)\right) F'_{jk}\big(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\big) - \\ -\frac{\tau\kappa^2}{N^2}\sum_{k=1}^{N}\sum_{\ell=1}^{N} F'_{jk}\big(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\big) F_{k\ell}\big(\phi_\ell\left(t_1, t_2, \ldots\right) - \phi_k\left(t_1, t_2, \ldots\right) - \varpi\tau\big). \tag{S.10}$$

Returning to the original time variable. The total time derivatives of ϕ_j are, up to $o(\varepsilon^2)$,

$$\frac{d\phi_j}{dt} = \varepsilon\frac{\partial \phi_j}{\partial t_1} + \varepsilon^2\frac{\partial \phi_j}{\partial t_2} + o(\varepsilon^2), \qquad \frac{d^2\phi_j}{dt^2} = \varepsilon^2\frac{\partial^2 \phi_j}{\partial t_1^2} + o(\varepsilon^2), \tag{S.11}$$

and substituting Eqs. (S.6) and (S.10) into (S.11) yields a closed second-order reduced model.

Carrying out this substitution produces, up to $o(\varepsilon^2)$,

$$\frac{d\phi_j}{dt} = \bar{\eta}_j(t) + \frac{\varkappa}{N}\sum_{k=1}^{N} F_{jk}\left(\phi_k - \phi_j - \varpi\tau\right) \\ -\frac{\tau\varkappa}{N}\sum_{k=1}^{N}\bar{\eta}_k(t) F'_{jk}\big(\phi_k - \phi_j - \varpi\tau\big) - \frac{\tau\varkappa^2}{N^2}\sum_{k=1}^{N}\sum_{\ell=1}^{N} F'_{j\ell}\big(\phi_\ell - \phi_j - \varpi\tau\big) F_{\ell k}\big(\phi_k - \phi_\ell - \varpi\tau\big). \tag{S.12}$$

In the last term, the dummy indices k and ℓ were interchanged for convenience.

This equation is a delay-free second-order phase model incorporating delay-induced inertial and higher-order interaction terms. Although exact, its explicit double-sum structure may be cumbersome for analytical work, especially in heterogeneous or random networks, thereby motivating the more compact formulation given in the main text.

Auxiliary Relations and an Alternative Second-Order Reduced Model with Inertia and Multibody Interaction

To obtain a more compact and analytically convenient second-order reduction, we make use of an auxiliary identity derived from the first-order slow-time equation (S.6). Differentiating Eq. (S.6) with respect to t_1 gives

$$\frac{\partial^2 \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_1^2} = \frac{\partial \bar{\zeta}_j\left(t_1, t_2, \ldots\right)}{\partial t_1} + \frac{\kappa}{N}\sum_{k=1}^{N} F'_{jk}\big(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\big)\frac{\partial \phi_k\left(t_1, t_2, \ldots\right)}{\partial t_1} \\ -\frac{\partial \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_1}\frac{\kappa}{N}\sum_{k=1}^{N} F'_{jk}\big(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\big).$$

Using this identity, Eq. (S.9) can be rewritten as

$$\frac{\partial \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_2} + \tau\frac{\partial^2 \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_1^2} = \tau\frac{\partial \bar{\zeta}_j\left(t_1, t_2, \ldots\right)}{\partial t_1} \\ -\frac{\tau\kappa}{N}\frac{\partial \phi_j\left(t_1, t_2, \ldots\right)}{\partial t_1}\sum_{k=1}^{N} F'_{jk}\big(\phi_k\left(t_1, t_2, \ldots\right) - \phi_j\left(t_1, t_2, \ldots\right) - \varpi\tau\big), \tag{S.13}$$

which makes explicit the coupling between the slow second derivative of ϕ_j and derivatives of the coupling and averaged external perturbations. This form is particularly convenient for constructing a delay-free reduced model of second-order accuracy.

Returning to the original time variable. Substituting relations (S.11) into Eq. (S.13), and replacing $\partial\phi_j/\partial t_1$ via Eq. (S.6), yields a closed second-order model for the slow phase ϕ_j:

$$\tau\frac{d^2\phi_j}{dt^2} + \frac{d\phi_j}{dt} = \varepsilon\left(\omega_j + \bar{\zeta}_j(t) + \frac{\kappa}{N}\sum_{k=1}^{N} F_{jk}\left(\phi_k - \phi_j - \varpi\tau\right)\right)\left(1 - \frac{\varepsilon\tau\kappa}{N}\sum_{k=1}^{N} F'_{jk}(\phi_k - \phi_j - \varpi\tau)\right) + \varepsilon\tau\frac{d\bar{\zeta}_j}{dt} \tag{S.14}$$

with all terms $o(\varepsilon^2)$ omitted. This equation represents the dynamics of the slow phase $\phi_j\,(t_1, t_2, \ldots)$ up to second order in ε. Notably, Eq. (S.14) is a second-order differential equation, where, along with the phase shift $\varpi\tau$, which is incorporated into the pairwise interaction function, the second derivative term, scaled by τ, arises due to the time delay in the original model. The right-hand side of Eq. (S.14) effectively represents the driving force for the slow phase evolution, modulated by a factor that depends on the derivative of the coupling function, reflecting the influence of the delay on the effective interaction strength.

Substitution of the compact notation for averaged disorder. Using $\bar{\eta}_j(t) = \varepsilon\omega_j + \varepsilon\bar{\zeta}_j(t)$ and $\varkappa = \varepsilon\kappa$, Eq. (S.14) becomes

$$\tau\frac{d^2\phi_j}{dt^2} + \frac{d\phi_j}{dt} = \left(\bar{\eta}_j(t) + \frac{\varkappa}{N}\sum_{k=1}^{N} F_{jk}\left(\phi_k - \phi_j - \varpi\tau\right)\right)\left(1 - \frac{\tau\varkappa}{N}\sum_{k=1}^{N} F'_{jk}(\phi_k - \phi_j - \varpi\tau)\right) + \tau\frac{d\bar{\eta}_j(t)}{dt}. \tag{S.15}$$

The quantity $\bar{\eta}_j(t)$ combines intrinsic frequency heterogeneity and the averaged external perturbation acting on oscillator j.

Equation (S.15) is precisely the second-order reduced model presented as Eq. (4) in the main text. It encapsulates the leading dynamical consequences of finite delay in the weak-coupling regime: an emergent inertial term, delay-induced renormalization of the effective coupling, and triadic interaction terms. This delay-free second-order phase description retains the essential structure of the underlying Kuramoto–Daido dynamics while offering a substantially more tractable framework, both analytically and numerically, than the original time-delayed system (S.1). As demonstrated in the main text, this reduced model accurately predicts complex collective dynamics and provides an efficient tool for studying high-dimensional patterns in time-delayed oscillator networks.