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Multistable randomly switching oscillators: The odds of meeting a ghost

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Abstract. We consider oscillators whose parameters randomly switch between two values at equal time intervals. If random switching is fast compared to the oscillator's intrinsic time scale, one expects the switching system to follow the averaged system, obtained by replacing the random variables with their mean. The averaged system is multistable and one of its attractors is not shared by the switching system and acts as a ghost attractor for the switching system. Starting from the attraction basin of the averaged system's ghost attractor, the trajectory of the switching system can converge near the ghost attractor with high probability or may escape to another attractor with low probability. Applying our recent general results on convergent properties of randomly switching dynamical systems [1,2], we derive explicit bounds that connect these probabilities, the switching frequency, and the chosen initial conditions.

1 Introduction

Stochastically driven dynamical systems have received a great deal of attention in the physical and mathematical literature [1–25]. Stochasticity can be brought into a system by driving noise and is capable of inducing various dynamical phenomena, including stochastic synchronization [3–6] and stochastic resonance [7–10].

Examples of stochastic dynamical systems also include oscillators and networks whose internal parameters or connections randomly change over time. The study of the interplay between the time-evolving structure and the overall dynamics of a

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system or a network is currently a hot research topic because of a variety of emerging applications [11–19].

In many realistic biological and man-made networks, collections of subsystems that are organized into a network interact only sporadically via fast on-off connections. Examples include stochastically switched engineering networks and circuits, such as power converters [2,20] and packet switched networks such as the Internet. In ecology, these are interconnected ecological metapopulations, under the realistic assumption that the migration among the patches is sporadic and due to rare and short term meteorological conditions. "Blinking" networks that were introduced in [12], are particularly relevant models for such sporadically interacting systems. Similar to the blinking of the eye, blinking connections rapidly switch on and off and switching occurs randomly and independently for different time intervals. The switching is assumed to be fast, compared to intrinsic time scale of the oscillators composing the network. Different types of synchronization in blinking networks were studied in [12,14,15].

Very recently, we developed a general, rigorous theory of stochastically blinking dynamical systems and networks [1,2]. More specifically, we considered a class of blinking dynamical systems whose parameters are switched within a discrete set of values at equal time intervals. Such blinking systems have two characteristic times, namely the characteristic time of the individual dynamical system and the characteristic time scale of the stochastic process. If switching is fast compared to the oscillator's intrinsic time scale, it is natural to expect the switching system to follow the averaged system, where the dynamical law is given by the expectation of the stochastic variables. However, this is not always the case, especially when the averaged system is multistable and its attractors are not invariant under the switching system. The attractors act as ghost attractors for the switching system; the trajectory of the switching system can only reach a neighborhood of the ghost attractor rapidly and remain close most of the time with high probability when switching is fast. In a multistable system, the trajectory may escape to another ghost attractor with low probability. In [1,2], we derived general bounds on such probabilities and their dependence on the switching period and the dynamical system's intrinsic parameters.

In this paper, we consider a multistable switching oscillator as an example of a blinking system. The oscillator randomly and rapidly switches its damping coefficient between two values, yielding an averaged system with two attractors: a stable equilibrium and a stable limit cycle, being a ghost attractor for the switching oscillator. We apply our general theory [1,2] to derive bounds on the probability that the trajectory of the switching oscillator converges to a neighborhood of the ghost attractor and on the remainder of the time interval it may stay in this neighborhood. We construct an appropriate Lyapunov function and show how to estimate its various bounds and decreasing rate to describe the probabilistic convergence to the ghost attractor. We also analyze the switching oscillator's system numerically within and beyond the fast switching limit.

The layout of this paper is as follows. First, in Sect. 2, we describe the randomly switching oscillator and the corresponding averaged system. Then, in Sect. 3, we construct a Lyapunov function for the switching system and derive the preliminary results, necessary for formulating the main theorem. In Sect. 4, we derive the main analytical result of the paper (see Theorem 1) and discuss its implications for the convergence properties of the switching oscillator. A brief discussion of the obtained results is given in the conclusions section. Finally, the appendix contains the calculations of upper bounds for first and second derivatives of the Lyapunov function, used in Theorem 1.

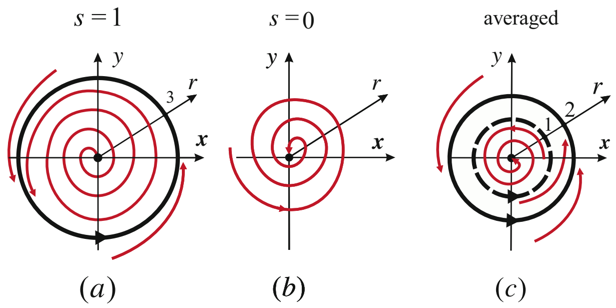

Fig. 1. The oscillator randomly switches between two systems: the one with closed switch $s = 1$ (a) and the one with open switch $s = 0$ (b). Its averaged system is multistable and has a stable equilibrium at the origin and a stable limit cycle at $r = 2$, separated by an unstable limit cycle at $r = 1$ (c). Note that the stable limit cycle of the averaged system is not an invariant set of the two systems with $s = 1$ and $s = 0$.

2 Multistable switching oscillator: The model

We consider the following randomly switching (blinking) system

$$\begin{aligned} \dot{x} &= -y - \tfrac{\lambda}{2}\left[\rho^2 - 2(2 - s(t))\rho + (7 - 10s(t))\right] x \\ \dot{y} &= x - \tfrac{\lambda}{2}\left[\rho^2 - 2(2 - s(t))\rho + (7 - 10s(t))\right] y, \quad \text{where } \rho = (x^2 + y^2)/2, \end{aligned} \tag{1}$$

$\lambda > 0$ is a damping parameter, and $s : [0, \infty) \to \{0, 1\}$ is a binary switching function. To define $s(t)$, we divide the time axis into intervals of length τ and let $s(t)$ take the value 1 with probability $p = 1/2$ and the value 0 with probability $q = 1 - p = 1/2$ in the time interval $t \in [(k-1)\,\tau,\ k\tau)$. The binary switching function $s(t)$ can be viewed as a switch in system (1); the switch is closed when $s_k = 1$ and open when $s_k = 0$. The sequence of binary vectors s_k, $k = 1, 2, ...$ is called the switching sequence as each component s_k switches on ($s_k = 1$) or off ($s_k = 0$) during the k-th time interval. These switching random variables are independent for different time intervals and identically distributed. In simple words, our stochastic model is as follows. During each time interval of length τ, the switch is closed with probability $p = 1/2$, independently of whether or not it has been closed during the previous time interval.

System (1) can be rewritten in polar coordinates with $\rho = (x^2 + y^2)/2$ and $\Theta = \arctan(y/x)$ as follows

$$\dot{\rho} = F(\rho, s(t)) = -\lambda\rho\left[\rho^2 - 2(2 - s(t))\rho + (7 - 10s(t))\right]; \quad \dot{\Theta} = 1. \tag{2}$$

When the switch is closed ($s = 1$), system (2) takes the form

$$\dot{\rho} = -\lambda\rho\left[\rho^2 - 2\rho - 3\right] = -\lambda\rho(\rho + 1)(\rho - 3); \quad \dot{\Theta} = 1. \tag{3}$$

System (3) has a unique stable limit cycle at $\rho = 3$, encircling an unstable fixed point at the origin (see Fig. 1a).

When the switch is open ($s = 0$), system (2) transforms into the following equations

$$\dot{\rho} = -\lambda\rho\left[\rho^2 - 4\rho + 7\right]; \quad \dot{\Theta} = 1. \tag{4}$$

System (4) has a unique globally stable fixed point at the origin (see Fig. 1b). Thus, the blinking system (1) randomly switches between system (3) with a stable limit cycle and system (4) with the globally stable origin.

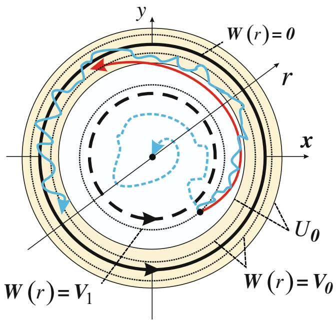

Fig. 2. Illustration of Theorem 1. The trajectory of the averaged system (regular red line) starts from an initial condition, corresponding to a level $W_\Phi(r) = V_1$ and converges to the stable limit cycle at which $W_\Phi(r) = 0$. This limit cycle acts as a ghost attractor for the switching system whose trajectory (solid irregular blue line) reaches a small neighborhood U_0 of the ghost attractor in time $T_{attraction}$ with probability $P^{direct}_{attraction}$. The trajectory oscillates around the ghost attractor and may eventually, after some time T_{remain}, diverge from the ghost attractor; however, this probability can be made arbitrarily small by decreasing the switching period τ (cf. (21)). There also is a non-zero probability P^{direct}_{escape} that the trajectory of the switching system may also escape from the attraction basin C_1 right away and converge to the origin (dashed irregular blue line). Note that this probability decreases exponentially fast when $\tau \to 0$.

If the switching period τ is small with respect to the characteristic times of systems (3) and (4), one can expect that the dynamics of the stochastically switching system (1) is close to that of the averaged system where the stochastic variable $s(t)$ is replaced by its mean value $p = 1/2$.

The averaged system associated with the switching system (1) reads as

$$\begin{aligned} \dot{\xi} &= -\eta - \tfrac{\lambda}{2}\left[r^2 - 3r + 2\right]\xi \\ \dot{\eta} &= \xi - \tfrac{\lambda}{2}\left[r^2 - 3r + 2\right]\eta, \quad \text{where } r = (\xi^2 + \eta^2)/2. \end{aligned} \tag{5}$$

Written in polar coordinates, it takes the form

$$\dot{r} = \Phi(r) \equiv E(F(\rho, s(t))) = -\lambda r(r-1)(r-2); \quad \dot{\Theta} = 1, \tag{6}$$

where $E(F(\rho, s(t)))$ is the expected value of $F(\rho, s(t))$. The averaged system has a stable limit cycle at $r = 2$ and a stable fixed point at the origin; an unstable limit cycle at $r = 1$ separates their basins of attraction (see Fig. 1c). It is worth noticing that while the trivial equilibrium at the origin is shared by both the switching and averaged systems, the stable limit cycle of the averaged system is not an invariant set of the switching system. Therefore, the trajectory of the switching system cannot converge to the stable limit cycle of the averaged system, it can only reach a neighborhood of the stable limit cycle and remain close most of the time with high probability when switching is fast (this statement will be made more precise later in Sect. 4). In this case, the stable limit cycle of the averaged system acts as a *ghost* attractor for the switching system (see Figs. 2 and 3).

The averaged system is *multistable*, therefore the main question in this study is whether or not the trajectory of the switching system will converge near the same (ghost) attractor as the averaged system when starting from the same initial state. In the following, we will derive explicit bounds that relate the probability of convergence near the ghost limit cycle, the switching period, and the choice of initial conditions to

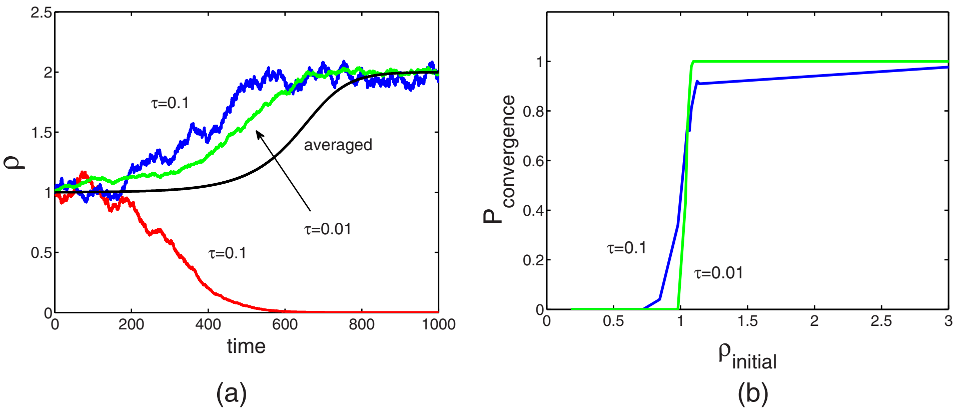

Fig. 3. (a) Trajectories of the averaged system (black smooth curve) and stochastic system with switching times $\tau = 0.1$ (blue and red) and $\tau = 0.01$ (green (light gray)). The trajectories start from the same initial condition $\rho = 1 + \delta$ with $\delta = 0.01$ in the attraction basin of the stable (ghost) limit cycle. For the given stochastic sequence the switching system follows the averaged system for $\tau = 0.01$ and oscillates about the ghost limit cycle. For slower switching with $\tau = 0.1$ (blue and red), two different stochastic sequences cause the trajectory of the switching system to converge near the ghost limit cycle (blue) or to escape to the wrong attractor (red). Note that the curves for $\tau = 0.1$ are more jagged. (b) Probability of converging to a neighborhood of the ghost limit cycle $\rho = 2$, starting from initial conditions with radius $\rho_{initial}$ for $\tau = 0.1$ (blue (dark gray)) and $\tau = 0.01$ (green (light gray)). Notice a sharp transition around $\rho = 1$, corresponding to the unstable limit cycle, separating the attraction basins of the ghost limit cycle and origin. As the analytical bounds (17), (18) suggest, decreasing τ significantly increases the odds of converging near the ghost. Probability calculations are based on 100 trials. The damping parameter equals $\lambda = 0.01$.

each other. In simple words, we will analyze the odds of converging to a neighborhood of the ghost attractor and link them to the switching period. Before formulating the main result regarding the switching system, we need to analyze the convergent properties of the averaged system.

3 Preliminary analysis: The Lyapunov function and its bounds

To apply the general theory developed in [1,2], we need to introduce and estimate the following functions and constants. To facilitate cross-paper reading, we shall use the same notation as in [1,2]

We first estimate the sizes of the compact absorbing (attracting) domain R for both the switching and averaged systems. Clearly, these domains are $0 \leq \rho \leq 3$ and $0 \leq r \leq 2$, respectively, such that the ultimate bound for both the systems is $R : 0 \leq \rho \leq 3$.

The averaged system (6) has a Lyapunov function that can be constructed as an integral of the system's (6) nonlinearity:

$$W_\Phi(r) = \int_0^r (q(q-1)(q-2))dq = r^2(r-2)^2/4. \tag{7}$$

The graph of Lyapunov function $W(r)$ is a forth-order parabola with two minima, also being the r-intercepts, $W(0) = 0$ and $W(2) = 0$, corresponding to the stable origin and the stable limit cycle, respectively. Its local maximum is at $r = 1$, corresponding to the unstable limit cycle. The absolute maximum of $W(r)$ in the absorbing domain $R : 0 \le r \le 3$ is reached at the endpoint $r = 3$.

The derivative of $W_\Phi(r)$ along the trajectories of the averaged system (6) becomes

$$\dot{W}_\Phi = -\lambda r^2 (r-1)^2 (r-2)^2. \tag{8}$$

Following the notations used in [2], this derivative represents $D_\Phi W$ in [2] such that $\dot{W}_\Phi \equiv D_\Phi W$. In what follows, we will use $\dot{W}_\Phi$ to calculate various quantities in Theorem 1.

Similarly, we introduce a Lyapunov function for the switching system (1)–(2)

$$W_F(\rho) = \rho^2(\rho-2)^2/4. \tag{9}$$

Upper bounds for the first and second derivatives of Lyapunov functions W_Φ and W_F in the absorbing domain $R : 0 \le \rho \le 3$ can be estimated as follows (the detailed calculations are given in the appendix):

$$\begin{aligned} B_{W\Phi} &= \max_{0\le r\le 3} \left|\dot{W}\right| = 36\lambda \\ LB_{W\Phi} &= \max_{0\le r\le 3} \left|\ddot{W}\right| = 792\lambda^2 \\ B_{WF} &= \max_{s\in\{0,1\}} \max_{0\le r\le 3} \left|\dot{W}_F\right| = 72\lambda \\ LB_{WF} &= \max_{s\in\{0,1\}} \max_{0\le r\le 3} \left|\ddot{W}_F\right| = 33264\lambda^2. \end{aligned} \tag{10}$$

4 The main result: The odds of converging near the ghost attractor

Let a trajectory of the averaged system (6) start from the initial condition $r(0) = 1+\delta$, where $0 < \delta < 1/2$ is a constant. This trajectory converges to the stable limit cycle of the averaged system at $r = 2$. Let $V_1 = (1-\varepsilon_1)/4$ be a level of the Lyapunov function, corresponding to $r(0) = 1+\delta$ (see Fig. 2). The constant ε_1 satisfies $W_\Phi(1+\delta) = (1-\varepsilon_1)/4$.

Let C_1 be a connected component of the level set $\{r \,|\, W_\Phi(r) \le V_1\}$. Note that C_1 is an annulus that contains the stable limit cycle of the averaged system and is bounded from below by $r = 1+\delta$ and from above by $r = 1+\sqrt{2}-\delta''$, where constant δ'' is chosen to satisfy $W_\Phi(1+\sqrt{2}-\delta'') = (1-\varepsilon_1)/4$.

Choose a neighborhood of the stable limit cycle of the averaged system $2-\delta < r < 2+\delta'$. Constants δ and δ' are chosen such that the two bounds $2-\delta$ and $2+\delta'$ satisfy the same level of Lyapunov function $V_0 = \varepsilon_0/4$, where ε_0 is a constant that depends on δ. This neighborhood $2-\delta < r < 2+\delta'$ will be related to a region around the ghost limit cycle in the switching system. Its trajectory should reach this neighborhood and stay inside with high probability (see Theorem 1).

Before formulating Theorem 1, we shall estimate the following minimal convergence speed of Lyapunov function W_Φ in C_1:

$$\gamma = \min_{r\in C_1,\; V_0\le W_\Phi\le V_1} |\dot{W}_\Phi|. \tag{11}$$

Simple analysis shows that this minimum in the annulus C_1 is reached at $r = 1+\delta$, the endpoint value closest to the unstable limit cycle at $r = 1$, separating the basins of attraction for the origin and the stable limit cycle. Therefore,

$$\gamma = |\dot{W}_\Phi(1+\delta)| = \lambda\delta^2(1-\delta^2)^2. \tag{12}$$

We introduce the following quantities to be used for deriving Theorem 1:

$$\begin{aligned} \Delta t &= \tfrac{\gamma}{2(LB_{WF}+LB_{W\Phi})} = \tfrac{\delta^2(1-\delta^2)^2}{68112\lambda} \\ \alpha &= B_{WF} + B_{W\Phi} = 108\lambda \\ c &= \tfrac{1}{64(LB_{WF}+LB_{W\Phi})B_{WF}^2} = \tfrac{1}{11298963456\lambda^4}, \end{aligned} \tag{13}$$

where constants $B_{W\Phi}, LB_{W\Phi}, B_{WF}, LB_{WF}$, and γ are given in (10) and (12), respectively.

The following theorem is obtained from Theorem 8.3 in [2] by plugging constants (12) and (13) into the conditions of Theorem 8.3 and therefore is given without the proof (the detailed proof of Theorem 8.3 can be found in [2]). Theorem 8.3 describes the behavior of a general blinking system in the most general case where the blinking system has multiple ghost attractors. The main merit of this study is in giving the first example of a randomly switching dynamical system with probabilities of converging to a neighborhood of a ghost attractor, explicitly calculated via the parameters of the switching system (the damping parameter λ and switching period τ) and the initial conditions, expressed via δ.

Theorem 1. *Choose the neighborhood V_0 of the ghost limit cycle in the averaged system such that*

$$V_1 - V_0 = \frac{1-6\delta^2+4\delta^3}{4} \geq \frac{3\delta^4(1-\delta^2)^4}{136224}. \tag{14}$$

Assume that the switching period τ is sufficiently small such that

$$2e\cdot\left(\frac{216\lambda}{\lambda\delta^2(1-\delta^2)^2}+1\right)\exp\left(-\frac{\delta^6(1-\delta^2)^6}{11298963456\lambda}\cdot\frac{1}{\tau}\right) \leq 1-\sqrt{\frac{e}{3}}, \tag{15}$$

where $e = 2.71828...$ is the universal constant.

Consider the two open regions

$$\begin{aligned} U_0 &= \left\{\rho \,\middle|\, W_F(\rho) < \tfrac{(2-\delta)^2\delta^2}{4} + \tfrac{\delta^4(1-\delta^2)^4}{68112}\right\}, \\ U_\infty &= \left\{\rho \,\middle|\, W_F(\rho) > \tfrac{(1-\delta^2)^2}{4} + \tfrac{\delta^4(1-\delta^2)^4}{68112}\right\}. \end{aligned} \tag{16}$$

Then the following inequalities hold:

1. *The probability P_{escape}^{direct} that the solution $\rho(t)$ of the switching system (1)–(2) reaches U_∞ before reaching U_0 (see Fig. 2) is bounded by*

$$P_{escape}^{direct} \leq \frac{419904}{\delta^4(1-\delta^2)^4}\exp\left(-\frac{\delta^6(1-\delta^2)^6}{11298963456\lambda}\cdot\frac{1}{\tau}\right). \tag{17}$$

Conversely, the probability that the solution $\rho(t)$ of the switching system reaches U_0 before reaching U_∞ is at least

$$P_{attraction}^{direct} \geq 1 - \frac{839808}{\delta^4(1-\delta^2)^4}\exp\left(-\frac{\delta^6(1-\delta^2)^6}{11298963456\lambda}\cdot\frac{1}{\tau}\right). \tag{18}$$

2. *Let $T_{attraction}$ be the time for the solution of the switching system to enter U_0 through its boundary and T_{remain} be the time it stays close to the ghost attractor and remains in*

$$\bar{U}_{0+} = \left\{\rho \,\middle|\, W_F(\rho) \leq \frac{(2-\delta)^2\delta^2}{4} + \left(\frac{3\delta^2(1-\delta^2)^2}{2}+108\right)\frac{\delta^2(1-\delta^2)^2}{68112}\right\} \tag{19}$$

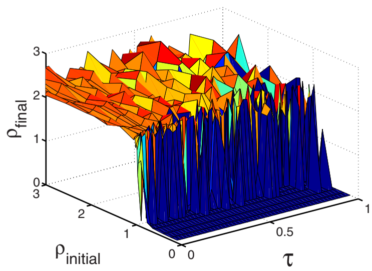

Fig. 4. The role of different initial conditions and stochastic frequencies. For each fixed τ, we vary $\rho_{initial}$ and observe where ρ_{final} ends, namely whether or not the switching system converges near the ghost or to the origin. Orange (light gray) corresponds to convergence near the ghost, while blue (dark gray) is convergence to the origin. The intermediate colors (yellow and green) indicate that the system is still oscillating in between the attractors (with yellow closer to the ghost and green closer to the origin). While this is only a single realization for each set of initial conditions, one notices the system becomes more unpredictable as τ increases. For larger values of τ, the system can converge to the origin despite initial conditions near the ghost (note blue (dark) "peaks" and "valleys" at $\rho_{initial} \approx 2$ and $\tau \approx 0.75$). The damping parameter equals $\lambda = 0.01$.

after reaching U_0. These times are random variables with the following properties

$$P\left(T_{attraction} \le \frac{1-6\delta^2+4\delta^3}{2\lambda\delta^2(1-\delta^2)^2}\right) > 1 - \frac{68112(1-6\delta^2+4\delta^3)}{\delta^4(1-\delta^2)^4} \times \times \exp\left(-\frac{\delta^6(1-\delta^2)^6}{11298963456\lambda}\cdot\frac{1}{\tau}\right) \tag{20}$$

and

$$P\left(T_{remain} > T\right) > 1 - T\frac{136224\lambda}{\delta^2(1-\delta^2)^2}\exp\left(-\frac{\delta^6(1-\delta^2)^6}{11298963456\lambda}\cdot\frac{1}{\tau}\right)\cdot \tag{21}$$

Remark 1. The switching period τ appears only in the denominator of the exponent in conditions (15), (17), (18). Therefore, the probability P^{direct}_{escape} of escaping from the ghost attractor's basin of attraction before reaching the ghost attractor's neighborhood U_0 can be made arbitrarily small by decreasing the switching period τ. Note that if the trajectory of the switching system converges to the "wrong" attractor (the origin), it has to pass first through U_∞. Therefore, P^{direct}_{escape} is an upper bound on the probability that the trajectory escapes from the ghost attractor and converges to the origin.

Remark 2. The bounds for the probabilities of escaping from and converging to the ghost attractor's neighborhood, given in Theorem 1, explicitly depend on the choice of initial conditions $\rho(0) = 1 + \delta$. These bounds indicate that the smaller parameter δ (i.e., the closer the initial condition is to the unstable limit cycle that separates the attraction basins), the higher the probability of escaping to the origin.

Remark 3. Note that probability bounds P^{direct}_{escape} and $P^{direct}_{attraction}$ do not sum to 1, as one would expect. Therefore, the trajectories of the switching system that never reach U_0 or U_∞ get trapped between the ghost attractor and the origin might have positive probability. However, this probability could not be larger than P^{direct}_{escape} and the case that this probability vanishes is compatible with the bounds [2].

Figure 3 presents numerical simulations of the switching system (1)–(2) for different switching periods and initial conditions. When the initial state of the switching system is close to the attraction basin boundary (see Fig. 3a), i.e., for small δ, there always is a small probability P_{escape}^{direct} of escaping to the wrong attractor. This can also be clearly viewed from the bound (17) and the expression for the speed of convergence γ (12), where δ is the distance from the attraction basin boundary, defined by the unstable limit cycle at $r = 1$. The smaller δ and the slower the switching, the bigger this probability is (see Fig. 3b).

In Fig. 4, we examine probability outcomes for different switching periods across a range of different initial conditions $\rho_{initial}$. We find that for larger switching periods τ the switching system can converge to the origin despite the initial conditions chosen in the attraction basin of the ghost attractor, far away from the attraction basin boundary. Though it is important to remember that Fig. 4 is only one realization (one trial from each initial condition) and not an average of numerous trials, it does show that there are certain "windows of opportunity" for which the optimal stochastic sequence/frequency can give the system enough kicks in the wrong direction at the appropriate time and to cause the system to rather consistently converge to the wrong attractor.

5 Conclusions

We have considered a multistable switching oscillator as an example of a stochastically blinking system. The switching oscillator has two attractors with the stable limit cycle being a "ghost". Switching is assumed to be a stochastic process (with identically distributed independent random switching variables), therefore the convergence properties of the switching oscillator have a probabilistic flavor. We applied our general theory [1,2] to prove weak convergence (with probability $P_{attraction}^{direct}$ approaching 1 when the switching period $\tau \to 0$) to a neighborhood of the correct (ghost) attractor. Using the Lyapunov function method, we gave explicit bounds on the probabilities of converging to and escaping from the ghost attractor's neighborhood and switching time. More precisely, we have derived the following result. We choose an initial condition for the switching and averaged systems in the attraction basin of the ghost attractor. For this initial condition, we identify V_1, a level of a Lyapunov function for the averaged system. We also choose a neighborhood U_0 of the ghost attractor of the switching system. We then find a bound for the minimum convergence speed γ of the Lyapunov function in the subregion of the ghost's attraction basin, bounded by V_1 and U_0. Consequently, we prove that the probability that the trajectory of the switching system rapidly reaches U_0 can be made arbitrarily close to 1 by decreasing the switching period. We find the reciprocal of τ to be a critical quantity, effectively controlling the probability of convergence.

We also showed that if switching is not sufficiently fast, the switching system may have "windows of opportunities", in terms of optimal switching periods and sets of initial conditions, within which the trajectory escapes to the wrong attractor against all the odds. Loosely speaking, the existence of the windows of opportunity in randomly switching multistable oscillators may be viewed somewhat similarly to stochastic resonance [7–10]. Although, the two phenomena have completely different dynamical origins; the coincidence of favorable initial conditions and stochastic sequences in the switching oscillator's case and a favorable condition to go over a threshold in a bistable system due to matching the time scales of a periodic driving force and driving noise in the stochastic resonance's case.

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6 Appendix

In this appendix, we give the details for the calculation of upper bounds (10) for the first and second derivatives of Lyapunov functions W_Φ and W_F.

The upper bound for the first derivative of Lyapunov function $\dot{W}_\Phi$ (8) on $R: 0 \leq \rho \leq 3$ falls on the endpoint $r = 3$ and equals

$$B_{W\Phi} = \max_{0\leq r\leq 3} \left|\dot{W}\right| = \max_{0\leq r\leq 3} \lambda r^2(r-1)^2(r-2)^2 = 36\lambda. \tag{22}$$

The second derivative $\ddot{W}(r) \equiv D^2_\Phi W$ (see the notation in [2]) reads as

$$\ddot{W}_\Phi = 2\lambda^2(3r^2 - 6r + 2)r^2(r-1)^2(r-2)^2 = 2\lambda(3r^2 - 6r + 2)\left|\dot{W}_\Phi\right|.$$

Hence, the absolute value of its bound on R, also at $r = 3$, is

$$LB_{W\Phi} = \max_{0\leq r\leq 3} \left|\ddot{W}_\Phi\right| < 2\lambda \cdot 11 \cdot \max_{0\leq r\leq 3} \left|\dot{W}_\Phi\right| = 792\lambda^2. \tag{23}$$

Similarly, we calculate the first derivative of Lyapunov function W_F (9) for the switching system (1)–(2)

$$\dot{W}_F = \operatorname{grad} W \cdot F = W_\rho\,(\rho_x \dot{x} + \rho_y \dot{y}) = -\lambda\rho^2(\rho-1)(\rho-2)(\rho^2 - 2(2-s)\rho + 7 - 10s).$$

Note that $\dot{W}_F$ depends on a particular sequence of $s(t) = 0$ and $s(t) = 1$, generated by the i.i.d. stochastic switching. In terms of notation in [2], $\dot{W}_F \equiv D_F W(x, s)$.

The absolute value of its bound on R can be estimated by

$$\begin{aligned} B_{WF} = \max_{s\in\{0,1\}} \max_{0\leq r\leq 3} \left|\dot{W}_F\right| &< \lambda \max_{0\leq\rho\leq 3} |\rho(\rho-1)(\rho-2)| \times \\ \times \max_{s\in\{0,1\}} \max_{0\leq r\leq 3} &\left|.\rho(\rho^2 - 2(2-s)\rho + 7 - 10s)\right|. \end{aligned} \tag{24}$$

The maximum of the first co-factor in (24), $|\rho(\rho-1)(\rho-2)|$, on R is reached at the endpoint $r = 3$ and equals 6. The second co-factor $\left|\rho(\rho^2 - 2(2-s)\rho + 7 - 10s)\right|$ depends on s which may take on values 0 and 1. When the switch is on and $s = 1$, it becomes $\left|\rho(\rho^2 - 2\rho - 3)\right|$, whereas when the switch is off and $s = 0$, it takes the form $\left|\rho(\rho^2 - 4\rho + 7)\right|$. Simple analysis shows that the maximum of the latter function for $s = 0$ on R is larger than that of the former function for $s = 0$. It is reached at $\rho = 3$ and therefore equals 12. Hence, the upper bound for estimate (24) becomes

$$B_{WF} = \lambda \cdot 6 \cdot 12 = 72\lambda. \tag{25}$$

Finally, we calculate the second derivative $\ddot{W}_F(\rho) \equiv D^2_F W$ (see the notation in [2]):

$$\begin{aligned} \ddot{W}_F = \operatorname{grad} \dot{W}_F \cdot F = -\lambda\rho^2(\rho-1)(\rho-2)(\rho^2 - 2(2-s)\rho + 7 - 10s)\times \\ \times\left(-\lambda\rho(\rho^2 - 2(2-\tilde{s})\rho + 7 - 10\tilde{s})\right), \end{aligned} \tag{26}$$

where s and $\tilde{s}$ are taken from two different stochastic sequences, corresponding to $\dot{W}_F$ and the switching system (1)-(2) (via F). Depending on the combination of random values $s = 0, 1$ and $\tilde{s} = 0, 1$, function (26) has four different expressions for $(s = 1, \tilde{s} = 1)$, $(s = 1, \tilde{s} = 0)$, $(s = 0, \tilde{s} = 1)$, and $(s = 0, \tilde{s} = 0)$. Our analysis indicates that the maximum value of $\ddot{W}_F$ on R is produced by the combination $s = 0, \tilde{s} = 0$, generating the following function:

$$\max_{s,\tilde{s}\in\{0,1\}} \ddot{W}_F = \lambda^2(6\rho^5 - 35\rho^4 + 84\rho^3 - 87\rho^2 + 28\rho)(\rho(\rho^2 - 4\rho + 7). \tag{27}$$

The first co-factor polynomial $P_1 = 6\rho^5 - 35\rho^4 + 84\rho^3 - 87\rho^2 + 28\rho$ can be bounded from above as follows: $\max\limits_{0\le r\le 3}|P_1| < \max\limits_{0\le r\le 3}|6\rho^5 - 35\rho^4| + \max\limits_{0\le r\le 3}|84\rho^3 - 87\rho^2 + 28\rho|$. The latter two maximum values are reached at the endpoint $\rho = 3$ so that $\max\limits_{0\le r\le 3}|P_1| < 1377 + 1395 = 2772$. The second co-factor polynomial $P_2 = (\rho(\rho^2 - 4\rho + 7)$ in (27) was estimated above (cf. (25)) and is bounded by 12. Thus,

$$LB_{WF} = \max_{s\in\{0,1\}} \max_{0\le r\le 3} \left|\ddot{W}_F\right| = \lambda^2 \max_{0\le r\le 3} |P_1| \max_{0\le r\le 3} |P_2| = 33264\lambda^2. \tag{28}$$

Collecting (22),(23),(25), and (28) yields bounds (10).

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