Breathing and switching cyclops states in Kuramoto networks with higher-mode coupling

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Cyclops states are intriguing cluster patterns observed in oscillator networks, including neuronal ensembles. The concept of cyclops states formed by two distinct, coherent clusters and a solitary oscillator was introduced by Munyayev et al. [Phys. Rev. Lett. 130, 107021 (2023)], where we explored the surprising prevalence of such states in repulsive Kuramoto networks of rotators with higher-mode harmonics in the coupling. This paper extends our analysis to understand the mechanisms responsible for destroying the cyclops’ states and inducing dynamical patterns called breathing and switching cyclops states. We first analytically study the existence and stability of cyclops states in the Kuramoto-Sakaguchi networks of two-dimensional oscillators with inertia as a function of the second coupling harmonic. We then describe two bifurcation scenarios that give birth to breathing and switching cyclops states. We demonstrate that these states and their hybrids are prevalent across a wide coupling range and are robust against a relatively large intrinsic frequency detuning. Beyond the Kuramoto networks, breathing and switching cyclops states promise to strongly manifest in other physical and biological networks, including coupled theta neurons.

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I. INTRODUCTION

Phase oscillator networks have emerged as a paradigmatic reduced model for describing emergent cooperative properties of diverse real-world systems, including neuronal networks [1–3], laser arrays [4–6], and power grids [7–9]. The celebrated Kuramoto model of one-dimensional (1D) oscillators [10,11] and its extension to two-dimensional (2D) oscillators with inertia [12] captures the essence of the phase networks and provides a concise framework to explore the richness of their cooperative dynamics [13–19]. These dynamics include full [20–24], partial [25,26], explosive [27–29] and asymmetry-induced synchronization [30,31], chimeras [32–39], solitary states [40–44], clusters [45–48], generalized splay [49], and cyclops states [50]. The cooperative dynamics of Kuramoto networks with attractive coupling have been studied more extensively than their counterparts in repulsive networks. While full synchronization is the simplest and most dominant rhythm in attractive networks, splay [51,52] and generalized and cluster splay states [48,49] are expected to be the most probable patterns in repulsive networks. Yet a complete understanding of rhythmogenesis in repulsive networks is still lacking. Two repulsively coupled oscillators tend to achieve antiphase synchronization; however, predicting an outcome of such interactions in large repulsive networks is often elusive. Notably, the role of repulsive connections can be counterintuitive, especially in networks with mixed attractive and repulsive coupling [53–56]. For example, adding pairwise repulsive inhibition to excitatory networks of bursting neurons can synergistically promote synchronization by facilitating transitions between different types of bursting [55].

In the context of Kuramoto-type networks, the prevailing approach is to model interactions by the first sinusoidal harmonic from a Fourier decomposition of a $2\pi$-periodic coupling function. This simplest choice of the coupling form adequately describes many dynamical features of real-world networks and is analytically tractable. However, higher-order coupling harmonics have been observed to play a significant role in rhythmogenesis in various scenarios. These encompass Kuramoto-type models of neuronal plasticity [57,58], coupled electrochemical oscillators [59], and Josephson junctions [60]. In particular, previous research has demonstrated that augmenting the classical Kuramoto model with higher-mode coupling can result in the emergence of multiple phase-locked states [61,62] and facilitate switching between synchrony clusters [63].

In a recent work [50], we studied rhythmogenesis in repulsive Kuramoto networks of identical 2D phase oscillators with phase-lagged first-mode and higher-mode coupling. We introduced the concept of cyclops states formed by two distinct, coherent clusters and a solitary oscillator reminiscent of the Cyclops’s eye. These cyclops states represent a particular class of three-cluster generalized splay states [49] with the solitary oscillator maintaining constant phase differences with the coherent clusters. We reported a surprising finding that adding the second or third harmonic to the Kuramoto coupling makes the cyclops state global attractors in a wide range of couplings’ repulsion. Beyond Kuramoto networks, we showed that the stabilization of cyclops states by the higher coupling harmonics is also robustly present in theta neurons with adaptive coupling.
This paper extends our previous analysis to reveal higher-mode coupling-induced mechanisms for emerging dynamical patterns termed breathing and switching cyclops states. Toward this goal, we derive analytical conditions on the existence and stability of cyclops states with constant intercluster phases in the presence of the second coupling harmonic. These conditions reveal two bifurcation scenarios for destabilizing the cyclops states. In the first scenario, the cyclops states with constant intercluster phases between its three clusters undergo an Andronov-Hopf bifurcation, preserving their intracircle formations but making the intercluster relative phase differences oscillate periodically. Similarly to breathing three-cluster patterns introduced in [47], we call these breathing cyclops states. These states can evolve into roto-breathers with intercluster phase differences governed by mixed-mode, oscillatory-rotatory phase difference dynamics. In the second bifurcation scenario, the cyclops state with constant intercluster phases loses its structural stability but quickly reforms into a new cyclops state with a reshuffled configuration. This repetitive switching process yields switching cyclops states. These states are similar to blinking chimeras, also characterized by a death-birth process in which the coherent cluster dissolves and is quickly reborn in a new configuration [64].

We show that breathing, roto-breathing, and switching cyclops states are stable in a wide range of the second harmonic coupling strength and phase lag parameter. Remarkably, breathing and roto-breathing cyclops states are dominant states, acting as the system’s global attractors in a large interval of the second harmonic’s phase lag parameter, corresponding to the overall repulsiveness of the combined first- and second-harmonic coupling. We also demonstrate that the cyclops states can robustly emerge in Kuramoto networks of nonidentical oscillators. In [50], we proved that the 2D Kuramoto model with the first and second-harmonic coupling is dynamically equivalent to a network of canonical theta neurons with adaptive coupling. Therefore, we expect breathing and switching cyclops states to manifest strongly in theta-neuron networks, pointing to the broader applicability of our results.

The layout of this paper is as follows. In Sec. II we introduce the oscillator network model and state the problem under consideration, and give formal definitions. In Sec. III we study the existence of cyclops states with constant intercluster phase differences, called stationary cyclops states. We derive an upper bound for the maximum number of stationary cyclops states with distinct intercluster phase differences. In Sec. IV we derive a four-dimensional (4D) system that governs the dynamics of the intercluster phase differences. We study the stability of the fixed point of the 4D system, which corresponds to constant intercluster phase differences. We derive the conditions under which the fixed point undergoes an Andronov-Hopf bifurcation, giving rise to a breathing cyclops state. In Sec. V we analyze the variational equations for the transversal stability of stationary cyclops states that determines the stability of their coherent clusters. In Sec. VI we numerically study breathing and switching cyclops states emerging from stationary cyclops states via two distinct bifurcation routes. We demonstrate the emergence of more complex, hybrid dynamical patterns that combine the properties of both breathing and switching cyclops states. We also study the prevalence of different cyclops states and show that they robustly appear from large sets of randomly chosen initial conditions. In Sec. VII we show the persistence of cyclops states against relatively large intrinsic frequency detuning. Section VIII contains concluding remarks and discussions. The Appendix contains the derivation of the upper bound for the maximum number of stationary cyclops states.

II. THE MODEL AND PROBLEM STATEMENT

We consider the Kuramoto-Sakaguchi network of 2D phase oscillators

\[
\mu \dot{\Theta}_k + \Theta_k = \omega + \sum_{n=1}^{N} \sum_{q=1}^{2} \bar{E}_n \sin[q(\Theta_n - \Theta_k) - \alpha_q],
\]

(1)

where the \( k \)th oscillator’s phase \( \Theta_k \) ranges from \(-\pi\) to \(\pi\), and the second-order Kuramoto-Sakaguchi coupling [65] represents a pairwise interaction function \( H(\Theta_n - \Theta_k) = \sum_{q=1}^{2} E_n \sin[q(\Theta_n - \Theta_k) - \alpha_q] \). The oscillators are assumed to be identical with frequency \(\omega\), inertia \(\mu\), and phase lag parameters \(\alpha_1\) and \(\alpha_2\). We consider the phase lag \(\alpha_1 \in (\pi/2, \pi)\), that makes the first-harmonic coupling repulsive and fix \(\epsilon_1 = 1\) that corresponds to a strong first-harmonic coupling. Throughout the paper, we also choose and fix a relatively strong inertia \(\mu = 1\) that is sufficient to make the dynamics of the 2D system qualitatively distinct from the 1D classical model and enable the emergence of breathing cluster dynamics [46]. We will consider a broader range of \(\alpha_2 \in (-\pi, \pi)\), so that the second harmonic may be pairwise attractive or repulsive. As a result, the overall combined coupling may be repulsive with \(H'(0) < 0\) or attractive with \(H'(0) > 0\). The latter is possible when the second-harmonic coupling \(\alpha_2\) is sufficiently strong to overcome the first-harmonic coupling contribution.

Phase coherence and cluster synchrony in the system (1) can be characterized via the 1th-order complex Kuramoto parameters [63,66]:

\[
R(t) = \frac{1}{N} \sum_{k=1}^{N} e^{i\Theta_k} = r(t) e^{i\psi(t)},
\]

where \(r(t)\) and \(\psi(t)\) define the magnitude and the phase of the 1th moment Kuramoto order parameter \(R(t)\), respectively. The first-order scalar parameter \(r_1 = |R_1|\) characterizes the degree of phase synchrony with \(r_1 = 1\) corresponding to full phase synchrony. Splay states or generalized splay states \(\Theta_k = \omega t + \phi_k, k = 1, \ldots, N\) with constant nonuniform relative phases \(\phi_k \in [-\pi, \pi]\) satisfy the condition \(r_1 = 0\) in the 2D Kuramoto model with the first-harmonic coupling (\(\epsilon_2 = 0\)). The second-order scalar parameter \(r_2 = |R_2|\) determines the degree of cluster synchrony. In the case of the first-harmonic coupling (\(\epsilon_2 = 0\), \(r_2\) controls the stability of generalized splay states so that increasing \(r_2\) enlarges their stability parameter regions [49,50]. It was shown in [50] that generalized splay states with a maximum \(r_2\) are (i) two-cluster symmetric splay states (for even \(N\)) and (ii) three-cluster splay states with the relative phases (for odd \(N\)):

\[
\begin{align*}
\varphi_1 &= \varphi_2 = \cdots = \varphi_{M-1} = \gamma, & \varphi_M &= 0, \\
\varphi_{M+1} &= \cdots = \varphi_{N} = -\gamma,
\end{align*}
\]

(2)
where \( \gamma = \arccos[1/(1-N)] \), \( M = (N+1)/2 \), and the choice of the reference zero phase for \( \phi_M \) is arbitrary. We termed three-cluster splay states (2) cyclops states. Adding the second-harmonic coupling with \( \varepsilon \neq 0 \) breaks their symmetry in \( \gamma \) and makes \( r_1 \) nonzero, albeit small. We demonstrated in [50] that the second- or higher-harmonic coupling can make these asymmetric patterns dominant states. In this paper, we generalize the definition of cyclops states (2) for the system (1) with second-harmonic coupling and odd \( N \) by relaxing the condition \( r_1 = 0 \). As a result, we refer to the following three-cluster state determined by the three-dimensional cluster manifold:

\[
D(3) = \left\{ \begin{array}{l}
\Theta_1(t) = \cdots = \Theta_{M-1}(t) = x + \Omega t, \\
\Theta_M(t) = \Omega t, \\
\Theta_{M+1}(t) = \cdots = \Theta_N(t) = y + \Omega t
\end{array} \right.
\]

as to a stationary cyclops state in which two equal clusters of \( M - 1 \) oscillators rotate with the common frequency \( \Omega \), preserving the stationary phase differences \( x = \gamma_1 \), and \( y = \gamma_2 \) with the \( M \)th solitary oscillator. Note that due to the asymmetry of the relative phases \( x \neq y \), the common rotational frequency \( \Omega \) differs from the oscillators’ intrinsic frequency \( \omega \) and can be derived explicitly from (1) as shown in the next section.

Similarly to (2), the stationary cyclops state defined by (3) can be conveniently represented within the reference frame \( \theta_k = \Theta_k - \Omega t, \ k = 1, \ldots, N \) where the relative constant phase of the \( M \)th oscillator, \( \theta_M \), is chosen at zero so that

\[
\theta_1 = \cdots = \theta_{M-1} = x, \quad \theta_M = 0, \quad \theta_{M+1} = \cdots = \theta_N = y. \tag{4}
\]

Due to the system’s global coupling symmetry and equal cluster sizes \( M - 1 \), the existence of the stationary cyclops states with intercluster phase differences \( x = \gamma_1 \) and \( y = \gamma_2 \) implies the existence of its counterpart with \( x = \gamma_2 \) and \( y = \gamma_1 \). Thus, cyclops states exist in symmetrical pairs. In the following, we will analyze the existence and stability of stationary cyclops states in the system (1) with odd \( N \). We will describe two main scenarios for destabilizing a stationary cyclops state that yield (i) a breathing cyclops state with periodically oscillating \( x(t), y(t) \) so that the intercluster phase differences are bounded as \( |x(t)| < \pi \), \( |y(t)| < \pi \) to produce no phase slips (Fig. 1) and (ii) a switching cyclops state, representing a repetitive death-birth process in which the clusters disintegrate to form a new cyclops state with a new reshuffled cluster configuration and a new solitary node.

We will also study how the breathing and switching cyclops states can (i) further evolve into more complex dynamical patterns, including hybrid switching-breathing states, (ii) become globally stable, and (iii) persist against intrinsic frequency detuning.

### III. POSSIBLE CONSTANT INTERCLUSTER PHASE DIFFERENCES

We seek to find permissible stationary cyclops states as a function of the system’s parameters. To determine the constant phase differences \( x, y \), and the rotational frequency \( \Omega \), we substitute the stationary cyclops state solution (3) into (1) and obtain the system of nonlinear transcendental equations:

\[
\omega - \Omega = -\sum_{q=1}^{N} \varepsilon_q \left( \sin(qx + \alpha_q) + \frac{N-1}{2} \sin(\alpha_q) \right) \\
+ \sin[q(x - y) + \alpha_q] = 0,
\]

\[
\omega - \Omega = -\sum_{q=1}^{N} \varepsilon_q \left( \cos(qy + \alpha_q) - \frac{N-1}{2} \sin(qx - \alpha_q) \right) \\
+ \sin[qy - \alpha_q] = 0,
\]

\[
\omega - \Omega = -\sum_{q=1}^{N} \varepsilon_q \left( \cos(qy + \alpha_q) - \frac{N-1}{2} \sin(qx - \alpha_q) \right) \\
+ \sin[q(y - x) + \alpha_q] = 0. \tag{5}
\]

We subtract the second equation from the first and third equations of (5) to eliminate \( \Omega \) and obtain the system of two equations for finding the unknown constants \( x \) and \( y \):

\[
\frac{N-3}{2} \sum_{q=1}^{N} \varepsilon_q \sin \alpha_q + \sum_{q=1}^{N} \varepsilon_q \sin(qx + \alpha_q) \\
+ \frac{N-1}{2} \left( \sum_{q=1}^{N} \varepsilon_q \sin(qx - \alpha_q) + \sum_{q=1}^{N} \varepsilon_q \sin(qy - \alpha_q) \right) \\
- \sum_{q=1}^{N} \varepsilon_q \sin[q(y - x) - \alpha_q] = 0,
\]

\[
\frac{N-3}{2} \sum_{q=1}^{N} \varepsilon_q \sin \alpha_q + \sum_{q=1}^{N} \varepsilon_q \sin(qy + \alpha_q) \\
+ \frac{N-1}{2} \left( \sum_{q=1}^{N} \varepsilon_q \sin(qx - \alpha_q) + \sum_{q=1}^{N} \varepsilon_q \sin(qy - \alpha_q) \right) \\
- \sum_{q=1}^{N} \varepsilon_q \sin[q(x - y) - \alpha_q] = 0. \tag{6}
\]
In turn, we find the rotational frequency $\Omega$ from the second equation of (5):

$$\Omega = \omega - \frac{1}{N} \sum_{q=1}^{N} \epsilon_q \sin \alpha_q + \frac{N-1}{2N} \left[ \sum_{q=1}^{2} \epsilon_q \sin(qx - \alpha_q) + \sum_{q=1}^{2} \epsilon_q \sin(qy - \alpha_q) \right]$$

(7)

with $x$ and $y$ calculated from (6).

Due to the complexity of system (6), its solution for $x$ and $y$ cannot be found in closed form. Yet we derive an upper bound for the maximum number of stationary cyclops states with distinct $x$ and $y$. To do so, we transform the real-valued system (6) into a system of complex polynomial equations and apply the Bernshtein theorem [67], a practical tool in algebra that bounds the number of nonzero complex solutions by the mixed volume of their Newton polytopes. The details of this analysis are quite technical and are delegated to the Appendix.

This analysis shows that the complex form of system (6) may have up to 17 possible solutions (including some nonphysical) corresponding up to 16 stationary cyclops states with distinct ordered pairs of constant phase differences $x$, $y$. As stationary cyclops states exist in pairs, there may be at most eight combinations of $x$, $y$ (up to the cluster permutation $x \leftrightarrow y$). It is worth noticing that there is a continuum of stationary cyclops states with a given pair $(x, y)$ due to an arbitrary choice of the reference solitary state’s phase $\theta_q$.

Figure 2 displays the number of different stationary cyclops states calculated by solving the complex polynomial equation (A1) using the NSolve function of Wolfram Mathematica. Note that this number equals two for small values of the second-harmonic amplitude $\epsilon_2$. This pair of stationary cyclops states with $x = y_1$ and $y = y_2$ ($x = y_2$ and $y = y_1$) emerges continuously from the symmetrical cyclops state (2) that exists in the system (1) in the absence of the second-harmonic coupling ($\epsilon_2 = 0$). As Figs. 2(a) and 2(b) indicate, increasing $\epsilon_2$ increases the number of coexisting stationary cyclops states and induces richer dynamics. In the following, we will derive general stability conditions for a permissible stationary cyclops state and specify these conditions to the cyclops state (4) with $x = y_1$ and $y = y_2$ from the dashed parameter region in Fig. 2 where no other cyclops states with distinct $x$ and $y$ (up to permutation of clusters $x \leftrightarrow y$) exist.

**IV. STABILITY OF THE INTERCLUSTER PHASE DIFFERENCES**

We seek to obtain the conditions for the stability of the constant intercluster phase differences to small perturbations of $x$ and $y$. The dynamics of the intercluster phase differences are governed by the system

$$\mu \ddot{x} + \dot{x} = \frac{2}{N} \sum_{q=1}^{N} \epsilon_q (\sin \alpha_q - \sin(qx + \alpha_q))$$

$$- \frac{N-1}{2} \left[ \sin(qx - \alpha_q) + \sin(qy - \alpha_q) + \sin \alpha_q + \sin[q(x - y) + \alpha_q] \right].$$

The 4D dynamical system (8) may be viewed loosely as a system of two nonlinearly coupled driven pendulum-like equations with the terms $\sin \alpha_q$ representing constant torques and the sine terms with $x$ and $y$ corresponding to pendulum-like nonlinearities and coupling. The presence of the second-harmonic coupling prevents transforming the system (8) into a more explicit system of two coupled pendula as was achieved for a three-cluster state in [47]. However, the pendulum-like structure of the 4D system (8) points to the possible existence of nontrivial dynamics related to oscillating and even chaotically evolving intercluster phase differences $x(t)$ and $y(t)$.

Fixed points of system (8) correspond to constant intercluster phase differences $x$, $y$ calculated from (6). We aim to study
the local stability of the fixed points and derive bifurcation conditions that induce oscillating phase differences \( x(t), y(t) \). Toward this goal, we consider small deviations \( \delta x(t) \) and \( \delta y(t) \) from a fixed point \( x = \gamma_1, y = \gamma_2 \) corresponding to a stationary cyclops state. So \( x(t) = \gamma_1 + \delta x(t), y(t) = \gamma_2 + \delta y(t) \). We linearize the system (8) in the vicinity of the fixed point state and obtain the following equations that govern the evolution of small deviations \( \delta x(t) \) and \( \delta y(t) \):

\[
\begin{align*}
\mu \delta \ddot{x} + \delta \dot{x} &= -\sum_{q=1}^{2} \frac{\epsilon_q \eta_q}{N} \left[ \cos(q \gamma_1 + \alpha_q) \delta x + \frac{N-1}{2} \left( \cos(q \gamma_2 - \alpha_q) \delta y + \cos(q \sigma + \alpha_q)(\delta x - \delta y) \right) \right], \\
\mu \delta \ddot{y} + \delta \dot{y} &= -\sum_{q=1}^{2} \frac{\epsilon_q \eta_q}{N} \left[ \cos(q \gamma_2 + \alpha_q) \delta y + \frac{N-1}{2} \left( \cos(q \gamma_1 - \alpha_q) \delta x + \cos(q \sigma - \alpha_q)(\delta y - \delta x) \right) \right].
\end{align*}
\]

(9)

where \( \sigma = \gamma_1 - \gamma_2 \).

Following the standard stability approach, we seek solutions \( \delta x(t) = A_1 e^{\mu t}, \delta y(t) = A_2 e^{\mu t} \) and derive a system of two characteristic equations for finding constants \( \lambda, A_1, \) and \( A_2 \):

\[
\begin{align*}
(\mu \lambda^2 + \lambda) A_1 &= -(p_{11} A_1 + p_{12} A_2), \\
(\mu \lambda^2 + \lambda) A_2 &= -(p_{21} A_1 + p_{22} A_2),
\end{align*}
\]

(10)

where

\[
\begin{align*}
p_{11} &= \sum_{q=1}^{2} \frac{\epsilon_q \eta_q}{N} \left[ \frac{N-1}{2} \left( \cos(q \gamma_1 + \alpha_q) + \cos(q \sigma + \alpha_q) \right) \right], \\
p_{12} &= \sum_{q=1}^{2} \frac{\epsilon_q \eta_q}{N} \left[ \frac{N-1}{2} \left( \cos(q \gamma_1 - \alpha_q) - \cos(q \sigma + \alpha_q) \right) \right], \\
p_{21} &= \sum_{q=1}^{2} \frac{\epsilon_q \eta_q}{N} \left[ \frac{N-1}{2} \left( \cos(q \gamma_2 - \alpha_q) - \cos(q \sigma - \alpha_q) \right) \right], \\
p_{22} &= \sum_{q=1}^{2} \frac{\epsilon_q \eta_q}{N} \left[ \frac{N-1}{2} \left( \cos(q \gamma_2 + \alpha_q) + \cos(q \sigma - \alpha_q) \right) \right].
\end{align*}
\]

(11)

Solving the characteristic system (10) of two coupled quadratic equations to explicitly find \( \lambda \) is out of reach. Instead, we introduce the variable \( \Lambda = \mu \lambda^2 + \lambda \) and turn the system (10) into the system of linear equations

\[
P(A_1, A_2)^T = \Lambda (A_1, A_2)^T,
\]

(12)

where

\[
P = \begin{pmatrix}
-p_{11} & -p_{12} \\
-p_{21} & -p_{22}
\end{pmatrix}.
\]

(13)

Therefore, the stability of (9) can be assessed from (13) in terms of its eigenvalues \( \Lambda \). To do so, we aim to determine the boundary of the stability region that is determined by \( \lambda = i \text{Im} \lambda \) and corresponds to an Andronov-Hopf bifurcation of the fixed point that induces oscillating phase differences \( x(t) \) and \( y(t) \). Therefore, we can set \( \text{Re} \Lambda + i \text{Im} \Lambda = -\mu (\text{Im} \lambda)^2 + i \text{Im} \lambda \) so that the real part equality \( \text{Re} \Lambda + \mu (\text{Im} \lambda)^2 = 0 \) defines the stability boundary \( \lambda = \text{Im} \lambda \). To ascertain what side of the stability boundary corresponds to stability, we select the test value \( \mu = 0 \). With this choice, we have \( \Lambda = \lambda \), and the inequality \( \text{Re} \Lambda < 0 \) ensures the stability of the fixed point. Extending this argument to nonzero \( \mu \), we can conclude that the inequality \( \text{Re} \Lambda + \mu (\text{Im} \lambda)^2 < 0 \) makes \( \text{Re} \lambda < 0 \) and determines the fixed point stability. Thus, we arrive at the following assertion.

**Statement 1.** [Internal stability of stationary cyclops states].

1. Constant intercluster phase differences \( x = \gamma_1 \) and \( y = \gamma_2 \) of the stationary cyclops state (3) are locally stable iff

\[
\text{Re}(\Lambda_{1,2}) + \mu (\text{Im} \Lambda_{1,2})^2 < 0,
\]

(14)

where \( \text{Tr} P \) and \( \text{det} P \) are, respectively, the trace and determinant of matrix \( P \) (13) whose coefficients are defined in (11).

2. The stability boundary

\[
\text{Re}(\Lambda_{1,2}) + \mu (\text{Im} \Lambda_{1,2})^2 = 0
\]

(15)

corresponds to an Andronov-Hopf bifurcation that destabilizes the stationary cyclops state, turning it into a breathing cyclops state with oscillating intercluster phase differences \( x(t) \) and \( y(t) \).

The stability of constant phase differences \( x \) and \( y \) defined via (14) can be interpreted as the internal (longitudinal) stability of the stationary cyclops state within the invariant three-cluster manifold \( D \) determined by (3) with arbitrary, possibly time-varying \( x(t) \) and \( y(t) \). The stability boundary (15) depicted by the blue solid curve in Fig. 3(a) corresponds to emerging instability of constant phase differences \( x \) and \( y \) that preserves the three-cluster cyclops formation on the invariant three-cluster manifold \( D \).

Having studied the stability of the constant intercluster phase differences, we proceed with the stability analysis of the synchronous clusters, each composed of \( M = 1 \) oscillators. These conditions, paired with the condition (14), shall indicate what stationary cyclops states can stably emerge in the network.
V. STABILITY OF SYNCHRONOUS CLUSTERS

We aim to derive the conditions for transversal stability of the stationary cyclops states (3) that amounts to the stability of the two synchronous clusters composing the stationary cyclops state. We introduce small deviations from the oscillators’ phases \( \theta_n \rightarrow \theta_n + \delta \theta_n \), composing the first cluster for \( n = 1, \ldots, M - 1 \) and the second cluster for \( n = M + 1, \ldots, N \). To study the local stability of each synchronous cluster, we consider the difference variables

\[
\xi_n = \delta \theta_{n+1} - \delta \theta_n, \quad n = 1, \ldots, M - 2, \tag{16}
\]

\[
\xi_n = \delta \theta_{n+1} - \delta \theta_n, \quad n = M + 1, \ldots, N - 1 \tag{17}
\]

that describe the phase difference dynamics within the first and second clusters, respectively. Therefore, from (1), (3), and (16), (17), we obtain two uncoupled variational equations with time-invariant coefficients. Each of the equations determines the local stability of the corresponding cluster within the cyclops state (3):

\[
\mu \ddot{\xi}_n + \dot{\xi}_n + \frac{1}{N} \sum_{q=1}^{2} \varepsilon_{q} \left\{ \cos(q \gamma_1 + \alpha_q) \right\} + \frac{N-1}{2} \left[ \cos \alpha_q + \cos(q \sigma + \alpha_q) \right]\dot{\xi}_n = 0, \tag{18}
\]

where \( n = 1, 2, \ldots, M - 2 \), and

\[
\mu \ddot{\xi}_n + \dot{\xi}_n + \frac{1}{N} \sum_{q=1}^{2} \varepsilon_{q} \left\{ \cos(q \gamma_2 + \alpha_q) \right\} + \frac{N-1}{2} \left[ \cos \alpha_q + \cos(q \sigma - \alpha_q) \right]\dot{\xi}_n = 0, \tag{19}
\]

where \( n = M + 1, \ldots, N - 1 \). The variational equations (18) and (19) are stable iff the time-invariant coefficients of the terms \( \xi_n \) and \( \dot{\xi}_n \) are positive. Therefore, we can formulate the stability conditions in the following assertion.

**Statement 2.** Transversal stability of stationary cyclops states. *Clusters of oscillators composing the stationary cyclops state (3) are locally stable if:*

\[
\sum_{q=1}^{2} \varepsilon_{q} \cos(q \gamma_1 + \alpha_q) + \frac{N-1}{2} \left( \sum_{q=1}^{2} \varepsilon_{q} \cos \alpha_q \right) > 0,
\]

\[
\sum_{q=1}^{2} \varepsilon_{q} \cos(q \gamma_2 + \alpha_q) + \frac{N-1}{2} \left( \sum_{q=1}^{2} \varepsilon_{q} \cos \alpha_q \right) > 0,
\]

where the left-hand sides of the inequalities (20) are the coefficients of the variational equations (18) and (19). It is also straightforward to show that the stationary cyclops state is always stable to the shift of all phases by a constant value \( \delta \theta_k = \delta \theta \) (\( k = 1, \ldots, N \)).

It is worth noticing that the eigenvalues \( \lambda_{\text{tran}}^{1,3} \) associated with the variational equations (18) and (19) have multiplicity \( M - 2 \). Thus, the eigenvalues \( \lambda_{\text{tran}}^{1} \) and \( \lambda_{\text{tran}}^{2} \) define the transversal stability of the first (\( n = 1, 2, \ldots, M - 2 \)) and second (\( n = M + 1, \ldots, N \)) clusters, respectively. Figure 3(a) displays their stability boundaries defined by the
conditions (20) with the left-hand sides set to 0 to correspond to $\Re \lambda_{1}^{\text{tran}} = 0$ (the red dashed line) and $\Re \lambda_{2}^{\text{tran}} = 0$ (the green dash-dotted line). To highlight the constructive role of the second-harmonic coupling with $\varepsilon_{2} \neq 0$, we chose the parameter values that yield unstable stationary cyclops in the network with only first-harmonic coupling with $\varepsilon_{2} = 0$ (see Fig. 3).

As Fig. 3(a) indicates, crossing the stability boundary (15) (the lower border of the region CS) induces breathing cyclops states in the region BCS (green) in accordance with Statement 1. In turn, crossing the upper border of the region CS, composed of the transversal stability boundaries $\Re \lambda_{1}^{\text{tran}} = 0$ (the red dashed line) and $\Re \lambda_{2}^{\text{tran}} = 0$ (the green dash-dotted line) can yield either switching cyclops states in the region SCS (yellow) or asymmetrical, two-cluster states with five- and six-oscillator synchronous clusters (white region 5:6). In the following, we will primarily focus on the properties of emerging breathing and switching cyclops states.

VI. EMERGING BREATHING AND SWITCHING CYCLOPS STATES

We performed numerical calculations using a widely adopted fifth-order Runge-Kutta scheme with a fixed time step 0.01 to further validate our analytical results and predictions. Figure 3 confirmed the two main bifurcation scenarios for destroying the stationary cyclops states and generating breathing and switching cyclops states described by Statements 1 and 2. In the first scenario, complex conjugate eigenvalues $\lambda_{1,2}$, that determine the stability of constant intercluster phase differences $x$ and $y$ via (10), become purely imaginary and induce oscillating $x(t)$ and $y(t)$ [Fig. 4(c)]. As a result, the stationary cyclops state becomes internally unstable; however, the stability of the clusters preserves and guarantees the emergence of a breathing cyclops state [see Figs. 4(a) and 4(d) and Supplemental Movie 1 [68] for the animation of the breathing cyclops state dynamics]. Periodic oscillations of the first two order parameters $r_{1}$ and $r_{2}$ depicted in Fig. 4(b) are a signature of such a breathing cyclops state. As the distance from the stability boundary of the CS region [solid blue line in Fig. 3(a)] increases when changing the second-harmonic coupling strength $\varepsilon_{2}$ and phase lag $\alpha_{2}$, the amplitudes of intercluster difference oscillations $x(t)$, $y(t)$, and order parameters $r_{1}(t)$, $r_{2}(t)$ increase. It is worth noticing that for the parameters $\alpha_{1}$ and $\varepsilon_{1}$ used in Fig. 3(a), the breathing cyclops state is also stable in the absence of the second-harmonic coupling ($\varepsilon_{2} = 0$).

In the second bifurcation scenario determined via Statement 2, the stationary cyclops state loses its transversal stability when one of the eigenvalues $\lambda_{1,2}^{\text{tran}}$ becomes positive [Fig. 5(c)]. Note that the real parts of the other eigenvalues controlling the internal stability of the intercluster differences remain negative, thereby preserving the stable component of the saddle dynamics. While the transversal instability of the cyclops state may lead to its complete destruction, it induces a switching cyclops state [Figs. 5(a) and 5(d)] when the transversal instability is weak [note the slightly positive eigenvalue, depicted by the red nabla in Fig. 5(c)]. This nonstationary cyclops state represents a two-stage repetitive process: During the first relatively long stage, the intercluster differences $x$ and $y$ practically do not change, and the synchronous clusters preserve their formation, i.e., the dynamical pattern is similar to a stationary cyclops state (Fig. 5). During the second short stage, one cluster reshuffles so that one node leaves the unstable cluster to become a new solitary oscillator, whereas the remaining oscillators from the cluster merge with the old solitary node. Figure 5(d) and Supplemental Movie 2 [68] illustrate this process. Accordingly, during the first stage, the magnitudes of the order parameters $r_{1}$ and $r_{2}$ are practically constant. They undergo an abrupt change during the second stage to return to a constant value [Fig. 5(b)]. As the parameter $\alpha_{2}$ increases, the duration of the first stage becomes shorter, and, hence, the period of oscillations in $r_{1}$ and $r_{2}$ gradually decreases, causing the switching cyclops state to eventually turn into a chaotically switching dynamical pattern (not shown in Fig. 5).

We also observe a hybrid of the switching and breathing cyclops states [Fig. 6(b)]. This hybrid state emerges when, in addition to the external instability of one cluster, there is an internal instability of the intercluster phase differences $x$ and $y$ (Fig. 3). In terms of the eigenvalue spectrum, this amounts to the presence of a pair of complex conjugate eigenvalues $\lambda_{1,2}$ (corresponding to the internal instability) and one real eigenvalue $\lambda_{1}^{\text{tran}}$ lying to the right from the...
FIG. 5. Switching cyclops state. (a) The colors depict the phase differences $\theta_k(t) - \theta_6(t)$. The strips with solid black borders indicate the reference solitary oscillator during the lifetime of a cyclops state configuration (the first stage). Note that clusters disintegrate to form a new cyclops state with a different solitary oscillator (the second stage). (b) The corresponding values of $r_1$ and $r_2$. The gray fragments correspond to the zoomed-in insets (right panels). (c) The eigenvalues associated with the destabilized stationary cyclops state. Some eigenvalues are repeated. The round (triangular) labels correspond to the internal (transversal) stability. Note a positive real eigenvalue (red) corresponding to the loss of the transversal stability of the stationary cyclops state due to Statement 2. (d) Phase distributions $\theta_k$ corresponding to a death-birth process in which a cyclops state existing at $t = t_1$ disintegrates to form a new cyclops state at $t = t_4$ (see Supplemental Movie 2 [68] for the details of this dynamical evolution). Parameters $N = 11, \mu = 1.0, \epsilon_1 = 1.0, \alpha_1 = 1.7, \epsilon_2 = 0.08, \alpha_2 = 0.78$ correspond to the diamond label in Fig. 3(a).

imagine axis [Fig. 6(c)]. We term this hybrid a switching-breathing cyclops state, which is effectively a switching cyclops state, which, during its first stage, has oscillating intercluster phase differences $x(t)$ and $y(t)$. Accordingly, the order parameter amplitudes $r_1$ and $r_2$ are time-periodic functions [Fig. 6(b)]. Figure 6(d) and Supplemental Movie 3 [68] detail the dynamical evolution of the switching-breathing cyclops state.

Breathing and switching cyclops states can also merge to form another hybrid cyclops state, termed rotobreathing cyclops states (Fig. 7) in the range of the second-harmonic phase shift with $|\alpha_2| > \pi/2$ (Fig. 8, the pink regions). Rotobreathing

FIG. 6. Switching-breathing cyclops state. The notations are as in Fig. 5. One cluster of the breathing cyclops state [depicted in orange in panel (a)] eventually disintegrates, forming a reshuffled synchronous cluster and a new solitary oscillator. Note the weak internal and transversal instability of the destabilized stationary cyclops state due to the three eigenvalues with small positive real parts [red circles and nabla in panel (c)]. Supplemental Movie 3 [68] animates the sequence given in panel (d). Parameters $N = 11, \mu = 1.0, \epsilon_1 = 1.0, \alpha_1 = 1.7, \epsilon_2 = 0.0578, \alpha_2 = 0.78$ correspond to the circled times label in Fig. 3(a).
cycles states, or simply roto-breathers, are also characterized by a two-stage repetitive process in which, during the first stage, an intercluster phase difference between one cluster and the solitary oscillator oscillates while the relative phase difference of the other cluster rotates. The clusters exchange their oscillatory and rotatory phase roles during the second stage. Figure 7 and Supplemental Movie 4 [68] give the full details of this two-stage process. Accordingly, the amplitudes of the order parameters \( r_1 \) and \( r_2 \) exhibit large periodic oscillations [Fig. 7(b)].

Figure 7 demonstrates the prevalence of cyclops states of various types. Remarkably, roto-breathers and breathing cyclops states, induced by nonzero second-harmonic phase lag \( \alpha_2 \) in the region where full synchronization is unstable, act as global attractors and emerge with a probability close to 1 [Figs. 8(b) and 8(c)]. Note that breathing and switching cyclops states can also emerge with a relatively high probability even when they coexist with presumably dominant full synchronization when the overall coupling is attractive with \( H'(0) > 0 \) [the region bounded by the black dashed vertical lines in Fig. 8(b)]; these lines correspond to the solid circles on the black dashed parabola in Fig. 8(a).

It is worth noting that the globally coupled network (1) with the first- and second-harmonic coupling admits any cluster partition. As a result, stationary cyclops states with a solitary oscillator and nonequally sized coherent clusters exist for even \( N \). However, our extensive simulations suggest that such stationary cyclops states are unstable for even \( N \) in the entire parameter range of \( \varepsilon_2 \) and \( \alpha_2 \) considered in this paper. Figure 9(a) supports this claim and demonstrates that, in the attractive coupling case, two-cluster states, coexisting with complete synchronization, are the only stable cluster patterns in the network with \( N = 10 \). The same claim holds for the repulsive coupling case, with the exception that complete synchronization is no longer stable so that the two-cluster states become prevalent and coexist with four-cluster states with rotating intercluster differences for \( N = 10 \) [Fig. 9(c)]. Similarly, stationary cyclops states (4) emerge as the only stable patterns along with complete synchronization for \( N = 11 \) in the attractive coupling case [Fig. 9(b)], and breathing cyclops states with equally sized coherent clusters appear as global attractors in the repulsive coupling case [Fig. 9(d)].

Remarkably, this distinct behavior in the emergence of two-cluster and cyclops states in networks of even and odd sizes carries over to large networks. Figure 10(a) demonstrates...
that stationary cyclops states (4) are stable in the odd-sized network with $N = 101$. Eliminating one oscillator from the network turns the cyclops state into a two-cluster state with equally sized clusters [Fig. 10(b)]. Removing an extra oscillator from this 100-node network transforms the two-cluster state back to a stationary cyclops state [Fig. 10(c)]. We expect this effect to persist in the thermodynamic limit of very large $N$.

VII. PERSISTENCE OF CYCLOPS STATES

In this section we demonstrate that cyclops states resist intrinsic frequency detuning. We mismatch the intrinsic frequency $\omega$ by choosing the $k$th oscillator’s frequency $\omega_k$, $k = 1, \ldots, N$ from a uniform random distribution in the interval $[\omega - \delta, \omega + \delta]$, where $\delta$ is a frequency detuning. We consider the parameter region where stationary cyclops states are stable [region CS in Fig. 3(a)]. Figure 11 demonstrates the persistence of three stationary cyclops states, each induced by a particular intrinsic frequency distribution. Note that although the oscillators’ phases within each synchronous cluster may not perfectly align due to the frequency detuning, they remain relatively close to each other (see Fig. 11). Additionally, the established frequencies of all oscillators are the same. The stationary cyclops state can lose the transversal stability similarly to their counterparts from the identical oscillator case (note the stationary cyclops state marked by the red labels in Fig. 11 that turns into a switching cyclops state at $\delta = \delta_1$). Remarkably, the frequency detuning can also induce a bifurcation scenario for disintegrating stationary cyclops states via a saddle-node bifurcation at $\delta = \delta_2$ and $\delta = \delta_3$. We did not observe such a bifurcation route in our extensive simulations of system (1) with identical frequencies reported in Figs. 3–8.

VIII. CONCLUSIONS

Building upon our recent study [50], this work has significantly advanced an understanding of rhythmogenesis in Kuramoto networks of 2D phase oscillators with first-mode and higher-mode coupling. A key focus of our work has been on the constructive role of higher coupling modes in inducing and stabilizing a unique class of dynamical states known as cyclops states. These states, characterized by two coherent clusters and a solitary oscillator resembling the Cyclops’s eye, represent a particular form of three-cluster generalized splay states [49].

Our initial findings in [50] revealed the unexpected result that adding the second or third harmonic to the Kuramoto coupling makes cyclops states global attractors, exhibiting remarkable stability over a substantial range of coupling’s repulsion. This paper delved deeper into the dynamic repertoire of cyclops states, introducing and systematically analyzing...
was determined in the time interval $0 \leq t \leq 10^4$ following the transition process. A stationary cycles state is indicated by overlapping circles and crosses of the same color, while separate circles and crosses denote a switching cycles state. The first cycles state (red) originates from the original intrinsic frequency distribution. Subsequently, modifying the distribution by exchanging the intrinsic frequency of the $M$th solitary oscillator, $\omega_M$, with $\omega_1$ (or $\omega_M+1$) induces the emergence of the second (or third) cycles state, depicted by orange and cyan, respectively. As the parameter mismatch $\delta$ increases, the distribution of mismatched frequencies widens. To induce each of the three cycles states as $\delta$ increases, a stationary cycles state from the identical frequency case was used as the initial condition for subsequent simulations at a higher value of $\delta$. The values of $\delta < \delta_1$ preserve all three stable stationary cycles states. Increasing $\delta > \delta_1$ destabilizes the first stationary cycles state (red) and turns it into a switching cycles state. Further increasing $\delta > \delta_2$ leads to disintegrating the second cycles state (orange) at $\delta = \delta_2$. The third stationary cycles state (cyan) persists to $\delta = \delta_3$. The cycles states are found from direct numerical simulations of system (1) for three sets of natural frequency distributions $\omega_0$ with a continuous increase in $\delta$ from zero. The inset shows instantaneous phase distributions $\delta_i$ for the third cycles state with nonidentical frequencies. Parameters are $N = 11$, $\omega_1 = 1.0$, $\omega_2 = 1.8$, $\omega_3 = 0.12$. The bifurcation parameter values are $\omega_2 = 0.2$, $\delta_1 = 0.034$, $\delta_2 = 0.105$, $\delta_3 = 0.183$.

breathing and switching cycles states and their hybrids, including switching-breathing cycles states and rotobreathe.

Through rigorous analytical derivations and numerics, we have identified conditions for the existence and stability of stationary cycles states, elucidating two distinct bifurcation scenarios. In both scenarios, the second coupling harmonic acts as a constructive agent, either inducing periodic oscillations in intercluster relative phase differences (breathing cycles states) or facilitating swift reconfigurations and transitions (switching cycles states). These dynamical patterns can be viewed as nontrivial hybrids of solitary states [41–44], generalized splay [49], clusters with breathing and rotatory intercluster phase shifts [46,47], and intermittent [37] and switching chimeras [64]. In particular, switching cycles states unite the properties of blinking chimeras [64] and three-cluster states [47].

Our extensive stability analysis has underscored the resilience and dominance of breathing, rotobreathing, and switching cycles states across wide parameter ranges, including the case of the overall attractive, two-harmonic coupling. Importantly, we have showcased that the constructive influence of higher coupling harmonics is not limited to networks of identical oscillators, as cycles states persist robustly in Kuramoto networks of nonidentical oscillators.

Importantly, our prior work [50] demonstrated the dynamic equivalence of the 2D Kuramoto model with first- and second-harmonic coupling to a network of canonical theta neurons with adaptive coupling. This equivalence also suggests the widespread manifestation of breathing and switching cycles states in theta-neuron networks, underscoring our results' broad applicability and significance in diverse physical and biological networks.

While it is crucial to differentiate between the higher-order harmonic coupling studied in this paper and the higher-order nonpairwise coupling [69–72], it is equally important to recognize their possible interplay and the richness they bring to the dynamics of networked systems. These concepts are not mutually exclusive; they can coexist, adding layers of complexity and fostering a diverse range of emergent behaviors. Recent research [73,74] analyzed the intricate interplay between pairwise-first-order harmonic and nonpairwise higher-order coupling in shaping collective dynamics in Kuramoto networks. The incorporation of both higher-order harmonics and nonpairwise interactions promises to induce even richer emerging dynamics, including various forms of cycles states, and may pave the way for a more holistic comprehension of complex networked systems.

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APPENDIX: MAXIMUM NUMBER OF STATIONARY CYCLOPS STATES

Here we provide the details for deriving an upper bound for the maximum number of stationary cycles states with distinct $x$ and $y$, given in Sec. III.

Finding all possible solutions of system (6) that determine the existence of stationary cycles states is elusive due to its complexity, and the number of solutions can vary depending on the parameters. In particular, it prevents locating all solutions of the system (6) by their continuation with respect to the parameters. However, this computational problem can be simplified by the change of variables $u = e^{i\alpha}$, $v = e^{i\beta}$ ($|u| = 1$, $|v| = 1$) that transforms the real-valued system (6) into the system of complex polynomial equations:

$$
(1-u)|v|^2 e^{i\delta} \left( u + v + \frac{2|v|}{N-1} \right) - e^{-i\delta} v \left( u + v + \frac{2|v|}{N-1} \right)
$$

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The analysis of system (A1) is more manageable, and the maximum number of its solutions (the roots of the complex polynomials) can be estimated by applying the classical Bernstein theorem from algebra. To facilitate the reading, we list this theorem below.

**Theorem** [Bernstein, 1975] [67]. Let a system of polynomials have a finite number of roots in $(C^*)^n$, where $C^* = C \setminus 0$. Then, the number of roots is bounded from above by the mixed volume $P_k$ of their Newton polytopes (the convex hull of polynomial supports $S_j$).

Before applying the theorem to (A1), we get rid of the factors $(1 - u)$ and $(1 - v)$ on the right-hand side of (A1) since we are interested only in solutions $u, v \neq 1$. By doing so, we have excluded the solutions that correspond to a one-cluster solution and two-cluster solutions of the form $(N - 1)/2 : (N + 1)/2$. It is worth mentioning that, in contrast to its real-valued counterpart (6), the complex polynomials may have either nonphysical solutions with $|u| \neq 1$ or $|v| \neq 1$, or solutions that do not correspond to stationary cyclops states. The latter solutions with $|u| = 1$ and $|v| = 1$, include a two-cluster $N - 1 : 1$ solitary state, corresponding to $u = v$, i.e., $x = y$.

The support of a polynomial $\sum_j \sum_l a_{jl} u^j v^l$ is the set of exponents $(j, l)$. Thus, the supports $S_1$ and $S_2$ of the resulting polynomials (degrees $u$ and $v$ in each of the first and second equations terms) have the form $S_1 = \{(0, 2); (0, 4); (1, 2); (1, 3); (1, 4); (2, 0); (2, 1); (2, 2); (3, 0); (3, 2)\}$, $S_2 = \{(0, 2); (0, 3); (1, 2); (2, 0); (2, 1); (2, 2); (2, 3); (3, 1); (4, 0); (4, 1)\}$.

Consider the mixed volume of Newtonian polytopes $P_j$ and $P_k$: $M(P_j, P_k) = \text{vol}(P_j \oplus P_k) - \text{vol}(P_j) - \text{vol}(P_k)$. As we need to handle only two equations, determining the mixed volume of the two Newtonian polytopes is straightforward and amounts to computing the area of the shaded regions in Fig. 12. This yields the following values: $\text{vol}(P_1) = 8$, $\text{vol}(P_2) = 8$, $\text{vol}(P_1 \oplus P_2) = 33$ (see Fig. 12). Therefore, $M(P_1, P_2) = 17$. Excluding the nonphysical solutions and solutions corresponding to noncyclops regimes from the sets of roots in (A1), we can always find the number of cyclops modes in the system (1), which is limited to 16 cyclops states.

Our numerical search for the roots of polynomials (A1) was performed using the NSolve function of Wolfram Mathematica. This search found 17 roots almost everywhere in the considered broad parameter regions, suggesting that our analysis effectively identified all possible solutions of (A1) and, therefore, all possible stationary states cyclops, identified from the 17 solutions by excluding the nonphysical solutions $(|u| \neq 1$ or $|v| \neq 1$) and noncyclops states ($u = 1$ or $v = 1$ or $u = v$).

![FIG. 12. The supports $S_1$, $S_2$ (black dots) and the corresponding Newton polytopes $P_1$, $P_2$ (shaded regions) of (a) the first and (b) second polynomials of system (A1). (c) The Minkowski sum $P_1 \oplus P_2$.](image)

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