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Beyond the Bristol book: Advances and perspectives in non-smooth dynamics and applications

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ABSTRACT

Non-smooth dynamics induced by switches, impacts, sliding, and other abrupt changes are pervasive in physics, biology, and engineering. Yet, systems with non-smooth dynamics have historically received far less attention compared to their smooth counterparts. The classic "Bristol book" [di Bernardo *et al.*, *Piecewise-smooth Dynamical Systems. Theory and Applications* (Springer-Verlag, 2008)] contains a 2008 state-of-the-art review of major results and challenges in the study of non-smooth dynamical systems. In this paper, we provide a detailed review of progress made since 2008. We cover hidden dynamics, generalizations of sliding motion, the effects of noise and randomness, multi-scale approaches, systems with time-dependent switching, and a variety of local and global bifurcations. Also, we survey new areas of application, including neuroscience, biology, ecology, climate sciences, and engineering, to which the theory has been applied.

This paper opens the 2022 Focus Issue on Non-Smooth Dynamics. We review advances in the theory of piecewise-smooth and non-smooth dynamical systems beyond the extensive coverage of the high-impact "Bristol book" that was published in 2008. We also highlight the contributions to this Focus Issue that articulate the role of non-smooth dynamics and cover a wide range of topics including Filippov systems, discontinuity-induced bifurcations, vibro-impact systems, pulse-coupled systems, switching networks, and applications in mechanics and biomechanics.

I. INTRODUCTION

Non-smooth dynamics, appearing as switches, impacts, sticking, sliding, and chatter require careful formulation and treatment due to the essential piecewise or discontinuous features. Piecewise-smooth and non-smooth dynamical systems represent a vast research area in nonlinear science, related to systematic mathematical analysis and modeling of non-smooth dynamics and bifurcations, possibly in the presence of uncertainty and stochasticity. The introduction of non-smoothness can generate nearly any type of behavior, via a huge range of discontinuity-induced bifurcations, some with smooth counterparts, like fold-type or Hopf-type bifurcations, but others specific to non-smooth phenomena, e.g., grazing and sliding. While the theory of smooth dynamical systems dates back to Poincaré's time, systematic efforts to understand non-smooth dynamics and bifurcations have only been performed more recently.

Notwithstanding valuable early contributions by Andronov *et al.*,[1] Neimark,[2] Filippov,[3,4] Feigin,[5,6] and others (see Sec. 1.7 of

Jeffrey[7]), progress on non-smooth dynamics underwent rapid acceleration in the 1990s, ignited by the fundamental work of Nordmark and collaborators on impact oscillators and discontinuity maps.[8–10] Researchers at Bristol, UK, and nearby Bath were central to many of these developments and took the extra step of collating the state-of-the-art theory at the time into a graduate-level textbook[11] published in 2008. The book was completely novel and has standardized terminology, and it made non-smooth dynamics mainstream by showing how standard dynamical systems ideas, when appropriately generalized, provide the key to understanding physical problems in diverse disciplines. As of this writing, the book has over 2000 citations in Google Scholar. In view of its lofty place in the non-smooth dynamics literature, we, and many others, refer to it simply as the "Bristol book."

However, the theory of non-smooth dynamics has developed further since 2008. The main purpose of this paper is to review these advances. The advances are diverse, some theoretical and others breaking into new areas of application. Some reviews and additional books have been compiled. Of particular note is the work of Jeffrey[7]—another Bristol book that extends Filippov's framework to systems with multiple switches and explains the occurrence of novel dynamics in physically motivated regularizations of nonsmooth models.

We also briefly survey articles in the present Focus Issue that brings together applied mathematicians, physicists, and engineers to display recent advances in the theory and applications of non-smooth dynamical systems. Topics covered range from the dynamics and bifurcations of piecewise-smooth and impacting systems, including non-classical sliding homoclinic and grazing bifurcations, to the constructive role of non-smoothness in the stability and control of switched networks with an eye toward applications in biology and engineering.

The idea of organizing this Focus Issue was inspired by a non-smooth dynamics minisymposium held at the virtual 2021 SIAM Conference on Applications of Dynamical Systems. This focus contains a collection of research papers from a broad spectrum of topics related to modeling, analysis, and control of non-smooth dynamical networks. We hope that this collection will generate significant interest among the mathematics, physics, and engineering audiences of the journal. Junior researchers might also find this collection useful as an inspiration to start graduate research in this exciting field of research.

II. HIDDEN DYNAMICS

Much of the Bristol book is dedicated to the dynamics and bifurcations of ordinary differential equation (ODE) systems of the form

$$\dot{x} = \begin{cases} f_L(x), & h(x) < 0, \\ f_R(x), & h(x) > 0. \end{cases} \tag{1}$$

Here, the system state $x(t) \in \mathbb{R}^n$ evolves according to one of two vector fields, f_L, and f_R, as governed by the sign of a smooth function $h : \mathbb{R}^n \to \mathbb{R}$. This represents the simplest formulation of state-dependent switching between two modes of evolution. Solutions can become constrained to the switching manifold $h(x) = 0$, Fig. 1. This is *sliding motion*, usually formulated as the solution to a convex combination of f_L and f_R in accordance with Filippov.[3,4]

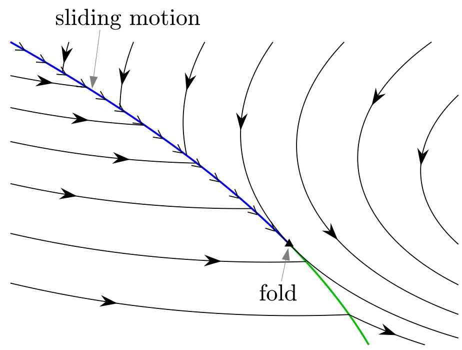

FIG. 1. A phase portrait of a two-dimensional non-smooth system of the form (1). Evolution on the switching manifold $h(x) = 0$ is termed sliding motion. Sliding motion usually ends when the system state reaches a point of tangency (visible fold).

A more realistic model might incorporate hysteresis or time-delay in the function h or smooth the vector field over a narrow region (boundary layer) containing the switching manifold. If the addition of such complexities has little or no bearing on the qualitative features of the dynamics, it is probably better to work with the simpler model (1). This is often indeed the case and serves to illustrate the importance of understanding the dynamics and bifurcations of such systems. However, in many situations, new dynamics arise.

Understanding the causes and consequences of such *hidden* dynamics has recently been championed by Jeffrey.[7] Here, things are clearer with (1) rewritten as

$$\dot{x} = [1 - H(h(x))]f_L(x) + H(h(x))f_R(x), \tag{2}$$

where H is the Heaviside function. Hidden dynamics can appear when H is replaced with a smooth approximation that is non-monotone.[12–14] This occurs, for example, in friction models to capture the shape of the Stribeck curve[15,16] that accounts for the extra break-away force that in-contact objects require to begin slipping.

The resulting non-monotone model typically involves dynamics that are qualitatively different to those of (2). The lack of monotonicity can cause a shift in bifurcation values or introduce new bifurcations.[17,18] It can introduce oscillations in the boundary layer in scenarios where (2) has roughly uni-directional sliding motion. Taken to an extreme, non-monotone smoothing can convert sliding motion into chaos.[19]

When multiple switching conditions are involved, monotone smoothing is sufficient to generate new dynamics.[19,20] As shown by Harris and Ermentrout,[21] this occurs for the Wilson–Cowan neuron model with discontinuous firing rate functions. The model is

$$\begin{aligned} \dot{u} &= -u + H(u - av - b), \\ \tau\dot{v} &= -v + H(u - cv - d), \end{aligned} \tag{3}$$

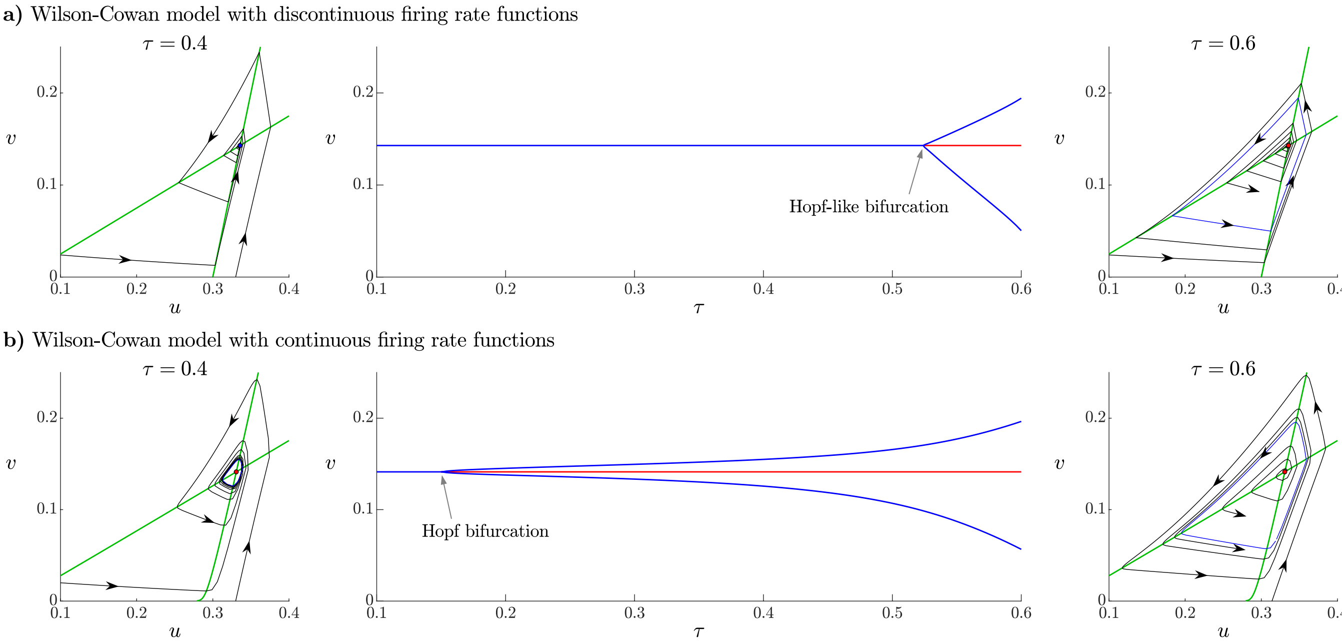

FIG. 2. The upper plots show a bifurcation diagram and representative phase portraits of the non-smooth system (3) with $a = 2$, $b = 0.05$, $c = 0.25$, and $d = 0.3$, as given by Harris and Ermentrout.[21] The lower plots are for the smooth system obtained by replacing the Heaviside functions with hyperbolic tangent functions as explained in the text (using $\varepsilon = 0.005$). Stable solutions are colored blue; unstable solutions are colored red. The green curves are switching manifolds in the upper plots and nullclines in the lower plots. The bifurcation diagrams show the v-value of the steady-state solution and minimum and maximum v-values of the limit cycle.

where $u(t)$ represents the average activity of a large neural network, $v(t)$ is a recovery variable, and H is again the Heaviside function. The intersection of the two switching manifolds, $u = av + b$ and $u = cv + d$, is a steady-state solution that loses stability and emits a stable limit cycle as the parameter τ is increased, Fig. 2(a). By replacing each $H(z)$ with $\frac{1}{2}\left(\tanh\left(\frac{z}{\varepsilon}\right) + 1\right)$ (here, z is a dummy variable), the system is now smooth but the analogous transition occurs much earlier in a (classical) Hopf bifurcation, Fig. 2(b). By taking $\varepsilon \to 0$, we recover (3), yet the Hopf bifurcation value converges to $\tau \approx 0.1373$, which is substantially earlier than the bifurcation value for (3) of $\tau \approx 0.5240$. We conclude that, for intermediate values of τ, arbitrarily steep monotone smoothing causes the steady-state solution to lose stability and stable, small-amplitude, oscillations to be created.

A deeper understanding of the dynamics and bifurcations of (1) can be gained by smoothing with monotone functions.[22,23] The smoothed model is inherently slow–fast and, in this way, slow–fast systems and piecewise-smooth systems are closely related (see also Sec. IX). For example, folded nodes[24] of slow–fast systems can, when the limit to the Heaviside function is taken, become *two-folds* of piecewise-smooth systems[25] [a two-fold of (1) is a point on $h(x) = 0$ at which both $f_L(x)$ and $f_R(x)$ have a tangency to $h(x) = 0$ and certain genericity conditions are satisfied[7]]. Two-folds were considered by Filippov[4] and Teixeira,[26] but only recently analyzed in more detail.[27–31] Single folds have been investigated by smoothing,[32] as have planar two-folds,[33] including the non-uniqueness of trajectories that enter two-folds.[34] A contraction analysis based on regularization was also used to study the stability of different classes of switched Filippov systems.[35]

III. GENERALIZATIONS AND EXTENSIONS OF SLIDING MOTION

The discontinuous neuron model (3) is one of many non-smooth models that involve multiple switching manifolds. To specify sliding motion along the intersection of two switching manifolds, Filippov's approach to constructing a tangent convex combination can fail to define a unique solution. Several ways to remedy this have recently been proposed. Jeffrey[36] identifies a "canopy" convex combination that is, in a sense, the simplest. Dieci and Difonzo[37] instead take the barycentric mean. Kaklamanos and Kristiansen[38] smooth the system, then define sliding motion by taking the nonsmooth limit. Jeffrey *et al.*[39] apply perturbations (hysteresis, time-delay, noise, and numerical discretization) and take the zero perturbation limit. Such a procedure gives different results for the different types of perturbations renewing the remarks of Utkin,[40] in the context of relay control, that the most appropriate definition for sliding motion depends critically on the physical properties of the system under consideration.

Related to this problem, friction models with sufficiently many degrees of freedom (DoF) naturally involve switching manifolds that are codimension-two (instead of codimension-one). Some theory for the dynamics and bifurcations of such systems has recently been developed by Antali and Stépán.[41,42]

IV. LOCAL BIFURCATIONS OF NON-SMOOTH ODES

As parameters are varied, interactions between invariant sets and switching manifolds produce a wide variety of novel bifurcations collectively known as *discontinuity-induced bifurcations*. The simplest type of discontinuity-induced bifurcation is arguably a *boundary equilibrium bifurcation* that occurs when an equilibrium of a smooth component of the system collides with a switching manifold. Discontinuity-induced bifurcations are described in Chap. 5 of the Bristol book, which, following Kuznetsov *et al.*,[43] chronicles ten topologically distinct, generic, boundary equilibrium bifurcations in the two-dimensional setting. Unfortunately, two cases were overlooked, shown in Fig. 3. These cases were only described later,[44,45] and they serve to illustrate the difficulty in attempting a comprehensive classification of bifurcations of non-smooth systems.[46]

Indeed, for systems with more than two dimensions, boundary equilibrium bifurcations can create chaotic attractors[47,48] and even multiple attractors.[49] This suggests that future developments in the bifurcation theory of high-dimensional non-smooth systems may benefit from focusing on weaker results that apply generally rather than a large number of strong results for particular situations.[50]

Boundary equilibrium bifurcations can mimic Hopf bifurcations by converting a stable equilibrium into a stable limit cycle,[43,51] but there are many other mechanisms, unique to non-smooth systems, that can achieve this transition.[52] Two-folds each shifting along a switching manifold as parameters are varied, can collide, interchange positions, and generate a limit cycle. As shown in Fig. 4, such a phenomenon occurs for the automatic pilot model

$$\ddot{\phi} + \dot{\phi} = -H(\phi + \beta\dot{\phi}), \tag{4}$$

given in the classic book of Andronov *et al.*[1] The desired heading of $\phi = 0$ for the ship or vessel is achieved when the control parameter β is positive. If the value of β is decreased through zero, two folds collide and a stable limit cycle is created. Only recently has this type of bifurcation been analyzed in a general setting.[53]

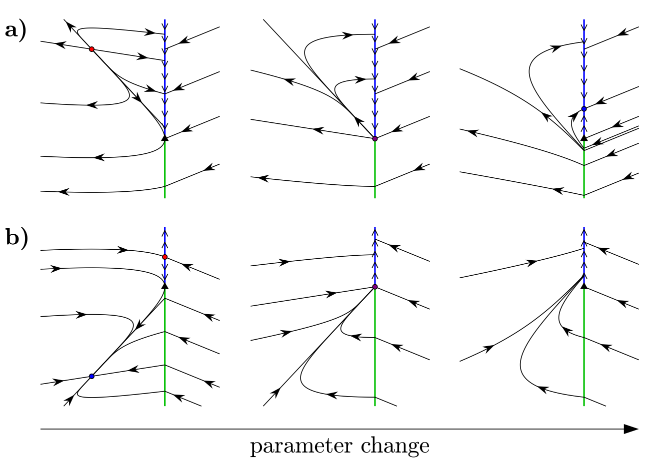

FIG. 3. Two boundary-equilibrium bifurcations not described in the Bristol book. In (a), an unstable node transitions to a stable pseudo-equilibrium. In (b), a stable node and a saddle pseudo-equilibrium collide and annihilate.

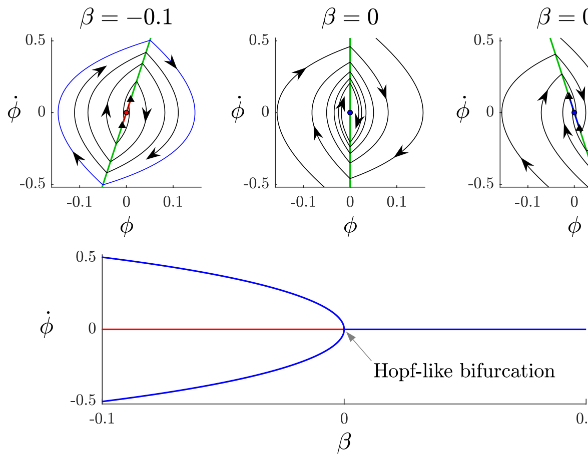

FIG. 4. Phase portraits and a bifurcation diagram of (4). This is a minimal model of an automatic pilot where the vessel heading $\phi(t)$ is controlled through a parameter β that governs how the rudder switches between two allowed positions. The switching manifold is colored green for crossing regions, blue for the attracting sliding region, and red for the repelling sliding region. These regions are bounded by folds shown as black triangles.

Every type of Hopf-like bifurcation involves a scaling law for the amplitude and period of the bifurcating limit cycle as a function of parameters.[54] The amplitude grows asymptotically linearly when the dynamics is piecewise-linear to leading order, while if two folds are involved, the amplitude is usually asymptotically proportional to the square-root of the parameter change, as in Fig. 4. An interesting exception is a two-fold perturbed by hysteresis, which gives a cube-root scaling law.[55] For non-smooth systems that are C^1 but not C^2, a modification to the standard Hopf bifurcation non-degeneracy coefficient is required.[56,57]

Also in recent years, there have been many studies that aim to count or bound that number of limit cycles possible in various classes of non-smooth systems; see Llibre and Zhang[58] and references within. The unfoldings of several codimension-two bifurcations have been derived,[59–61] as has the three-dimensional unfolding of the simultaneous occurrence of Hopf, saddle-node, and boundary equilibrium bifurcations.[62]

V. GLOBAL BIFURCATIONS

The global bifurcation theory for systems with discontinuities remains quite undeveloped. Di Bernardo and Hogan[63] provided an extensive review in 2010. Perhaps, the first major focus of existing studies is on deriving conditions under which global bifurcations in non-smooth ODEs are qualitatively similar to their classical (smooth) counterparts.[64] Novaes and Teixeira[48] derived a version of Shilnikov saddle-focus theorem whereby a sliding saddle-focus homoclinic loop yields a countable infinity of sliding saddle periodic orbits. Belykh *et al.*[65] constructed an analytically tractable non-smooth system with a Lorenz-type attractor whose global bifurcations could be rigorously characterized and explicitly connected to the system parameters. A second major focus concerns global discontinuity-induced bifurcations of limit cycles[67] and homoclinic orbits.[47,66,68] For instance, a saddle in a Filippov system can attain a homoclinic connection involving sliding motion that generates a stable limit cycle independently of the sign of a saddle value.[43,66] Figure 5 contrasts classical and non-classical homoclinic butterfly bifurcations for Lorenz-type systems.[66] Remarkably, the emergence of sliding motion induces a non-classical homoclinic bifurcation in which an unstable homoclinic orbit gives birth to a stable period-doubled limit cycle.

The characterization of other global discontinuity-induced bifurcations in three dimensions or higher remains a challenging problem. For example, non-smooth systems can merge saddle-focus homoclinic orbits with sliding motion and boundary equilibrium bifurcations, generating a fundamentally different and complicated bifurcation set[47] which calls for the development of new rigorous methods.

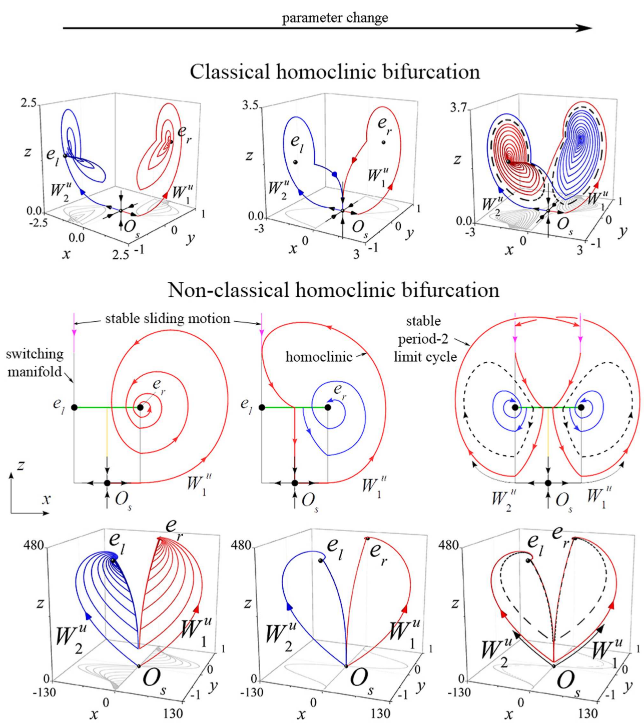

FIG. 5. The role of stable sliding motions for shaping the outcome of the classical homoclinic butterfly bifurcation in a piecewise-smooth version of the Lorenz system.[66] Top row: the classical homoclinic bifurcation in the absence of sliding motion. Two stable foci e_l and e_r attract opposite branches of the unstable manifold of a saddle O_s (left); the unstable homoclinic orbit (middle) gives birth to two saddle cycles (dashed curves) (right). Middle row: sketches for the homoclinic bifurcation in the presence of sliding motion. A one-sided diagram (left) similar to that above; the homoclinic orbit tangent to the switching manifold (middle); the emergence of a stable period-2 limit cycle (red) with sliding motion fragments (pink) and two saddle limit cycles (dashed) (right). Bottom row: corresponding phase portraits. Figure modified from Belykh *et al.*[66]

VI. GRAZING, SLIDING, AND BOUNDARY-INTERSECTION CROSSING BIFURCATIONS

For non-smooth ODEs and hybrid systems (combining ODEs and maps), a bifurcation occurs when a limit cycle collides with a switching manifold. Such bifurcations can be grouped broadly into three classes, see Fig. 6. *Grazing bifurcations* are common in vibro-impacting systems and occur most simply when a non-impacting oscillatory solution grows to hit the impacting surface. *Sliding bifurcations* involve sliding motion, for example, a limit cycle may gain or lose a sliding segment.[69] Lastly, the limit cycle may, without additional codimension, collide with a switching manifold at a corner (kink)[70] or reach an intersection of switching manifolds (common in power converter models[71]). The Bristol book refers to these as *boundary-intersection crossing bifurcations*.

The local dynamics can be investigated by constructing and analyzing Poincaré maps, or stroboscopic maps in the case of periodically forced systems. A robust way to achieve this is to follow Nordmark's approach of composing of a smooth global map, which covers the reinjection, with a local piecewise-smooth map, termed a *discontinuity map*, that incorporates the effect of the switching manifold. The composed map is piecewise-smooth and the bifurcation occurs when the fixed point that corresponds to the limit cycle collides with the switching manifold of the map. In the context of the map, the bifurcation is termed a *border-collision bifurcation*.

A large portion of the Bristol book (Chaps. 6–8) is dedicated to deriving the form of such maps for different types of grazing, sliding, and boundary-intersection crossing bifurcations. In most cases, one piece of the map admits a Taylor series expansion, while, as a consequence of a quadratic tangency between the grazing trajectory and the switching manifold, the other piece of the map has an expansion in terms of powers of the square root of the displacement from the

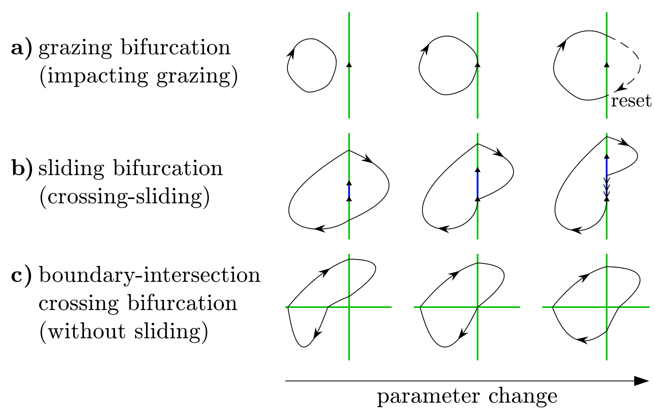

FIG. 6. Sketches illustrating a grazing bifurcation, a sliding bifurcation, and a boundary-intersection crossing bifurcation. In each case, a limit cycle (or more generally a distinguished trajectory[69]) encounters a fold (black triangle) or intersection of switching manifolds. There are several different types of bifurcations within these three classes depending on the local geometry of orbits. Names for the particular bifurcations shown here are indicated in brackets; for details refer to the Bristol book.

switching manifold. As shown in Chap. 4 of the Bristol book, one-dimensional square-root maps can have chaotic attractors. In higher dimensions, square-root maps can behave in a one-dimensional fashion, but this does not prove they exhibit chaos. Proofs of chaos have now been achieved for two-dimensional square-root maps in an ergodic sense[72] and a topological sense.[73] Also, the existence of a Smale horseshoe has been established under certain conditions.[74]

As normal forms, truncated square-root maps such as the Nordmark map

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \tau x_n + y_n - \chi H(x_n)\sqrt{x_n} \\ -\delta x_n + \mu \end{bmatrix} \tag{5}$$

capture the local dynamics of a system in a neighborhood of a grazing bifurcation. However, it is now well recognized that the size of this neighborhood is often hopelessly small. A recent study[75] of a prototypical impact oscillator model, using parameter values based on experiments,[76,77] found that when the model is interpreted as a perturbation of (5), a period-doubling bifurcation of a period-2 solution occurs at $\mu \approx 0.000\,18$. Consequently, the neighborhood of $\mu = 0$ in which the Nordmark map reproduces the local dynamics qualitatively does not extend past this value. There is a pressing need to derive extensions to (5) that better capture near-impact dynamics. Some advances have already been made in this direction.[78–82]

VII. BORDER-COLLISION BIFURCATIONS

Piecewise-smooth Poincaré maps that do not have a square-root term are usually piecewise-linear to leading order. Such maps are most interesting when they are non-differentiable, otherwise fixed points simply pass through border-collision bifurcations without a change to their stability. If the map is continuous, it can be truncated and transformed into the piecewise-linear form

$$x_{n+1} = \begin{cases} A_L x_n + b\mu, & c^{\mathsf{T}} x_n \le 0, \\ A_R x_n + b\mu, & c^{\mathsf{T}} x_n \ge 0, \end{cases} \tag{6}$$

where A_L and A_R are $d \times d$ matrices and $b, c \in \mathbb{R}^d$. The assumption that (6) is continuous on the switching manifold implies A_L and A_R differ by a rank-one matrix.

The dynamics created in border-collision bifurcations of (6) are the subject of Chap. 3 of the Bristol book. Here, they reproduce the results of Feigin[5,6] that show there are at most five cases for the number of fixed points and period-2 solutions that exist on each side of a generic border-collision bifurcation. However, due to the constraint on the difference between A_L and A_R, one of these cases cannot occur.[83] It is simply not possible for two fixed points to exist on one side of a generic border-collision bifurcation and a period-2 solution to exist on the other side of the bifurcation.

Shortly after 2008 the codimension-two coincidence of border-collision bifurcations with either saddle-node, period-doubling, or Neimark–Sacker bifurcations were unfolded in a general setting.[84,85] The results show that in two-parameter bifurcation diagrams curves of saddle-node bifurcations emanate tangentially from curves of border-collision bifurcations, whereas curves of period-doubling and Neimark–Sacker bifurcations emanate transversally, Fig. 7.

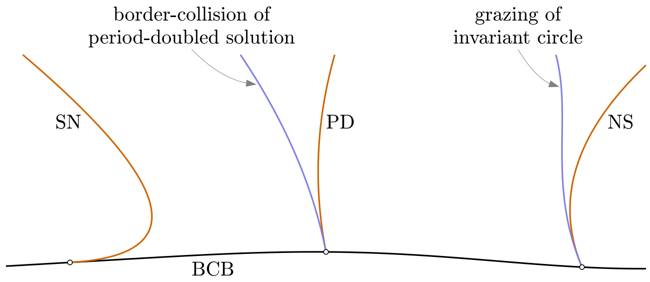

FIG. 7. Sketches of codimension-one bifurcation curves near codimension-two points at which a non-hyperbolic fixed point undergoes a border-collision bifurcation (BCB: border-collision bifurcation; SN: saddle-node bifurcation; PD: period-doubling bifurcation; NS: Neimark–Sacker bifurcation). The period-doubled solution created in the period-doubling bifurcation collides with the switching manifold on a curve that is tangent to the period-doubling curve at the codimension-two point (and similarly for the invariant circle created in the Neimark–Sacker bifurcation).

It has long been observed that periodicity regions of piecewise-linear maps exhibit a distinctive "sausage-string" structure in two-parameter bifurcation diagrams. This is illustrated in Fig. 8 for the following integrate-and-fire neuron model with square-wave forcing:

$$\begin{gathered} \dot{V} = -V + I + A\,\mathrm{sgn}\big(\sin\big(\tfrac{2\pi t}{T}\big)\big), \\ \text{if } V = 1 \text{ then} V \mapsto 0 \end{gathered} \tag{7}$$

studied by Tiesinga[86] and later Granados *et al.*[87] The voltage $V(t)$ is reset to 0 whenever it reaches the value 1 and corresponds to the neuron firing, while the forcing models the pulsatile release of hormones, for example. The stroboscopic map of (7) is a piecewise-linear circle map due to the combined effect of the reset law and discontinuous forcing. The theory of circle maps can be used to establish the uniqueness and continuity of the rotation number, which relates directly to the average firing rate.[88]

The sausage-string structure is characterized by the presence of *shrinking points*[89] where periodicity regions have zero width. Recent asymptotic results for maps of the form (6) characterize the geometry of the periodicity regions near any generic shrinking point.[90,91] In particular, the results show how the thickness of the regions can differ near different shrinking points.

Also, it has long been known that border-collision bifurcations can create multiple attractors simultaneously.[92,93] It is now known that in fact any number of attractors can be created.[94–96] Multiple chaotic attractors can be created,[97–99] and chaotic attractors can be high-dimensional.[100–104] Chaotic attractors are often created robustly in the sense that if some instance of (6) exhibits a chaotic attractor, then so does (6) for any sufficiently small perturbations to the entries of A_L, A_R, b, and c.[105–109] The chaotic attractor also persists when the higher-order terms that were removed to obtain (6) from a mathematical model are added back in. This was recently used to prove that chaos is created in a prototypical power converter model, which has been conjectured for some time.[110] Other studies have explored stronger notions of robustness, such as the continuity of the attractor with respect to Hausdorff metric[111] or with respect to

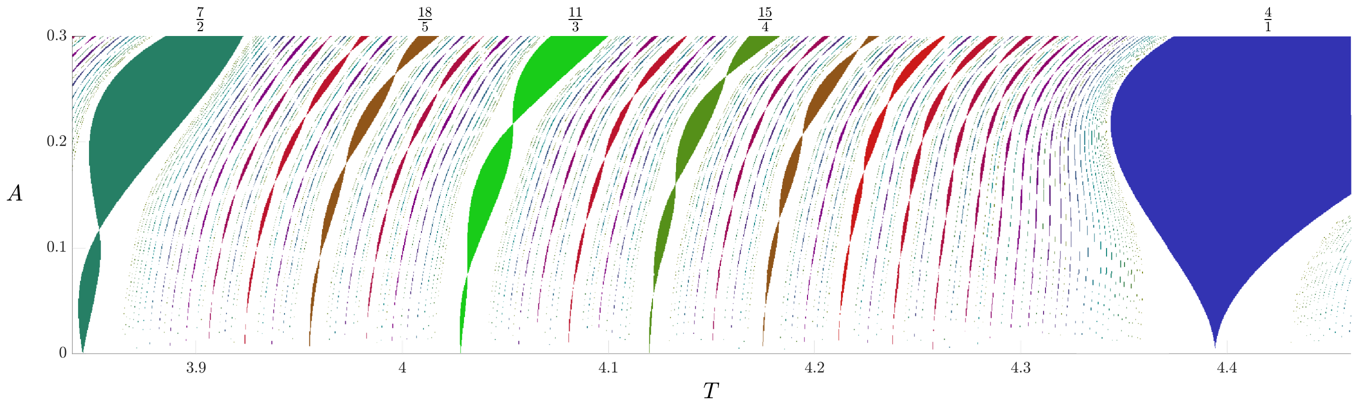

FIG. 8. Mode-locking regions of the integrate-and-fire model (7) with $I = 1.5$. Any periodic solution of (7) involves p firings per q periods (of length T) for some $p, q \geq 1$, making for an average firing rate of $\frac{p}{q}$. Each sausage-string corresponds to a fixed irreducible fraction $\frac{p}{q}$, and here these are shown for all $q \leq 30$. The values of some fractions are indicated.

measure.[112] While the robustness of chaotic attractors provides a stark difference to chaotic attractors of smooth systems created in period-doubling cascades for which periodic windows are dense in parameter space,[113] it is important to recognize that robust chaos does generically occur in smooth systems with sufficiently many dimensions.[114–116]

VIII. STOCHASTICS

A variety of questions have been explored with newly developed techniques in the stochastic context. Buckdahn *et al.*[117] studied how stochastic solutions converge to Filippov solutions in the zero-noise limit. The effects of randomness on orbits with sliding segments in canonical relay control systems were also studied via a combination of boundary layer analyses.[118–120] It was shown how these analyses determine statistics for the entry, sojourn, and exit dynamics of random trajectories near sliding segments of an underlying deterministic orbit. More recently, Hill *et al.*[121] described the paths that random trajectories are most likely to take by using a path integral framework.

Noise may dominate steep nonlinearities in the equations of motion. This was demonstrated in a model of a prototypical oscillator subject to friction[122,123] suggesting that the Stribeck effect could be ignored if there is sufficient noise. There are a variety of approaches for analyzing friction models in the presence of noise, including path integral methods, forward and backward Kolomogorov equations, and spectral exploration of Langevin dynamics.[124–127] Another study[128] uses a Fokker–Planck approach together with the Kramers' escape rate to compare numerical and experimentally observed transitions in a non-smooth Duffing-type circuit. The interplay of stochastic forcing and parametric noise with time-periodic delays in an act-and-wait control model has been studied analytically,[129] developing densities for the eigenvalues of the matrices characterizing the dynamics over an act-and-wait cycle and contrasting key statistics for the state variables in both act and wait periods.

For piecewise-linear maps with a randomly varying switching value, conditions for attractors and instabilities were obtained by Glendinning.[130] For square-root (Nordmark-type) maps (see Sec. VI) with noise, invariant densities may be approximately Gaussian, in which case analytical approximations are available, or highly skewed.[131] In the context of single-degree-of-freedom impact oscillators, different noise sources yield different stochastic maps that display fundamentally different densities and dynamics.[132] More recent works of Staunton and Piiroinen[133–136] provide some understanding for the destabilization of periodic states via the non-monotonic influence of noise in a square-root map. Their analyses include the derivation of stochastic zero-time discontinuity maps that track boundary interactions and their influence on the resulting dynamical sequences. This approach is valuable for efficient computation, model reduction, and an overall understanding of stochastic dynamics that cross-switching surfaces. Stochastic effects from noisy forcing and random components in the discontinuity boundaries were also studied in the contexts of the basins of attraction, multi-stability, deviation of trajectories, and loss or gain of stability for certain (nearly) periodic orbits. Also, Rounak and Gupta[137] propose computational measures for stochastic bifurcations and shifts in basins of attraction for a harmonically excited bilinear impact oscillator with a soft barrier. With a random component included in the excitation, qualitatively different dynamical behaviors appear for different parameter values due to the presence of multiple underlying attractors.

Lawley considered a series of theoretical and applied (usually biological) scenarios with stochastic switching and random environments, with earlier work[138] providing equations for relevant statistics and distributions. Later work includes neural and molecular systems with stochastic gating and both theoretical and applied work with switching in diffusion and cellular structures.[139,140]

IX. MULTI-SCALE

The analysis of slow–fast or singularly perturbed settings for non-smooth systems has considered reduced slow manifolds in a number of settings.[14] In some simpler settings, boundary equilibrium bifurcations appear,[141,142] while others consider more complex structures such as canards. A wealth of complex phenomena arise for canards in piecewise-linear systems,[25,143–151] including mixed-mode oscillations, Hopf-like bifurcations, bursting, and super-explosions, with various references indicating similarities and differences with analogous canards in smooth systems. Regularization via geometrical singular perturbation theory of a stiction oscillator model exhibiting canards was recently used to resolve the non-uniqueness of solutions.[152] Also Cardin and others[153,154] have studied the stability and persistence of typical singularities and periodic orbits in singularly perturbed Filippov systems.

Recently, new approaches have been developed for dynamic bifurcations in non-smooth, non-autonomous systems, where a parameter varies slowly through a critical point (bifurcation). As in smooth dynamics, this slow dynamic variation results in a lag in the critical transition or tipping value, but with different functional dependence on the rate parameter. For a non-smooth fold-type dynamic bifurcation, Budd *et al.*[155] obtain expressions for the tipping value that captures the competition between this lag with the advance driven by external oscillatory forcing. Slow passage through Hopf-like bifurcations has also been studied.[156] Their analysis captures features analogous to the smooth counterpart, illustrating how non-smooth systems can facilitate a somewhat simplified approach through canonical (linear) slow manifolds. They give conditions for the dynamic Hopf-like bifurcation, connecting zones of linearity with attracting and repelling slow manifolds. This yields a full description of the way-in/way-out functions that describe transitions, such as in bursting-type phenomena.

A series of works have studied a vibro-impact nonlinear energy sink based on an impact pair, a ball moving within a cavity in a larger externally forced mass, using a multi-scale analysis to analytically capture the reduced envelope dynamics of a regular periodic orbit of alternating impacts on either end of the cavity.[157,158] Recent semi-analytical solutions for the full system with feedback yield different families of periodic orbits and their eigenvalues, providing the validity and conditions for the multi-scale reduction.[159]

X. SWITCHED SYSTEMS

Dynamical systems with time-dependent switching arise naturally as models in many research fields, including physics, biology, and engineering.[160] For example, the temporal patterning of interactions in active matter discontinuously evolves as their comprising units change their spatial organization.[161–164] Similarly, synchronized patterns in brain networks emerge from pulsatile interactions between spiking and bursting neurons.[165] Integrate-and-fire networks have proven to be remarkably useful for analyzing synchrony in pulsatile neural networks. Such integrate-and-fire systems combine two types of discontinuities emerging from the intrinsic reset process and jump interactions. These discontinuous processes can occur simultaneously, thereby leading to an ordering problem[166] that has no direct analogs in smooth oscillator networks. The tools developed for overcoming this problem and proving the stability of neural synchronization include network saltation matrices[166] and the generalized master stability function.[167] However, in the broader context, a remaining open question is whether switching the coupling between agents can trigger synchronization in a network of piecewise-smooth oscillators. Progress has been made in addressing this question via distributed discontinuous coupling.[168]

Non-smooth switching dynamics is also a key property of various engineering systems, such as power converters and packet-switched communication networks.[169,170] Of particular interest are *blinking* networks,[171–173] in which connections switch on and off randomly, modulating the ability of the collective dynamics of the interacting nodes. Different aspects of synchronization and consensus in stochastically blinking networks of continuous-time[171–174,174–178] and discrete-time[179–185] oscillators have been studied in the fast-switching limit where the dynamics of a stochastically switching network is close to the dynamics of a static network with averaged, time-independent connections (see also the review by Belykh *et al.*[186]). Beyond synchronization, a rigorous theory for the behavior of stochastic switching networks of continuous-time oscillators in the fast switching limit was developed in Hasler *et al.*[187,188] These studies have clarified counter-intuitive relationships between the stochastic network and its time-averaged counterpart, where the dynamical law is given by the expectation of the stochastic variables.

Beyond fast switching, a number of studies documented a strong sensitivity of dynamics to the switching frequency.[189–195] For example, non-fast switching connections yield a plethora of unexpected dynamical phenomena, including bounded windows of intermediate switching frequencies (*windows of opportunity*) in which synchronization becomes stable even though the network switches between unstable modes.[178,196–198] Another striking discontinuity-induced effect is the ability of switching systems that alternate between stable modes to display unstable behavior at select switching frequencies.[199–201] Our understanding of dynamical systems and networks with non-fast switching connections is elusive, and even simple planar systems can defeat our intuition. As an example, we briefly examine the variation of a Stuart–Landau oscillator considered by Porfiri *et al.*,[201] which alternates between two modes according to a binary signal $s(t)$. The system periodically switches at period T with a duty cycle $\delta = 0.5$ so that the switch is on for $T/2$ units of time and is off for $T/2$ units of time. Using a complex representation of the form $z(t) = x(t) + iy(t)$, we have

$$\dot{z} = \begin{cases} (1 + i\omega)z + i\omega \dfrac{z^3}{|z|^2}, & s(t) = 0, \\ (1 + i\omega)z - i\omega \dfrac{z^3}{|z|^2}, & s(t) = 1, \end{cases} \tag{8}$$

where ω is the radian frequency.

The system is non-differentiable at the origin, which acts as a hybrid fixed point. Although unstable, the origin has features of an unstable node and a stable focus, see Fig. 9. A trajectory will first rotate about the origin approaching it (like a stable focus) and then will diverge along the principal unstable direction (like an unstable node). The principal unstable directions of the two modes are orthogonal, causing the emergence of non-smooth dynamics during each switching event. The interplay between the antagonistic

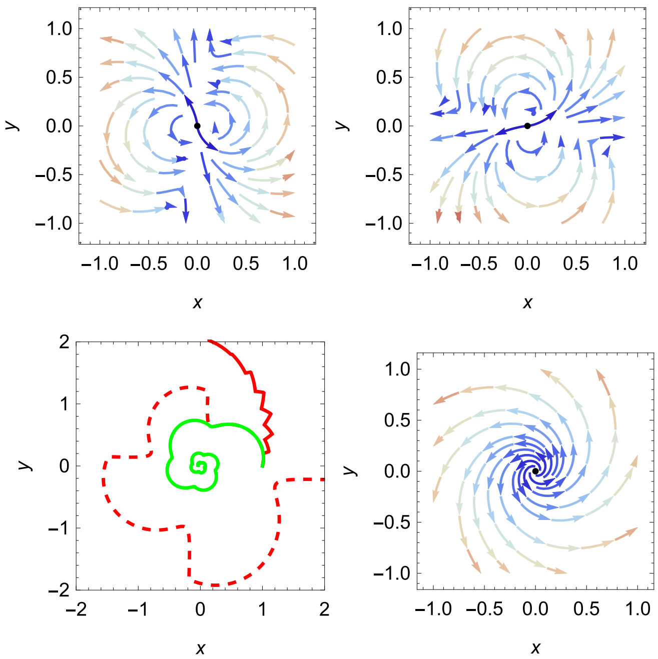

FIG. 9. (a) Streamlines of the two unstable modes of the switched system (8) for $\omega = 2$, with color indicating the intensity of the vector field for $s = 0$ (left) and $s = 1$ (right). (b) Trajectories of the periodically switched system in (8) for $\omega = 2$ and $\delta = 0.5$ and $T = 0.1$ (solid red), $T = 2$ (green), and $T = 5$ (dashed red) with initial condition $(1, 0)$ (left); streamlines of the averaged system in (8) for $T \ll 1$ (right). Figure modified from Porfiri *et al.*[201]

characteristics of the modes underpins the stability of the switched system.

With $T \gg 1$, the dynamics of the switched system is unstable as it would spend a large fraction of time in one of the unstable modes before it could switch to the other.[190] After spiraling toward the origin for almost an entire quadrant, each trajectory will approach the principal unstable direction of the mode and travel away from the origin. Switching will cause the trajectory to experience a sudden turn and after each period, the distance from the origin will increase [see the dashed red line in Fig. 9(b)(left)]. Similarly, the trajectory will also escape to infinity in the fast-switching case $T \ll 1$. This result can be explained by examining the averaged system $\dot{z} = (1 + i\omega)z + i(1 - 2\delta)\omega\frac{z^3}{|z|^2}$, obtained from (8) with $T \ll 1$. Similar to the individual modes, all the trajectories of the averaged system, except for the origin, spiral out to infinity [Fig. 9(b)(right)]. Surprisingly, there is a wide range of intermediate switching periods that induce the stability of the switched system so that the origin becomes a stable focus [see the green line in Fig. 9(b)(left)]. The stabilization mechanism relies on switching slowly enough to be attracted toward the origin during the rotation, but fast enough to avoid reaching the unstable principal directions. Rigorous proof for the existence of this antiresonance window is given in Porfiri *et al.*[201]

XI. OTHER APPLICATIONS

The Bristol book made plain the fact that discontinuities arise in seemingly every area of science. Indeed, we believe it has spurred researchers in diverse disciplines to employ non-smooth models to help answer their research questions. Several applications have been mentioned above; in this section, we survey several more.

Reduced or conceptual climate models regularly employ switches as simple descriptions for transitions that are fast relative to longer time scales typical for climate dynamics. Examples that have been analyzed dynamically and computationally are Budyko-type models for glacial cycles that use a discontinuous function for the albedo,[202–205] energy models for Arctic ice melt with a discontinuity capturing the transition to an ice-free state,[206,207] PP04 models for glaciation–deglaciation events[208] analyzed recently as a non-smooth system,[209] and threshold models for rainfall and convection.[210] Bistability in Stommel-type models for ocean circulation, where the relative size of temperature and salinity is the basis for a switch,[211] has been studied using stochastic methods[212,213] and multi-scale methods.[155] Figure 10 illustrates the non-smooth orbits that can be observed in a periodically forced Stommel model, near transitions from salinity- to temperature-dominated behaviors. It also shows how noise-driven tipping between these states is advanced as the regularity of the underlying fold bifurcation is reduced.

Non-smooth ODEs (particularly Filippov systems) have been used as models for population dynamics since at least the work of Gause *et al.*[214–217] in the 1930s, but it seems that they have been used more widely to model ecological systems only since around 2008 and the work of Dercole and others.[43,218,219] Such models, that are non-smooth when they assume a species switches between

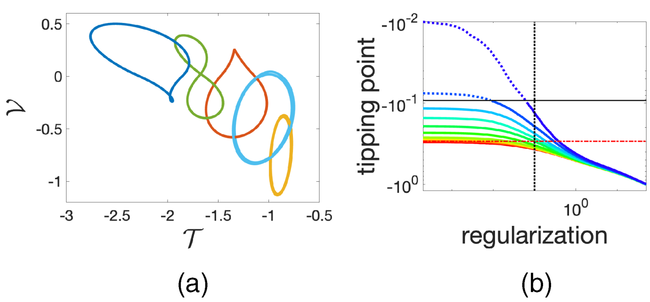

FIG. 10. Left panel: Limiting orbits in the phase plane for a periodically forced Stommel-type ocean circulation model, which can cross the switching surface $\mathcal{V} = 0$ between temperature (rescaled $\mathcal{T}$)- and salinity ($\mathcal{T} - \mathcal{V}$)-dominated behavior. Curves from right to left correspond to a decreasing fresh water forcing parameter. Right panel: Log–log plot (shifted) of noise-driven tipping—the departure from a salinity-dominated branch of equilibria—as a function of the regularity of the underlying static fold on this branch in a 1D Stommel-like model. Additive white noise ranges from zero noise (red curve) to the largest noise (violet curve), greater than the square root of the slow rate (black vertical dotted line) of the variation of the bifurcation parameter that passes dynamically through the static fold point. Dotted (solid) curves indicate an advance (lag) of tipping, where the noise-induced advance exceeds (is less than) the lag due to a dynamic bifurcation parameter. Dashed-dotted red line: asymptote of noise-free tipping (dynamic bifurcation) near a non-smooth fold. The black solid line is the cut-off between advance and lag relative to the static fold.

different habitats or food sources,[220] use a Hollings Type I functional response,[221] or, in the case of pest control, assume that control measures are applied only when some measure of the pest population exceeds a threshold.[222–226] Similarly, the plant disease model of Zhao *et al.*[227] assumes infected plants are removed only when their number is sufficiently high. Models of diseases and epidemics may be non-smooth due to an assumption that control measures (such as mask use and limitations on gatherings) are applied only when case numbers exceed a threshold.[228–230] Non-smooth models have also been used for flu epidemics[231] and HIV therapy.[232] Recent work on switched models of sleep/wake cycles employs a multi-scale analysis to obtain discontinuous circle maps in the slow–fast limit, which are used to explain the behavior of sleep patterns in relation to circadian rhythms.[233–235]

Several studies of single DoF vibro-impact systems, such as balls bouncing on moving surfaces and pendulums impacting barriers, appeared in the 1990s and 2000s and are documented in the Bristol book. More recently multi-DoF vibro-impacting systems have been studied,[157,236–238] and there are now books[239,240] and at least one review paper.[241] Complex bifurcation sequences are discussed in recent analytical work on vibro-impact pairs[242] and degenerate bifurcations for single DoF impact oscillators show fold and/or period-doubling bifurcations coinciding with the grazing point.[243] Combined analytical and numerical studies[240] illustrate the potential for complex behaviors and transitions with various sequences of smooth and non-smooth bifurcations across canonical models of single and multiple DoF systems, while sequences of period-doubling and grazing bifurcations in a vibro-impact-slider pair have also been studied.[244] Mason and Piiroinen[245,246] focused on the vibro-impact dynamics of gears whose motion resembles that of a point mass impacting two moving barriers. They explored gaps between chaotic regions and various codimension-two bifurcations involving grazing. The advances in studying non-smooth phenomena in more complex settings has supported closer comparisons between analytical and experimental work, such as in rotors,[247] mechanisms for targeted energy transfer,[248] and soft impact systems.[249]

Non-smooth systems have been extensively used in neuronal modeling.[250] At the individual neuron level, piecewise approximations are common practice for deriving reduced, analytically tractable models of excitable cells, including integrate-and-fire,[251–253] spiking,[254,255] and neural mass models.[256] At the network level, chemical synapses are often modeled via pulsatile on–off coupling, which sharply activates upon the arrival of a spike from a pre-synaptic cell.[257] Such excitatory and inhibitory networks have dynamical and synchronization properties[166,252,258–263] that are drastically different from models with gap junctions described by smooth functions.[264] Also, models of gene networks can be usefully simplified when Hill functions are replaced by Heaviside functions.[265]

The ubiquity of thresholds and affine functions in machine learning (ML) neural networks has also motivated studies of the dynamical behavior of ML algorithms with these elements. There have been a range of computational and theoretical studies related to their convergence and stability, such as the bifurcation analysis of maps for recurrent neural networks,[266] dynamical analysis of piecewise-linear ReLu networks,[267] and algorithm design based on the influence of threshold-induced chatter.[268] Of course, there remain many open questions related to their dynamical behavior.

Non-smooth systems naturally model various bio-mechanical applications, including animal, human, and robot locomotion. These include vibro-impact capsules[269,270]—self-propelled mechanisms driven by an autogenous force that move progressively in a resistive medium.[271] Such devices have the potential to be used for controllable endoscopic procedures.[272] The on–off “drift and act” control strategy for human balance[273] has been studied with delayed feedback to capture physiologically relevant sensory dead zones and switching control.[274] Milton and Insperger’s extensive work in this area is collected in a recent book[275] and a bifurcation analysis[276] has been performed for a data-inspired model of Asai *et al.*[277] Non-smooth inverted pendulum models also have been used successfully to capture the lateral motion of pedestrians in response to ground movement.[278–283] These models are based on the parsimonious assumption that walking is fundamentally a process in which the stance leg acts as a rigid strut, causing the body’s center of mass to act like an inverted pendulum in the frontal plane during each footstep. Rather than fall over, the step ends when the other leg strikes the ground and, ignoring the brief double-stance phase seen in realistic gaits, the pedestrian switches to an inverted pendulum on that leg. Such models have been used to study bidirectional interactions between pedestrians and a lively bridge. These interactions can yield complex dynamics, including bistable pedestrian gaits[281] and the emergence of bridge instability without crowd synchronization.[283]

XII. CONTRIBUTIONS TO THIS FOCUS ISSUE

Guo and Luo[284] study periodic motions and homoclinic orbits in a discontinuous dynamical system on a single domain with two vector fields. This simple discontinuous dynamical system has energy-increasing and energy-decreasing vector fields. The authors derive analytical conditions of bouncing, grazing, and sliding motions at the two energy boundaries and suggest a procedure for constructing complex motions of interest in engineering applications.

Zhusubaliyev *et al.*[285] contribute to the theory of bifurcations of closed invariant curves in piecewise-smooth maps. The authors discuss a border-collision bifurcation of a repelling resonant closed invariant curve (a repelling saddle-node connection) colliding with the switching manifold by a point of the repelling cycle. This bifurcation is unique to non-smooth systems and leads to the creation of a new attractor as well as a new repelling closed invariant curve.

Su *et al.*[286] propose a universal non-smooth coordinate transformation for general bilateral rigid vibro-impact systems. The essence of this coordinate transformation is the mirror image of state variables and non-uniform stretching of velocity, making the trajectories remain continuous in auxiliary phase space. The developed transformation especially has no requirements on whether the positions of the barriers are symmetrical and whether the restitution coefficients of the barriers on each side are consistent.

Ghosh and Simpson[287] study the border-collision normal form having an attractor satisfying Devaney’s definition of chaos. This strengthens existing results on the robustness of chaos in piecewise-linear maps. The authors show that the stable manifold of a saddle fixed point, despite being a one-dimensional object, densely fills an open region containing the attractor. The authors also identify a heteroclinic bifurcation at which the attractor undergoes a crisis and may be destroyed.

Llibre and Teixeira[288] analyze limit cycles of discontinuous differential systems formed by two pieces of linear differential centers separated by a circle. The authors prove that these differential systems can have at most three limit cycles and that there are differential systems having exactly zero, one, two, and three limit cycles. This line of research is in part motivated by Hilbert's 16th problem which remains unsolved.

Nagy and Insperger[289] study the application of feedback control concepts in a mechanical model of human stick balancing. The authors provide a control model by creating a transition between delayed state feedback and predictor feedback in the presence of sensory dead zones. The resulting mismatched predictor feedback, while being a relatively simple model, generates non-smooth dynamics that resembles actual human stick-balancing characteristics, such as limitations caused by delayed reactions.

Li *et al.*[290] extend the Melnikov method for proving homoclinic orbits to hybrid systems with impulsive effects and noise excitation. The authors consider a non-smooth Hamiltonian system with a homoclinic orbit as an unperturbed system. The homoclinic orbit continuously crosses one switching manifold and then jumps across a second by an impulsive effect. In particular, the results demonstrate that changing the periodic excitation coefficient or noise intensity can induce or suppress chaos.

Zhang and Chen[291] study a non-smooth model for the interaction of two neurons, where the non-smoothness enters as impulse effects. The model is described by a modified Smale horseshoe and represented by a one-sided symbolic dynamical system. The authors stress that simple neuronal models can present extremely complex chaotic dynamics, which nevertheless can be rigorously analyzed.

Klinshov and D'Huys[292] study non-smooth dynamics and noise-induced switching in a phase oscillator with pulse-delayed feedback subject to two types of noise: additive phase noise acting on the oscillator and stochastic fluctuations of the coupling delay. Pulse-coupled oscillators are a well-known example of non-smooth dynamics since their state undergoes abrupt jumps when they receive pulses. The authors use an event-based approach to derive a stochastic map that describes the system evolution from one pulse to the next and offers a qualitative explanation for the switching.

Szaksz and Gabor[293] investigate the dynamics of a fuzzy controlled polishing machine when Stribeck-type damping is present between the tool and the workpiece, introducing a destabilizing effect. The authors use a one-dimensional piecewise-linear map to examine transient chaotic motion and derive closed-form expressions for the expected value of the kickout number and the corresponding standard deviation. These quantities provide useful information for engineers during the tuning process of the control parameters.

Costa *et al.*[294] develop a soft (compliant) impact model for a vibro-impact energy harvester, calibrating it with the relevant hard (instantaneous) impact model for large stiffness as a foundation for systematic comparisons of the compliant dynamics. For example, varying the natural frequency of the membranes that form the impact boundaries shifts the bifurcation structure for increased softness parameters. Complementary computational and analytical results reveal new stable and unstable periodic orbits, co-existing non-smooth behaviors, and symmetry-breaking bifurcations not captured by hard impact models.

Peng *et al.*[295] establish a multi-domain framework for analyzing and controlling switched electromechanical dynamics in servomotor systems including their transient phases. The switched electromechanical dynamics is derived from the individual models of internal DC motor, gear train, and H-bridge circuit. The authors introduce a combination of cycle averaging with piecewise analytical solutions of the non-smooth dynamics to handle different temporal scales from high-frequency electrical to low-frequency mechanical variables.

At the time of writing this review, we anticipate that additional papers may be included in the Focus Issue. The topics of these papers include (i) synchronization of networked systems in the presence of limited resources via edge snapping and (ii) the construction of a piecewise-smooth dynamical system with a double-scroll attractor whose existence and chaotic properties can be proven rigorously.

XIII. OPEN PROBLEMS AND CHALLENGES

The area of non-smooth dynamical systems is ripe with open problems and challenges. We hope this review, along with the collection of research papers included in this Focus Issue, will encourage and motivate junior readers to enter this field. Several open problems are described above and in the Focus Issue. Here, we list others we believe are central to this research domain.

While there have been significant advances in the analysis of global bifurcations and stochastic dynamics in non-smooth systems, arguably they are presently limited to specific settings. By developing novel methods and then capitalizing on emerging commonalities that appear within different frameworks, one should aim to build generalized methodologies for studying the global non-smooth dynamics of systems within uncertain or stochastic contexts. For instance, an important yet undeveloped research problem in the treatment of switched and impacting systems is an analytical understanding of the constructive role of non-smoothness in stabilizing and controlling their dynamics. Such instances include the emergence of stable ghost attractors,[188,192,296] windows of opportunity for synchronization in switched systems,[196–198] and improved performance of impacting mechanical systems.[11,297] Similarly, it would be extremely useful to develop a rigorous theory for understanding and controlling switching networks whose evolution is governed by a Markov process, instead of sequences of independent random variables. The machinery developed for synchronization in networks with non-fast Markov switching[298] could give clues on how to proceed.

While stochastic bifurcations and the influence of noise on critical transitions have been studied in a number of specific non-smooth settings and applications, more general results that capture the influence of noise on non-smooth phenomena are largely unexplored. For example, under what circumstances does noise enhance or disrupt grazing bifurcations? When might it play a regularizing role, causing a non-smooth system to exhibit dynamics similar to a smooth system? Under what circumstances might noise heighten the non-smooth behavior? Likewise, techniques for mapping out basins of attractions and global stability are important ingredients for understanding stochastic effects, yet general methodologies are limited or non-existent in non-smooth settings. Recently, new computational methods[137,299] explore characterizations of basins of attraction for stochastic impacting systems.

The sausage-string structure of periodicity regions shown in Fig. 8 is well-understood in the context of piecewise-linear maps,[90,91] but it arises more broadly: in the asymptotic behavior of heteroclinic networks,[300] in delay differential equations with pulsatile forcing,[301] and homogeneous maps.[302] It would be nice to explain the structure in a way that encompasses such alternate settings.

Llibre and Zhang[58] list several open problems regarding limit cycles. For example, two-dimensional piecewise-linear ODEs with a single linear switching manifold can exhibit three limit cycles, but, surprisingly (since analytical expressions are available for the flow in each piece of the system), it is not known whether or not they can exhibit four or more.

Solving the above problems will presumably require overcoming a number of technically challenging issues and even developing entirely new analytical methods. We believe future advances will demand a truly interdisciplinary approach, integrating nonlinear, stochastic, and computational methods. We hope this review will contribute to further igniting interest in non-smooth systems and promoting interdisciplinary collaborations.

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Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Igor Belykh: Writing – original draft (equal). **Rachel Kuske:** Writing – original draft (equal). **Maurizio Porfiri:** Writing – original draft (equal). **David J. W. Simpson:** Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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