When multilayer links exchange their roles in synchronization

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Real world networks are best described by the multiple layers of links whose interactions frequently lead to extraordinary collective dynamical behaviors, including synchronization. The fundamental problem of assessing how structural changes in interaction layers affect synchronization in multilayer networks remains open due to serious limitations of the existing stability methods. Towards removing this obstacle, we propose an approximation method which significantly enhances the predictive power of the master stability function and its extension, simultaneous block diagonalization, for stable synchronization in multilayer networks. For coupled Rössler oscillators, our method reduces the complex stability analysis to simply solving a lower-dimensional set of linear algebraic equations. Using the method, we analytically predict surprising effects due to multilayer coupling. In particular, we prove that two coupling layers - one of which would alone hamper synchronization and the other would foster it - reverse their roles when used in a multilayer network. We also analytically demonstrate that increasing the size of a globally coupled layer, that in isolation would induce stable synchronization, makes the multilayer network unsynchronizable.

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Introduction. Many natural and engineering networks contain units that are coupled through multiple interaction layers [1, 2]. Neurons are often coupled via excitatory, inhibitory, and electrical synapses whose co-action may lead to counterintuitive synergistic effects [3, 4]. Interpersonal interactions via multiple types of social networks also affect the speed and the scale of disease propagation [5]. Multiplex and multilayer networks can exhibit rich cooperative dynamics [1, 6-8], including complete synchronization [9-12], clusters of synchrony [13-15], explosive [16], inter-layer/intra-layer [17, 18], and relay synchronization [19]. The role of multilayer network topologies in promoting or hampering synchronization is significantly less understood, compared to single-layer networks [20-36]. Two hallmark methods, the master stability function [20] and the connection graph method [25, 26], are generally used to predict the stability of synchronization in a single-layer network. However, the predictive power of the master stability function [9-11] is severely impaired in multilayer networks. This is due to the fact that the connectivity matrices that represent interaction layers typically cannot be diagonalized simultaneously and thus their eigenvalues are not informative. The most successful application of the master stability function to multilayer networks was performed in [10]. This approach consists in simultaneous block diagonalization (SBD) of the connectivity matrices [10] that can reduce the problem of assessing synchronization in a large network to a smaller network. However, it remains a limited approach as the results of the reduction can remain difficult to analyze. Reductions typically result in networks with weighted positive and negative connections as well as self-loops.

An an alternative, the connection graph-based method for assessing the impact of multilayer network topology on synchronization was recently developed in [12]. This method connects the stability of synchronization with traffic loads on critical edges. However, this method is restricted to Type I oscillators such as Lorenz systems that retain synchronization for any coupling strengths exceeding a coupling threshold [25]. As a result, there is a critical gap in research methods that can explicitly relate the stability of synchronization to structural changes in multilayer networks of Type II oscillators [20, 37]. When coupled via a single-layer network, these oscillators remain synchronized only in a bounded region of coupling strength [20]. The $p$-coupled Rössler systems that are widely used as the test bed oscillators [20, 22, 32] for probing the master stability function belong to this Type II class. Therefore, to date, the synchronization properties of multilayer networks of Rössler and other Type II oscillators remain poorly understood and are typically studied on a case-by-case basis via full-scale simulations of all transversal Lyapunov exponents of the high-dimensional networked system [11].

In this Letter, we close this gap by offering an approximation method that can significantly improve the predictive power of the SBD methods [10, 15] or any other possible generalization of the master stability function. Studying multilayer networks of Rössler oscillators, we show that the contribution of the chaotic saddle-focus oscillator dynamics into the stability of synchronization can be approximated by a 2D linear focus system rather precisely. As a result, our method reduces time-varying stability equations to a lower-dimensional linear time invariant system whose stability can be analytically evaluated via the Routh–Hurwitz criterion. The application of this method analytically predicts counter-intuitive effects...
caused by multilayer coupling.

The multilayer network model. We consider a two-layer undirected network of $N$ Rössler oscillators

\[
\begin{aligned}
\dot{x}_i &= -y_i - z_i + \varepsilon_x \sum_{j=1}^{N} d_{ij}(x_j - x_i), \\
y_i &= x_i + ay_i + \varepsilon_y \sum_{j=1}^{N} g_{ij}(y_j - y_i), \\
\dot{z}_i &= b + (x_i - c)z_i, \quad i = 1, \ldots, N,
\end{aligned}
\]

(1)

where $a = 0.2$, $b = 0.2$, and $c = 9$ are the standard parameter values that yield chaotic behavior of uncoupled Rössler oscillators. $D = (d_{ij})$ and $G = (g_{ij})$ are $n \times n$ symmetric Laplacian connectivity matrices with zero-row sums and off-diagonal elements $d_{ij} = d_{ji}$ and $g_{ij} = g_{ji}$, respectively. These connectivity matrices $D$ and $G$ correspond to the $x$ and $y$ coupling layers, and non-zero $d_{ij} = 1$ and $g_{ij} = 1$ define links in the two layers. $\varepsilon_x$ and $\varepsilon_y$ are coupling strengths. Complete synchronization in network (1) is defined by synchronization manifold $S = \{x_1(t) = x_2(t) = \ldots = x_N(t) = s(t)\}$, where $x_1 = (x_i, y_i, z_i)$, and synchronous solution $s(t) = (x(t), y(t), z(t))$ is governed by the uncoupled Rössler oscillator. Our main objective is to determine the role exchange effect in the multilayer network model (1), i.e. synchronization is stable only in a bounded region of coupling ($\varepsilon_x^*, \varepsilon_y^*$), so that the coupling becomes desynchronizing for $\varepsilon > \varepsilon^*$. On the contrary, the $y$ coupling enables the single-layer network with Type I synchronization properties, so that the network is stably synchronized when $\varepsilon_y > \varepsilon_y^*$.

A puzzle: the role exchange effect. To illustrate the complexity of assessing multilayer connections in inducing or hindering the synchronization even in small networks, we consider the simplest three-node multilayer network (1) with one $x$ link $d_{12} = 1$ and one $y$ link $g_{23} = 1$, depicted in Fig. 1. Remarkably, two striking effects appear. First, the $x$ coupling switches its type, from Type II to Type I and supports stable synchronization for any sufficiently large values of $\varepsilon_x$, provided that $\varepsilon_y$ is in some range of intermediate coupling strength (see the dark horizontal stripe in Fig. 1). Second, the $y$ coupling switches its synchronizing role from Type I to Type II and destabilizes synchronization for sufficiently large values of $\varepsilon_y$, provided that $\varepsilon_x$ is sufficiently large. This puzzle calls for an explanation and ultimately motivates the development of an effective approach that can predict the stability bounds at which the layers reverse their synchronizing and desynchronizing roles.

Following the standard stability approach [20], we linearize three-node system (1) around synchronous solution

\[
\begin{aligned}
\dot{\xi}_{12} &= -\eta_{12} - \zeta_{12} - 2\varepsilon_x \xi_{12}, \\
\eta_{12} &= \xi_{12} + a\eta_{12} + \varepsilon_y \eta_{23}, \\
\dot{\zeta}_{12} &= z(t)\xi_{12} + (x(t) - c)\zeta_{12}, \\
\dot{\xi}_{23} &= -\eta_{23} - \zeta_{23} + \varepsilon_x \xi_{12}, \\
\eta_{23} &= \dot{\xi}_{23} + a\eta_{23} - 2\varepsilon_y \eta_{23}, \\
\dot{\zeta}_{23} &= z(t)\xi_{23} + (x(t) - c)\zeta_{23},
\end{aligned}
\]

(2)

where $\xi_{ij} = x_i - x_j$, $\eta_{ij} = y_i - y_j$, and $\zeta_{ij} = z_i - z_j$, $i = 1, j = 2$ and $i = 2, j = 3$ are transversal perturbations. Note that connectivity matrices $D = (d_{ij})$ ($x$ layer) and $G = (g_{ij})$ ($y$ layer) of this simplest two-layer network do not commute. Therefore, the master stability function [20] cannot be applied to diagonalize and decouple system (2) into two 3D systems whose stability would be controlled by the eigenvalues of the connectivity matrices. Technically, the simultaneous block-diagonalization [10] can handle this case of the non-commuting matrices; unfortunately, its application transforms the 6D system (2) into a more complex 6D system [43] without reducing its dimensionality. Therefore, one has to rely on numerical simulations of the full 6D system that offer little insight into the underpinnings of the role exchange effect.

Instead, we propose to constructively exploit structural
intrinsic properties of the Rössler oscillator to simplify
and transform stability equation (2) into an analytically
tractable, predictive tool. The synchronous trajectory
s(t) is governed by the Rössler system which exhibits
chaotic dynamics centered around a saddle-focus at
the origin. The origin is unstable in the xy-plane which
responds to the unstable focus part while the z direction
indicates the 1-D stable manifold of the saddle-focus [39].
The synchronous trajectory is an outward spiral which
spends most of the time on or close to the xy-plane and
then makes a large excursion along the vertical z di-
rection to return back to the xy-plane (Fig. 2). This
important property suggests that the overall transver-
sal stability of the synchronous trajectory is essentially
controlled by the focal part of the synchronous trajec-
tory that lies in the xy-plane. Our numerical calcu-
lations of the instantaneous Lyapunov exponent corre-
sponding to the transversal stability of the synchronous
solution confirm this claim and indicate that synchroniza-
tion becomes stable as long as the instantaneous transver-
sal Lyapunov exponent becomes negative along the focal
part of s (Fig. 2). Therefore, stability equation (2) can
be reduced to the linear system with constant coefficients

\[
\begin{aligned}
\dot{\xi}_1 &= -\alpha_1 \xi_1 - 2\epsilon_x \xi_1, \\
\dot{\eta}_1 &= \alpha_1 + \alpha_2 (N - R) \epsilon_y \eta_1 + \gamma \epsilon_y \eta_2, \\
\dot{\zeta}_1 &= z(t) \xi_1 + (x(t) - c) \zeta_1, \\
\dot{\xi}_2 &= -\alpha_2 \zeta_2, \\
\dot{\eta}_2 &= \alpha_2 + \alpha_3 (N - R) \epsilon_y \eta_2 + \gamma \epsilon_y \eta_1, \\
\dot{\zeta}_2 &= z(t) \xi_2 + (x(t) - c) \zeta_2,
\end{aligned}
\] (5)

by ignoring the \(\xi_{ij}\) perturbations corresponding to non-
zero values of \(z(t)\). The stability of linear system (3)
yields stable synchronization and can be assessed via the
characteristic equation

\[
\lambda^4 + \alpha_1 \lambda^3 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4 = 0,
\] (4)

where \(\alpha_1 = 2(\epsilon_x + \epsilon_y - a), \alpha_2 = a^2 - 4a \epsilon_x - 2a \epsilon_y +
4 \epsilon_x \epsilon_y + 2, \alpha_3 = 2(a^2 \epsilon_x - 2a \epsilon_x \epsilon_y - a + \epsilon_x + \epsilon_y),
\alpha_4 = -2a \epsilon_x + \epsilon_x \epsilon_y + 1.\) By the Routh-Hurwitz stability
criterion, all eigenvalues \(\lambda\) have negative real parts if
the principal diagonal minors of the Hurwitz matrix are
positive so that \(M_1 = \alpha_1 > 0, M_2 = \alpha_1 \alpha_2 - \alpha_3 > 0,
M_3 = \alpha_1 \alpha_2 \alpha_3 - a^2 \alpha_4 - \alpha_0 \alpha_2 > 0, M_4 = \alpha_4 > 0.\) Figure 1
shows that analytical stability bounds \(M_{1,2,3,4} = 0\) coincide
with the actual bounds remarkably well. More
specifically, the lower border of the stability region
(dark) in Fig. 1 is predicted by the curve \(\epsilon_y = 2a - \frac{1}{2}\)
that follows from \(M_4 = 0.\) The upper border of the
stability region in Fig. 1 is bounded by the upper
curve governed by the condition \(M_3 = 0\) and defined
by implicit function \(f(\epsilon_x, \epsilon_y) = 0 [44].\) Considered
together, these analytical curves effectively predict the
role exchange effect of the \(x\) and \(y\) coupling in stabilizing
and destabilizing synchronization and resolve the puzzle.

Larger networks. To further illustrate the predictive
power of our approximation method, we consider a well-
known example of a two-layer 2N-node network which
consists of two fully connected subnetworks with \(N\) nodes
within each subnetwork and \(R\) links between the sub-
networks, as demonstrated in Fig. 3. It was previously
shown that the variational equations for traversal sta-
bility of synchronization manifold \(S\) in this network can
be reduced via SBD to a two-node transversal mode and
(2\(N - 3\)) one-node transversal modes where the latter are
represented by two distinct sets of identical systems [10].
For the network of Rössler oscillators (1), the variational
equations for the two-node transversal mode are

\[
\begin{aligned}
\dot{\xi}_1 &= -\eta_1 - \xi_1 - 2\epsilon_x \xi_1, \\
\dot{\eta}_1 &= \xi_1 + \alpha_1 + (N - R) \epsilon_y \eta_1 + \gamma \epsilon_y \eta_2, \\
\dot{\zeta}_1 &= z(t) \xi_1 + (x(t) - c) \zeta_1, \\
\dot{\xi}_2 &= -\eta_2 - \zeta_2, \\
\dot{\eta}_2 &= \xi_2 + \alpha_2 + (N - R) \epsilon_y \eta_2 + \gamma \epsilon_y \eta_1, \\
\dot{\zeta}_2 &= z(t) \xi_2 + (x(t) - c) \zeta_2,
\end{aligned}
\] (5)

where \(\gamma = \sqrt{(N - R)R}\), and \((\xi_1, \eta_1, \zeta_1)\) and \((\xi_2, \eta_2, \zeta_2)\)
are transversal perturbations associated with eigen-like
modes 1 and 2. The variational equations for $2N - 2 - R$ identical one-node transversal modes $(\xi, \eta, \zeta)$, and $R-1$ identical one-node transversal modes $(\xi, \eta, \zeta)$ are given in [45] and [46]. To derive the variational equations, we used a numerical SBD algorithm [40] that yielded an outcome different from [10]; however, the stability argument is essentially the same. While the dimensionality reduction from the 2N-node network is significant, to assess the stability of synchronization, one has to numerically analyze the 6D variational system for the two-node mode system (5) and two 3D systems for each one-node system. As a result, the underpinnings of emergent stability and instability of synchronization as a function of intra- and interlayer connections and the network size remain difficult to identify. To resolve this problem, we apply our approximation method and transform the 6D system (5) into the 4-D system

$$
\begin{align*}
\dot{x}_1 &= -y_1 - 2\varepsilon_x \xi_1, \\
\dot{y}_1 &= x_1 + \alpha x_1 - (N - R)\varepsilon_y y_1 + \gamma \varepsilon_y \eta_1, \\
\dot{x}_2 &= \varepsilon_x, \\
\dot{y}_2 &= x_2 + \alpha x_2 - R\varepsilon_y y_2 + \gamma \varepsilon_y \eta_1.
\end{align*}
(6)
$$

The stability of this linear system with constant coefficients can be determined through the characteristic equation (4) with new coefficients $\alpha_1 = N\varepsilon_y + 2\varepsilon_x - 2/5$, $\alpha_2 = 2N\varepsilon_x \varepsilon_y - N\varepsilon_y/5 - 4\varepsilon_x/5 + 51/25$, $\alpha_3 = -2N\varepsilon_x \varepsilon_y + N\varepsilon_y + 5\varepsilon_x/5 - \frac{2}{5}$, and $\alpha_4 = 2N\varepsilon_x \varepsilon_y - 2R\varepsilon_x \varepsilon_y - \frac{2}{5} + 1$. Bounds for the stability of linear system (6) determined by minors $M_{1,2,3,4} = 0$ of the corresponding Hurwitz matrix are plotted in Fig. 3. Notice that analytical curves $M_1, M_2, M_3, M_4$ are given by minors $M_{1,2,3,4} = 0$ of the corresponding Hurwitz matrix that indicates that the stability of the focal part of the synchronous trajectory determined by a linear system implies the overall stability of synchronization. As a result, our approach reduces the dimensionality of the stability problem and replaces numerical calculations of Lyapunov exponents with a lower-dimensional set of linear algebraic equations amenable to analytical treatments. In particular, the application of this method reveals and analytically predicts a counterintuitive “role exchange” effect in which one layer coupling that would destabilize synchronization in a single layer network reverses its role in a two-layer network. At the same time, the other layer coupling, that would alone foster synchronization, becomes destabilizing. Our preliminary analysis indicates that this effect is not limited to Rössler oscillators and is common among other Type II saddle-focus oscillators, including multilayer networks of coupled tritrophic Rosenzweig-MacArthur models [38].

Beyond synchronization in multilayer networks, our approach exploiting the structural intrinsic oscillator properties can enable analytical stability treatment of cluster synchronization [13, 41] and synchronization in simplicial complexes [42] by further reducing typically multi-dimensional variational stability equations. Re-
turning to the celebrated master stability function originally developed for single-layer networks of Rössler oscillators [20], this reduction can also eliminate the need of numerical simulations, making the master stability function practically analytical.

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The simultaneous block diagonalization transforms variational equations (2) into the 6D coupled system

\[
\dot{\eta}_1 = -\eta_1 + \eta_2 - \eta_3 - \eta_5 + \eta_6,
\]

\[
\dot{\eta}_2 = \eta_1 + \eta_2 - \eta_3 + \eta_5 - \eta_6,
\]

\[
\dot{\eta}_3 = 2\eta_2 + 2\eta_3 - \eta_4 - \eta_5 + \eta_6.
\]

The equations (1) are then reduced to the following form:

\[
\dot{\xi}_1 = -\xi_1 + \xi_2 + \xi_3 - \xi_4 + \xi_5 - \xi_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6.
\]

\[
\dot{\xi}_2 = -\xi_1 + \xi_2 + \xi_3 - \xi_4 + \xi_5 - \xi_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6.
\]

\[
\dot{\xi}_3 = -\xi_1 + \xi_2 + \xi_3 - \xi_4 + \xi_5 - \xi_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6.
\]

\[
\dot{\xi}_4 = -\xi_1 + \xi_2 + \xi_3 - \xi_4 + \xi_5 - \xi_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6.
\]

\[
\dot{\xi}_5 = -\xi_1 + \xi_2 + \xi_3 - \xi_4 + \xi_5 - \xi_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6.
\]

\[
\dot{\xi}_6 = -\xi_1 + \xi_2 + \xi_3 - \xi_4 + \xi_5 - \xi_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6 - \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \eta_6.
\]