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Network Synchronization Through Stochastic Broadcasting

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***Abstract*—Synchronization is often observed in interacting dynamical systems, comprising natural, and technological networks; however, seldom do we have precise knowledge and control over the synchronous trajectory. In this letter, we investigate the possibility of controlling the synchronization of a network by broadcasting from a single reference node. We consider the general case in which broadcasting is not static, but stochastically switches in time. Through an analytical treatment of the Lyapunov exponents of the error dynamics between the network and the reference node, we obtain an explicit dependence of synchronization on the strength of the broadcasting signal, the eigenvalues of the network Laplacian matrix, and the switching probabilities of broadcasting. For coupled chaotic tent maps, we demonstrate that: 1) time averaging fails to predict the onset of controlled synchronization and 2) the success of broadcasting depends on the network topology, where the more heterogeneous the network is, the more difficult it is to control.**

***Index Terms*—Lyapunov exponents, random network, scale-free network, switching control, tent map.**

I. Introduction

COLLECTIVE behavior within networks has received a considerable amount of attention in the literature, from animal grouping to robotic motion [1], [2]. One type of collective behavior, synchronization, is particularly important in how prevalent it is in real-world systems [3]. Synchronization occurs when all of the nodes act in unison.

Synchronization typically emerges from local interactions within a network and is determined by the interplay between the node dynamics and the network structure [3]. A variety of mathematical tools have been proposed to assess the synchronizability of a network by studying local and global stability of the synchronous solution. The master stability function has been proposed as a powerful approach to predict the role of the individual node dynamics and the network structure in terms of a single master equation [4].

In many applications, there is a need for controlling the synchronous solution toward a designed common trajectory [5]. For example, schooling fish may align their swimming directions toward a group leader to improve foraging success or escape from a predator [6]. Similarly, teams of mobile robots are often tasked with the goal of coordinating their motion to maintain a given formation while sampling environmental variables and identifying potential targets [2].

In this letter, we study the feasibility of broadcasting from a common reference node to control synchronization of a network of coupled dynamical systems. Within this approach, every node in the network has access to the same information from the reference node at a given time, but we allow this information to stochastically change in time. For example, the reference node could share information with the network only sporadically in time or could alternate between several conflicting messages. With respect to schooling fish, for example, leaders may use sudden changes in swimming direction as a visual cue to elicit followership of the group [7].

The problem of controlled synchronization through broadcasting shares similarities with the literature on pinning control, where a control action is applied to a small selected subset of nodes to tame the dynamics of the entire network to the trajectory of a reference node [8], [9]. Building on the standard approach developed a decade ago [10]–[12], most of the literature considers the case in which pinning control is statically applied to the network, such that control gains are held constant in time. This is in contrast with the broadcasting approach, in which all nodes are controlled but for a limited fraction of time and with a varying gain. Only a few studies have explored the possibility of time-varying pinning control, but we still have limited analytical insight into the interplay between the internal nonlinear dynamics of the nodes, the evolution of the control gains, and the network structure [8].

Here, we seek to close this gap in the context of stochastic broadcasting. Toward this aim, we extend our recent work [13], [14], where we established a rigorous methodology for assessing the mean square stability of the synchronous solution in a pair of coupled discrete-time oscillators. Specifically, we derive a master equation for the mean square stability of the error dynamics, parametrized in terms of the eigenvalues of the graph Laplacian of the network. We apply ergodic theory [15] to study the Lyapunov exponent of the

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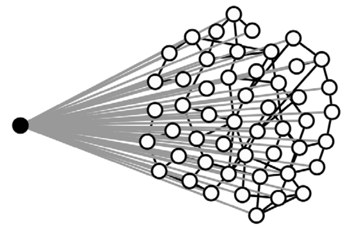

Fig. 1. Illustration of the problem: the reference oscillator, x, (black) stochastically broadcasts to a network of N oscillators, $y_1, \ldots, y_N$ (white).

master equation, which we specialize to chaotic tent maps. In an effort to compare our analysis with the state of the art on switching networks [16], we also challenge the practical use of time-averaging to predict the onset of stochastic synchronization of discrete-time oscillators.

II. Problem Formulation

We study the synchronization of a network of N coupled discrete-time oscillators ($y_i \in \mathbb{R}$, where $i = 1, 2, \ldots, N$), interacting with a reference oscillator ($x \in \mathbb{R}$) that broadcasts to the entire network, as shown in Fig. 1. The network nodes are interconnected by an arbitrary, undirected and unweighted, topology, which is represented by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set. The evolution of all the oscillators (network and reference nodes) is governed by the same nonlinear function $F : \mathbb{R} \to \mathbb{R}$. That is, in the absence of coupling, $y_i(k+1) = F(y_i(k))$ and $x(k+1) = F(x(k))$. The broadcasting process is independent and identically distributed (i.i.d.) in time, such that, at each time step k, the control gain of the reference node, $\varepsilon(k)$, is randomly drawn from a set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$ with probabilities $p_1, p_2, \ldots, p_m$, respectively (with $\sum_{i=1}^{m} p_i = 1$). As broadcasting is global, the value of the switching gain is common to all nodes. We say that the reference node and the network are synchronized if $y_1(k) = y_2(k) = \cdots = y_N(k) = x(k)$ for $k \in \mathbb{Z}^+$.

To summarize the setup of the problem, the system of equations governing the discrete-time evolution is

$$\begin{aligned} x(k+1) &= F(x(k)), \\ y_i(k+1) &= F(y_i(k)) + \varepsilon(k)(x(k) - y_i(k)) \\ &\quad + \sum_{j=1, ij \in \mathcal{E}}^{N} \mu\big(y_j(k) - y_i(k)\big), \end{aligned} \tag{1}$$

for $i = 1, 2, \ldots N$, where μ is a network coupling used to scale the node-to-node interactions versus the strength of the broadcasting signal. System (1) is a stochastic dynamical system whose study requires tools from both stability and ergodic theories [15]. More specifically, system (1) describes a switched nonlinear linear system, with an underlying memoryless switching.

In general, the network dynamics can be written in the following vector form

$$y(k+1) = F(y(k)) - \mu L y(k) - \varepsilon(k) I_N (y(k) - x(k) 1_N), \tag{2}$$

where $y(k)$ is the vector comprised of all the network states, $F(y)$ is the vector-valued extension of the mapping function $F(y)$, 1_N is the vector of ones of length N, and L is the Laplacian matrix of $\mathcal{G}$ [17], that is,

$$L = \begin{cases} L_{ij} = -1 & ij \in \mathcal{E} \\ L_{ii} = \sum_j 1 & i = 1, 2, \ldots, N \end{cases},$$

with eigenvalues $\gamma_1 = 0 \leq \gamma_2 \leq \cdots \leq \gamma_N$.

To understand the role of broadcasting in inducing synchronization with the reference trajectory $x(k)$, we look at the evolution of the error dynamics $\xi(k) = x(k)1_N - y(k)$. To study the stability of synchronization, we linearize the system about the reference trajectory:

$$\xi(k+1) = [DF(x(k))\, I_N - \mu L - \varepsilon(k)\, I_N]\xi(k), \tag{3}$$

where I_N is the $N \times N$ identity matrix and $DF(x(k))$ is the Jacobian of F at $x(k)$. In (3), we have assumed infinitesimal perturbations $\xi(k)$ in the directions transversal to the synchronous solution thereby, allowing for the linearization and application of the Jacobian DF. The linearized discrete-time system in (3) is a first order Markov chain, due to the presence of the switching gain $\varepsilon(k)$. Although $\varepsilon(k)$ is drawn from i.i.d. distribution, (3) describes a linear time-varying switching system, since the reference trajectory $x(k)$ generally varies in time. Stochasticity and time-dependence, however, only appear as a compound multiplier of the identity matrix, thereby affording the possibility of diagonalizing the system in terms of the eigenspaces of the Laplacian matrix. In other words, to a first linear approximation, the error dynamics on the eigenspaces of the Laplacian evolve independently of each other, allowing for the use of a single stochastic master equation.

Specifically, diagonalizing (3), we obtain the following scalar equation for each transversal mode

$$\zeta(k+1) = \big[DF(x(k)) - \mu\gamma - \varepsilon(k)\big]\zeta(k), \tag{4}$$

where $\gamma \in \{\gamma_1, \ldots, \gamma_N\}$ encapsulates the role of the network topology on the evolution of the error dynamics along the eigenvectors of L, identified by $\zeta(k)$. Master equation (4) can be parametrically studied as a function of $\mu\gamma$ to illuminate the influence of the strength of the broadcasting signal and the switching probabilities of broadcasting on the stability of synchronization along each eigenvector of L. This equation reduces to the traditional, deterministic master stability equation in [4] in the absence of stochastic broadcasting ($\varepsilon(k) = 0$).

III. Master Stability Function

While there are many criteria that one can contemplate when examining stochastic stability of a network about a synchronous solution, we use the lens of mean square stability for its practicality of implementation and inclusiveness with other criteria [18], [19]. For example, as shown in [20], mean square asymptotic stability of switching linear systems with an underlying time-homogenous finite-state Markov chain is equivalent to exponential second moment stability and implies almost sure stability. Several studies have demonstrated the feasibility of using mean square stability in the study of synchronization of discrete-time systems, and we build on this literature toward an analytical treatment of broadcasting [21], [22]. Through this lens, the error dynamics is controlled by both the mean and the variance of the switching signal, different from fast-switching approaches in continuous-time systems that only rely on the mean [23], [24].

Definition 1: The synchronous solution $y_1(k) = y_2(k) = \cdots = y_N(k) = x(k)$ in the stochastic system (1) is locally asymptotically mean-square stable, if $\lim_{k\to\infty} E[\xi^2(k)] = 0$ for any initial condition $\xi(0)$ of (3), where $E[\cdot]$ denotes expectation with respect to the σ-algebra generated by the stochastic process of switching.

Analyzing mean square stability of the stochastic system (3), and therefore of the master equation (4), corresponds to studying the deterministic evolution of the second moment of $\zeta(k)$ toward the derivation of a rigorous convergence criterion. To this end, we take the expectation of the square of both sides in (4), to obtain

$$E[\zeta^2(k+1)] = \Big[(DF(x(k)) - \mu\gamma)^2 + 2(\mu\gamma - DF(x(k))) \times E[\varepsilon(k)] + E\Big[\varepsilon^2(k)\Big]\Big]E[\zeta^2(k)]. \quad (5)$$

The stability of the deterministic system in (5) can be inferred through the study of its Lyapunov exponent [25]. If the limit exists, the Lyapunov exponent is computed from the error dynamics for a non-zero initial condition $\zeta(0)$ as

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \ln\left[\frac{E\big[\zeta^2(k)\big]}{\zeta^2(0)}\right] = \lim_{j\to\infty} \frac{1}{j} \sum_{k=1}^{j} \ln E\Big[\zeta^2(j)\Big]. \quad (6)$$

The study of the stochastic stability of the N-dimensional error dynamics in (3) reduces to monitoring the sign of λ as a function of γ.

Lemma 1: The synchronous solution $x(k)$ of the nonlinear stochastic system (1) is locally asymptotically mean-square stable if the Lyapunov exponent λ in (6) is negative for every $\gamma \in \{\gamma_1, \ldots, \gamma_N\}$.

Proof: The proof is trivial. The negativenesses of the Lyapunov exponent implies the convergence of $E[\zeta^2(k)]$ to zero, and therefore guarantees local asymptotical mean-square stability, according to Definition 1. ■

Given the strength of the node-to-node interaction μ, the mean and variance of the broadcasting signal $E[\varepsilon(k)]$ and $E[\varepsilon^2(k)]$, and the individual dynamics F, one can numerically compute λ for a range of values of γ to generate a so-called master stability function. From the master stability function, one can then infer which network topology will support synchronization to the reference trajectory.

Remark 1: For a linear system, $F(x) = \alpha x$, where $\alpha \in \mathbb{R}$, the Lyapunov exponents can be easily computed from the limit in (6), such that

$$\lambda = \ln\Big[(\alpha - \mu\gamma)^2 + 2(\mu\gamma - \alpha)E[\varepsilon(k)] + E\Big[\varepsilon^2(k)\Big]\Big]. \quad (7)$$

Remark 2: For nonlinear discrete-time systems, numerical computation of Lyapunov exponents may be a challenging task, potentially leading to false predictions on stochastic synchronization. For example, stable dynamics may lead to $E[\zeta^2(k)]$ attaining values below numerical precision in a few steps, thereby hampering the evaluation of the Lyapunov exponent; and similarly, unstable dynamics may lead to sudden numerical overflow.

Toward overcoming potential confounds associated with numerical computation of (6), we adopt Birkoff's ergodic theorem [15] to derive the main general statement of this letter.

Proposition 1: The synchronous solution $x(k)$ of the stochastic system (1) with the invariant density $\rho(x)$ of F is locally mean square stable if

$$\lambda = \int_B \ln\Big[(DF(z) - \mu\gamma_i)^2 + 2(\mu\gamma - DF(z))E[\varepsilon(k)] + E\Big[\varepsilon^2(k)\Big]\Big]\rho(z)\mathrm{d}z, \quad (8)$$

where $B \subseteq \mathbb{R}$ is where the invariant density $\rho(x)$ is defined.

Proof: The key step in the proof lies in the introduction of the invariant density function of F, which measures the probability that a typical trajectory will visit a neighborhood of the state x – for example, for a periodic trajectory of period two, such that $x(k) = x(k+2)$, the invariant density $\rho(x)$ will correspond to two Delta distributions of equal intensity at $x(0)$ and $x(1)$ and for more complex, possibly chaotic, systems the invariant density may become a continuous function. By virtue of Birkoff's ergodic theorem [15], using $\rho(x)$, one can replace the summation over time in (6) with the integration over the probability space in (8), following the line of argument in [13] and [14]. ■

Remark 3: With knowledge of the invariant density, (8) can be analytically or numerically evaluated to establish a master stability function for controlled stochastic synchronization through broadcasting, without incurring in the computational challenges indicated in Remark 2.

For non-switching broadcasting, such that the switching gain ε is constant in time and equal to $\bar{\varepsilon}$, (8) reduces to

$$\lambda = 2\int_B \ln|DF(t) - \mu\gamma - \bar{\varepsilon}]|\rho(z)\mathrm{d}z. \quad (9)$$

Notice that the dependence of the argument of the logarithm on γ is linear, different from (8), where we find linear and quadratic dependencies. Equation (9) will be used to elucidate the predictive power, or lack thereof, of the averaged system on stochastic synchronization. Specifically, we will examine the sign of the Lyapunov exponent in (9) with $\bar{\varepsilon} = E[\varepsilon(k)]$ and compare with (8).

Remark 4: Since 0 is necessarily an eigenvalue of the Laplacian matrix (due to the zero row-sum property), one of the Lyapunov exponents is always given by $\lambda_1 = \int_B \ln[DF(z)^2 - 2DF(z)E[\varepsilon(k)] + E[\varepsilon^2(k)]]\rho(z)\mathrm{d}z$. This Lyapunov exponent indicates that the stability of synchronization of an individual oscillator to the reference oscillator is *necessary* for the stability of synchronization of the entire network to broadcasting. Therefore, the network can not facilitate synchronization as it introduces further constraints on the switching gain beyond those implied by a direct one-to-one coupling between an isolated node and the reference node.

IV. Application to Tent Maps

To illustrate the implications of (8), we consider the case of the chaotic tent map [26] with parameter equal to 2, which has a known density function $\rho(x) = 1$ on the interval $B = [0, 1]$. We limit the analysis to two control gains for the broadcasting signal, $\varepsilon_i \in \{\varepsilon_1, \varepsilon_2\}$. These gains could exemplify a single broadcasting message (if $\varepsilon_2 = 0$), or two conflicting messages

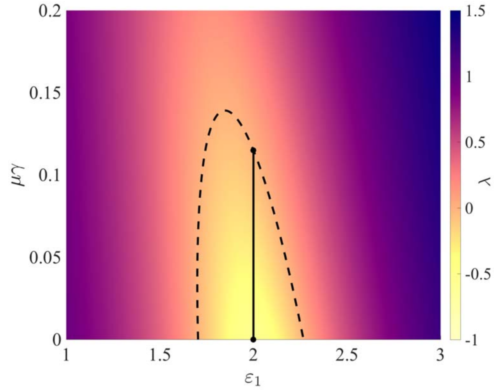

Fig. 2. (Color online) Master stability function for stochastic synchronization of chaotic tent maps, for $\varepsilon_2 = 2.2$ and $p_1 = p_2 = 0.5$. For synchronization to be stable, each eigenvalue of the Laplacian matrix must correspond to a negative Lyapunov exponent (indicated by the yellow color, isolated by the black dashed curve). For example, the black vertical line shows the range of admissible values of $\mu\gamma$ that would guarantee stability at $\varepsilon_1 = 2$.

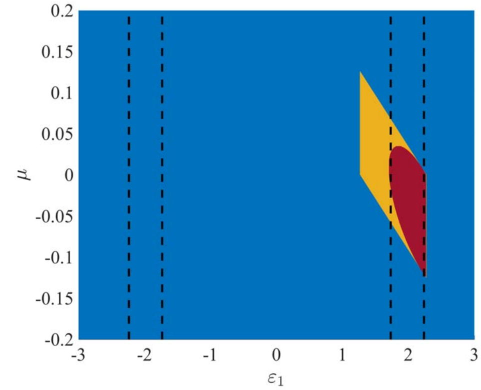

Fig. 3. (Color online) Verification of predictions from the averaged system on the stochastic synchronization of a ring of 100 chaotic tent maps, for $\varepsilon_2 = 2.2$ and $p_1 = p_2 = 0.5$. The red (dark gray) region indicates when averaged system correctly identifies the stability of synchronization, the yellow (light gray) region indicates when the averaged system predicts stability of synchronization against the master stability function that posits unstable synchronization, and the blue (large gray) region indicates when the averaged system correctly anticipates unstable synchronization. The regions enclosed by the dashed vertical lines identify the values of ε_1 that could support stable synchronization of an isolated node, that is, $\mu = 0$.

(if $\varepsilon_1 \neq \varepsilon_2$). In our numerical demonstrations, we parametrically vary ε_1 and μ, with $\varepsilon_2 = 2.2$ and $p_1 = p_2 = 0.5$. The value of ε_2 is chosen such that a single, isolated map would synchronize to a non-switching broadcasting signal [26]. At the same time, ε_1 may take a value such that it would destabilize synchronization in the non-switching case. Therefore, ε_1 and ε_2 can be assimilated to conflicting messages broadcasted by the reference node to the network.

A. Master Stability Function

Substituting the invariant density function into the integral in (8), we can compute the Lyapunov exponent in closed form for the mean square stability of the error dynamics, thereby arriving at the following application of Proposition 1 to the chaotic tent maps.

Corollary 1: The Lyapunov exponent for the mean square stability of the synchronous solution in the system (1) of chaotic tent maps is

$$\lambda = \ln\Big[(2-\mu\gamma)^2 + 2(\mu\gamma - 2)E[\varepsilon(k)] + E\Big[\varepsilon^2(k)\Big]\Big] \times \Big[(-2-\mu\gamma)^2 + 2(\mu\gamma + 2)E[\varepsilon(k)] + E\Big[\varepsilon^2(k)\Big]\Big]. \tag{10}$$

where $E[\varepsilon(k)] = p_1\varepsilon_1 + p_2\varepsilon_2$ and $E[\varepsilon^2(k)] = p_1\varepsilon_1^2 + p_2\varepsilon_2^2$.

This Lyapunov exponent demonstrates the explicit dependence of the stability of stochastic synchronization on the node-to-node coupling strength, the eigenvalues of the Laplacian matrix, and the stochastically switching coupling strengths along with their respective probabilities.

Figure 2 illustrates the dependence of λ on ε_1 and $\mu\gamma$. The dashed curve in Fig. 2 indicates the boundary between positive and negative Lyapunov exponents, identifying the onset of mean square stability of the error dynamics. In order for the network to synchronize to the reference node, the point $(\varepsilon_1, \mu\gamma)$ must fall within the dashed curve for *every* eigenvalue in the spectrum of the Laplacian matrix. In agreement with our predictions, we find that as $\mu\gamma$ increases the range of values of ε_1 which affords stable synchronization becomes smaller and smaller. This suggests that the resilience of the network to synchronize improves with $\mu\gamma$.

Remark 5: While the nonlinear dependence of the stability boundary on ε_1 and $\mu\gamma$ is modulated by the nonlinearity in the individual dynamics, it should not be deemed as a prerogative of nonlinear systems. As shown in Remark 1, the stochastic stability of synchronization in the simplest case of a linear system is also nonlinearly related to the spectrum of the Laplacian matrix and to the expectation and variance of the broadcasting signal – even for classical consensus with $\alpha = 1$ [8].

Remark 6: In this example of a chaotic tent map, the stability boundary is a single curve, defining a connected stability region. To ensure stable synchronization of a generic network, it is thus sufficient to monitor the largest eigenvalue of the Laplacian matrix, γ_N, such that $(\varepsilon_1, \mu\gamma)$ will fall within the stability region. This is in contrast with the master stability function for uncontrolled, spontaneous synchronization [4], which would typically require the consideration of the second smallest eigenvalue, often referred to as the algebraic connectivity [17]. However, similar to master stability functions for uncontrolled, spontaneous synchronization [27], we would expect that for different maps, one may find several disjoint regions in the $(\varepsilon_1, \mu\gamma)$-plane where stable stochastic synchronization can be attained.

B. Comparing the Static and Stochastic Systems

Next, we wish to gain insight into the ability of the averaged system to predict the onset of synchronization on the broadcasting trajectory. From (9), we obtain the following closed-form for the Lyapunov exponent:

$$\lambda = \ln|(2 - \mu\gamma - E[\varepsilon(k)])(-2 - \mu\gamma - E[\varepsilon(k)])|. \tag{11}$$

In Fig. 3, we summarize predictions on the stability of stochastic synchronization for a ring of 100 nodes, gathered through the master stability function depicted in Fig. 2 and

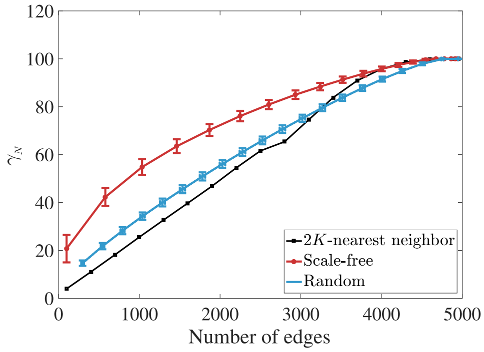

Fig. 4. (Color online) Largest eigenvalue γ_N of the Laplacian matrix as a function of the number of edges for three different types of networks of 100 nodes: a $2K$-regular network (black curve with square markers), scale-free (red curve with circle markers), and random Erdös-Renyi (blue curve) networks. Scale-free and random networks are run 10000 times to compute means and standard deviations, reported herein – note that error bars are only vertical for scale-free networks since the number of edges is fully determined by q, while for random networks additional horizontal error bars can be seen due to the process of network assembly.

the averaged system described through (11). Figure 3 identifies distinct regions of the parameter space spanned by μ and ε_1, where predictions from the averaged system with respect to the exact result from the master stability function should be considered valid or invalid.

Remark 7: Our results suggest the presence of a wide region of the parameter space in which the averaged system fails to predict the stability of synchronization, while we find no evidence of invalid predictions on the instability of synchronization by the averaged system. Hence, in this particular example, the averaged system seems to provide a necessary condition. Care should be placed in extending this claim to other settings, whereby, as shown in [13] and [14], one can construct examples for $N = 1$ in which unstable synchronization in the averaged system does not imply unstable synchronization for stochastically coupled systems.

C. Role of Network Topology

The master stability function in Fig. 2 shows that both μ and γ contribute to the resilience of the network to synchronization induced by stochastic broadcasting. For a given value of the node-to-node coupling strength μ, different networks will exhibit different residences based on their topology. Based on the lower bound by Grone and Merris [28] and the upper bound by Anderson and Morley [29], for a graph with at least one edge, we can write $\max\{d_i, i = 1, \ldots, N\} + 1 \leq \gamma_N \leq \max\{d_i + d_j, ij \in \mathcal{E}\}$, where d_i is the degree of node $i = 1, \ldots, N$. While these bounds are not tight, they suggest that the degree distribution has a key role on γ_N. For a given number of edges, one may expect that networks with highly heterogeneous degree distribution, such as scale-free networks [30], could lead to stronger resilience to broadcasting as compared to regular or random networks, with more homogenous degree distributions [30].

In Fig. 4, we illustrate this proposition by numerically computing the largest eigenvalue of the graph Laplacian for three different network types:

(i) A $2K$-regular network, in which each node is connected to $2K$ nearest neighbors, such that the degree is equal to $2K$. As K increases, the network approaches a complete graph.

(ii) A scale-free network which is grown from a small network of q nodes. At each iteration of the graph generation algorithm, a node is added with q edges to nodes already in the network. The probability that an edge will be connected to a specific node is given by the ratio of its degree to the total number of edges in the network. Nodes are added until there are N nodes in the network. When q is small, there are a few hub nodes that have a large degree and many secondary nodes with small degree, whereas when q is large, the scale-free network is highly connected and similar to a complete graph.

(iii) A random Erdös-Renyi network which takes as input the probability, p, of an edge between any two nodes. When p is small, the network is almost surely disconnected, and when p approaches 1, it is a complete graph.

We fix N to 100 and vary K, q, and p in (i), (ii), and (iii), respectively, to explore the role of the number of edges.

As expected from the bounds in [28] and [29], for a given number of edges, the scale-free network tends to exhibit larger values of γ_N. This is particularly noticeable for networks of intermediate size, whereby growing the number of edges will cause the three network types to collapse on a complete graph of N nodes. As the largest eigenvalue of the Laplacian matrix fully controls the resilience of the network to broadcasting-induced synchronization (in the case of linear and chaotic tent maps), we may argue that, given a fixed number of edges, the network can be configured such that it is either more conducive (regular graph) or resistant (scale-free graph) to synchronization. The increased resilience of scale-free networks should be attributed to the process of broadcasting-induced synchronization, which globally acts on all nodes simultaneously, without targeting critical nodes (low or high degree) like in pinning control [8], [9].

V. Conclusion

Much attention recently has been placed on controlling the synchronization of networks, though the vast majority of the literature considers cases in which the control is continuously applied on selected network sites. Here, we have taken a different approach, by addressing the problem of broadcasting-induced synchronization of a network of oscillators, using a single, external, reference node. The reference node is stochastically coupled with the network, such that control actions are intermittently applied over time, switching over a set of potentially conflicting messages.

In the context of mean square convergence, we have examined the stochastic stability of the error dynamics of the network oscillators with respect to the reference trajectory. By decomposing the error dynamics on independent components along the eigenvectors of the Laplacian matrix, we have established a master stability equation to predict the onset of stable synchronization. From the Lyapunov exponent of the master stability equation, we posit a master stability function, which can be used to systematically study the role of the mean and variance of the broadcasting control gain on synchronization. In a principled manner, we have applied elements of ergodic theory to cast the computation of the Lyapunov exponent in terms of an integration in a probability space, which is amenable to analytical and numerical treatments. We have illustrated the approach for chaotic tent maps, for which we have clarified the predictive power of time-averaging and systematically analyzed the role of network topology.

Our general approach is not limited to one-dimensional maps but directly applicable to higher dimensional node dynamics, provided that the invariant ergodic measure of the given map can be calculated. In rare cases which include Anosov maps (two-dimensional diffeomorphisms on tori) [31], invariant density functions can be assessed analytically. In some examples of two- and higher-dimensional chaotic maps, the invariant density can be calculated numerically and approximated by an explicit continuous function. Known examples of such numerically-assisted approximations include volume-preserving two-dimensional standard maps and the four-dimensional Froeschlé map [32].

In contrast to classical master stability functions for uncontrolled, spontaneous synchronization, where both the algebraic connectivity and the largest eigenvalue of the Laplacian matrix determine the onset of synchronization, we report that the algebraic connectivity has no role on broadcasting-induced synchronization of linear maps and chaotic tent maps. Specifically, the resilience of the network to broadcasting synchronization increases with the value of the largest eigenvalue of the Laplacian matrix. Heterogenous topologies with hubs of large degree should be preferred over homogenous topologies, when designing networks that should be resilient to influence from a broadcasting oscillator. On the contrary, homogenous topologies, such as regular or random topologies, should be preferred when seeking networks that could be easily tamed through an external broadcasting oscillator. Interestingly, these predictions would be hindered by a simplified analysis based on averaging, which could lead to false claims regarding the stability of synchronous solutions.

Two of the key assumptions of the current setup are the lack of a memory in switching and the need for switching at every time step. Both these assumptions could be relaxed by building on our recent work on synchronization of two coupled maps [13], [14], where we have demonstrated potential advantages of memory and non-fast switching. The analysis presented therein corresponds to broadcasting-induced synchronization for $N = 1$. We anticipate that combining those findings with the methodology proposed in this letter could lay the foundation for a general theory of non-fast broadcasting with memory, which could translate into control strategies with improved energy efficiency and performance.

References

[1] D. J. Sumpter, *Collective Animal Behavior*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[2] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*. London, U.K.: Springer, 2008.

[3] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, "The synchronization of chaotic systems," *Phys. Rep.*, vol. 366, nos. 1–2, pp. 1–101, 2002.

[4] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Phys. Rev. Lett.*, vol. 80, no. 10, pp. 2109–2111, 1998.

[5] A. E. Motter, "Networkcontrology," *Chaos Interdiscipl. J. Nonlin. Sci.*, vol. 25, no. 9, 2015, Art. no. 097621.

[6] A. J. King, D. D. P. Johnson, and M. Van Vugt, "The origins and evolution of leadership," *Current Biol.*, vol. 19, no. 19, pp. R911–R916, 2009.

[7] D. Köhler, "The interaction between conditioned fish and naive schools of juvenile carp (*Cyprinus carpio*, pisces)," *Behav. Process.*, vol. 1, no. 3, pp. 267–275, 1976.

[8] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Trans. Ind. Informat.*, vol. 9, no. 1, pp. 427–438, Feb. 2013.

[9] Y. Tang, F. Qian, H. Gao, and J. Kurths, "Synchronization in complex networks and its application—A survey of recent advances and challenges," *Annu. Rev. Control*, vol. 38, no. 2, pp. 184–198, 2014.

[10] M. Porfiri and M. Di Bernardo, "Criteria for global pinning-controllability of complex networks," *Automatica*, vol. 44, no. 12, pp. 3100–3106, 2008.

[11] F. Sorrentino, M. di Bernardo, F. Garofalo, and G. Chen, "Controllability of complex networks via pinning," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 75, no. 4, 2007, Art. no. 046103.

[12] T. Chen, X. Liu, and W. Lu, "Pinning complex networks by a single controller," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 54, no. 6, pp. 1317–1326, Jun. 2007.

[13] O. Golovneva, R. Jeter, I. Belykh, and M. Porfiri, "Windows of opportunity for synchronization in stochastically coupled maps," *Phys. D Nonlin. Phenomena*, vol. 340, pp. 1–13, Feb. 2017.

[14] M. Porfiri and I. Belykh, "Memory matters in synchronization of stochastically coupled maps," *SIAM J. Appl. Dyn. Syst.*, vol. 16, no. 3, pp. 1372–1396, 2017.

[15] L. Arnold, *Random Dynamical Systems*. Heidelberg, Germany: Springer, 2013.

[16] C. Pedicini, F. Vasca, L. Iannelli, and U. Jonsson, "An overview on averaging for pulse-modulated switched systems," in *Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf. (CDC ECC)*, Orlando, FL, USA, 2011, pp. 1860–1865.

[17] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York, NY, USA: Springer-Verlag, 2001.

[18] H. J. Kushner, *Introduction to Stochastic Control*. New York, NY, USA: Holt, Rinehart & Winston, 1971.

[19] Y. Fang, "Stability analysis of linear control systems with uncertain parameters," Ph.D. dissertation, Dept. Syst. Control Ind. Eng., Case Western Reserve Univ., Cleveland, OH, USA, 1994.

[20] T. Morozan, "Optimal stationary control for dynamic systems with Markov perturbations," *Stochastic Anal. Appl.*, vol. 1, no. 3, pp. 299–325, 1983.

[21] J. Zhou and Q. Wang, "Convergence speed in distributed consensus over dynamically switching random networks," *Automatica*, vol. 45, no. 6, pp. 1455–1461, 2009.

[22] M. Porfiri, "A master stability function for stochastically coupled chaotic maps," *Europhys. Lett.*, vol. 96, no. 4, 2011, Art. no. 40014.

[23] M. Porfiri, D. J. Stilwell, and E. M. Bollt, "Synchronization in random weighted directed networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 55, no. 10, pp. 3170–3177, Nov. 2008.

[24] R. Jeter and I. Belykh, "Synchronization in on-off stochastic networks: Windows of opportunity," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 62, no. 5, pp. 1260–1269, May 2015.

[25] A. Pikovsky and A. Politi, *Lyapunov Exponents: A Tool to Explore Complex Dynamics*. Cambridge, U.K.: Cambridge Univ. Press, 2016.

[26] M. Hasler and Y. L. Maistrenko, "An introduction to the synchronization of chaotic systems: Coupled skew tent maps," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 44, no. 10, pp. 856–866, Oct. 1997.

[27] A. Stefański, P. Perlikowski, and T. Kapitaniak, "Ragged synchronizability of coupled oscillators," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 75, no. 1, 2007, Art. no. 016210.

[28] R. Grone and R. Merris, "The Laplacian spectrum of a graph II," *SIAM J. Discr. Math.*, vol. 7, no. 2, pp. 221–229, 1994.

[29] W. N. Anderson, Jr., and T. D. Morley, "Eigenvalues of the Laplacian of a graph?" *Linear Multilinear Algebra*, vol. 18, no. 2, pp. 141–145, 1985.

[30] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: Structure and dynamics," *Phys. Rep.*, vol. 424, nos. 4–5, pp. 175–308, 2006.

[31] B. Hasselblatt and A. Katok, Eds., *Handbook of Dynamical Systems*. Amsterdam, The Netherlands: Elsevier, 2006.

[32] Z. Levnajić and I. Mezić, "Ergodic theory and visualization. I. Mesochronic plots for visualization of ergodic partition and invariant sets," *Chaos Interdiscipl. J. Nonlin. Sci.*, vol. 20, no. 3, 2010, Art. no. 033114.