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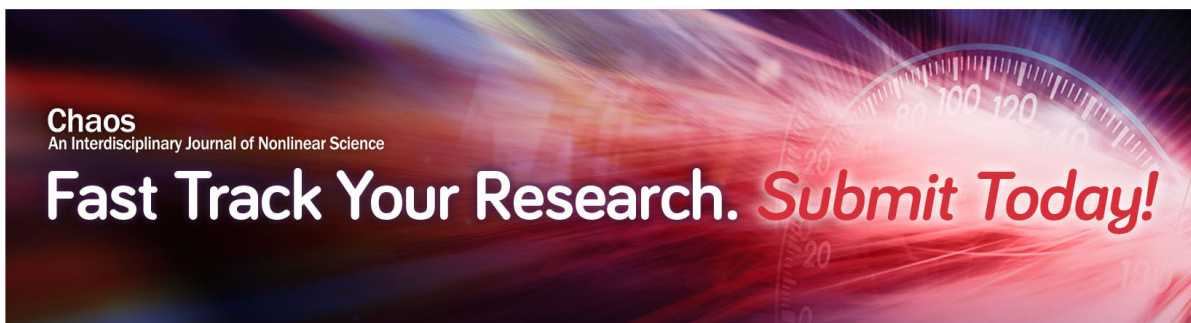
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# Overcoming network resilience to synchronization through non-fast stochastic broadcasting

Russell Jeter,<sup>1</sup> Maurizio Porfiri,<sup>2</sup> and Igor Belykh<sup>1</sup>

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Stochastic broadcasting is an important and understudied paradigm for controlling networks. In this paper, we examine the feasibility of on-off broadcasting from a single reference node to induce synchronization in a target network with connections from the reference node that stochastically switch in time with an arbitrary switching period. Internal connections within the target network are static and promote the network's resilience to externally induced synchronization. Through rigorous mathematical analysis, we uncover a complex interplay between the network topology and the switching period of stochastic broadcasting, fostering or hindering synchronization to the reference node. We derive a criterion which reveals an explicit dependence of induced synchronization on the properties of the network (the Laplacian spectrum) and the switching process (strength of broadcasting, switching period, and switching probabilities). With coupled chaotic tent maps as our test-bed, we prove the emergence of "windows of opportunity" where only non-fast switching periods are favorable to synchronization. The size of these windows of opportunity is shaped by the Laplacian spectrum such that the switching period needs to be manipulated accordingly to induce synchronization. Surprisingly, only the zero and the largest eigenvalues of the Laplacian matrix control these windows of opportunities for tent maps within a wide parameter region. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5044420>

**Broadcasting propaganda is a manipulative approach used to promote a particular political cause or influence public opinion. Similarly to this abused art of persuasion, driving a technological or biological network towards some desired behavior via global broadcasting from an external node is an effective tool for controlling networks. Examples include a robotic leader influencing the behavior of a school of fish, or a small group of neurons which can form an epileptic focus and cause an epileptic seizure. In this paper, we study the conditions under which a reference broadcasting node can synchronize a target network by stochastically transmitting sporadic, possibly conflicting signals. We demonstrate that manipulating the rate at which the connections between the broadcasting node and the network stochastically switch can overcome network resilience to synchronization. Through a rigorous mathematical treatment, we discover a nontrivial interplay between the network properties that control this resilience and the switching rate of stochastic broadcasting that should be adapted to induce synchronization. Unexpectedly, non-fast switching rates controlling the so-called windows of opportunity guarantee stable synchrony, whereas fast or slow switching leads to desynchronization, even though the networked system spends more time in a state favorable to synchronization.**

real-world systems.<sup>1-3</sup> Synchronization is one of the most basic instances of collective behavior, and one of the easiest to diagnose: it occurs when all of the nodes in a network act in unison. Typically, it manifests in ways similar to a school of fish moving as one larger unit to confuse or escape from a predator<sup>4</sup> or a collection of neurons firing together during an epileptic seizure.<sup>5</sup> Significant attention has been devoted to the interplay between node dynamics and network topology which controls the stability of synchronization.<sup>6-9</sup> Most studies have looked at networks whose connections are static; networks with a dynamically changing network topology, called temporal or evolving networks, are only recently appearing in the scientific literature<sup>10-16</sup> (see the recent book<sup>17</sup> for additional references). A particular class of evolving dynamical networks is represented by on-off switching networks, called "blinking" networks,<sup>18,19</sup> where connections switch on and off randomly and the switching time is fast, with respect to the characteristic time of the individual node dynamics.

As summarized in a recent review,<sup>20</sup> different aspects of synchronization, consensus, and multistability in stochastically blinking networks of continuous-time and discrete-time oscillators have been studied in the fast-switching limit where the dynamics of a stochastically switching network is close to the dynamics of a static network with averaged, time-independent connections. While a mathematically rigorous theory of synchronization in fast-switching blinking networks is available, the analysis of synchronization in non-fast switching networks of continuous-time oscillators has proven to be challenging and often elusive.

## I. INTRODUCTION

Network synchronization presents a challenging, yet fundamental problem in the theoretical and empirical study of

Non-fast switching connections yield a plethora of unexpected dynamical phenomena, including (1) the existence of a significant set of stochastic sequences and optimal frequencies for which the trajectory of a multistable switching oscillator can converge to a “wrong” ghost attractor<sup>21</sup> and (2) bounded windows of intermediate switching frequencies (“windows of opportunity”) in which synchronization becomes stable even though the network switches between unstable states.<sup>22,23</sup> Found numerically in networks of continuous-time Rössler oscillators and food chain models, the emergence of windows of opportunity calls for a rigorous explanation of unexpected synchronization from non-fast switching.

Blinking networks of discrete-time systems (maps) with non-fast switching offer such a mathematical treatment.<sup>24</sup> More precisely, the switching period in discrete-time networks can be quantified as a number of the individual map’s iterates such that rescaling of time yields a new, multi-iterate map that is more convenient to work with. Using the simplest network of two stochastically coupled tent maps, we have derived explicit conditions for the emergence of windows of opportunities and provided a rigorous basis for understanding the dynamics of non-fast switching networks of discrete-time oscillators.

In this paper, we go further and address an important problem of how non-fast switching can be used to control synchronization in a target network through stochastic broadcasting from a single external node. This problem of controlling synchronous behavior of a network towards a desired common trajectory<sup>25</sup> arises in many technological and biological systems where agents are required to coordinate their motion to follow a leader and maintain a desired formation.<sup>26</sup> In our setting, each node of the target network, implemented as a discrete-time map, is coupled to the external node with connections that stochastically switch in time with an arbitrary switching period. The network is harder to synchronize than its isolated nodes, as its structure contributes to resilience to controlled synchronization probed by the externally broadcasting node.

Combining ideas from our previous work on mutual synchronization of two coupled maps via non-fast switching<sup>24</sup> and controlled synchronization in fast-switching networks,<sup>27</sup> we reveal a complex interplay between the structure of the target network that provides resilience to controlled synchronization and the switching period of stochastic broadcasting that can be adapted to induce synchronization.

We examine the mean square stability of the synchronous solution in terms of the error dynamics and provide an explicit dependence of the stability of controlled synchronization on the network structure and the properties of the underlying broadcasting signal, defined by the strength of broadcasting connections and their switching period and probability. Via an analytical treatment of the Lyapunov exponents of the error dynamics and the use of tools from ergodic theory, we derive a set of stability conditions that provide an explicit criterion on how the switching period should be manipulated to overcome network resilience to synchronization as a function of the Laplacian spectrum of the network.<sup>28</sup>

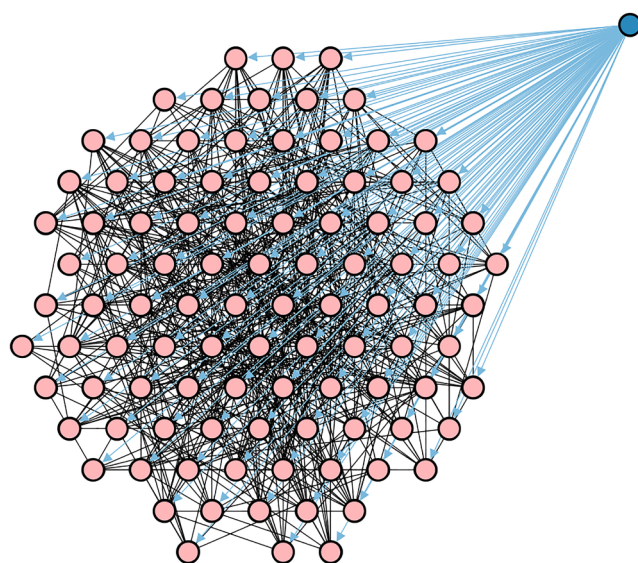


FIG. 1. The reference node (blue) stochastically broadcasts a signal to each of the nodes in a network of  $N$  oscillators (pink). The network has a complex topology of static connections.

Through the lens of chaotic tent maps, we discover that the network topology shapes the windows of opportunity of favorable non-fast switching in a highly nonlinear fashion. In contrast to mutual synchronization with a network whose stability is determined by the second smallest and largest eigenvalue of the Laplacian matrix via the master stability function,<sup>6</sup> controlled synchronization by the external node is defined by all its eigenvalues, including the zero eigenvalue. In the case of chaotic tent maps, the zero and the largest eigenvalue appear to effectively control the size of these windows of opportunity. This leads to the appearance of a persistent window of favorable switching periods where all network topologies sharing the largest eigenvalue become more prone to controlled synchronization.

## II. GENERAL PROBLEM

We study the synchronization of a network of  $N$  discrete-time oscillators given by the state variables  $y_i \in \mathbb{R}$  for  $i = 1, 2, \dots, N$ <sup>29</sup> that are driven by an external reference node given by  $x \in \mathbb{R}$  via a signal that is stochastically broadcasted to all of the nodes in the network. The topology of the network is undirected and unweighted. It is described by the graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  is the set of vertices and  $\mathcal{E}$  is the set of edges. The broadcaster-network system is depicted in Fig. 1. The evolution of the oscillators in the network and the reference node are given by the same mapping function  $F : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $x(k+1) = F[x(k)]$ . The switching of the broadcasted signal is an independent and identically distributed (i.i.d.) stochastic process that re-switches every  $m$  time steps. That is, the coupling strength of the reference node  $\varepsilon(mk) = \varepsilon(mk+1) = \dots = \varepsilon[m(k+1) - 1]$  is drawn randomly from a set of  $n$  coupling strengths  $\{\varepsilon_1, \dots, \varepsilon_n\}$  with probabilities  $p_1, \dots, p_n$ , respectively ( $\sum_{l=1}^n p_l = 1$ ).

The evolution of the discrete-time broadcaster-network system can be written compactly as

$$\begin{aligned} x(k+1) &= F[x(k)], \\ \mathbf{y}(k+1) &= \mathbf{F}[\mathbf{y}(k)] - \mu L \mathbf{y}(k) \\ &\quad - \varepsilon(k) I_N [\mathbf{y}(k) - x(k) \mathbf{1}_N], \end{aligned} \tag{1}$$

where  $\mathbf{F}$  is the natural vector-valued extension of  $F$ ,  $\mu$  is the coupling strength within the network,  $\mathbf{1}_N$  is the vector of ones of length  $N$ ,  $I_N$  is the  $N \times N$  identity matrix, and  $L$  is the Laplacian matrix of  $\mathcal{G}$ , i.e.,  $L_{ij} = -1$  for  $ij \in \mathcal{E}$ ,  $L_{ii} = -\sum_{j=1, j \neq i}^N L_{ij}$ ,  $i = 1, 2, \dots, N$ . Without loss of generality, we order and label the Laplacian spectrum of  $L$ :  $\gamma_1 = 0 \leq \gamma_2 \leq \dots \leq \gamma_N$ .

We study the stability of the stochastic synchronization of the network about the reference node’s trajectory, or  $y_1(k) = y_2(k) = \dots = y_N(k) = x(k)$ . Towards this goal, it is beneficial to re-format the problem and examine the evolution of the error dynamics  $\xi(k) = x(k) \mathbf{1}_N - \mathbf{y}(k)$ . When all of the nodes  $y_i(k)$  have converged to the reference trajectory,  $\xi(k) = x(k) \mathbf{1}_N - \mathbf{y}(k) = \mathbf{0}_N$ . To study the stability of synchronization, we linearize the system about the reference trajectory

$$\xi(k+1) = \{DF[x(k)]I_N - \mu L - \varepsilon(k)I_N\} \xi(k), \tag{2}$$

where  $DF[x(k)]$  is the Jacobian of  $F$  evaluated along the reference trajectory  $x(k)$ . As is typical of linearization, we assume that the perturbations  $\xi_i(k)$  in the variational equation (2) are small and in directions transversal to the reference trajectory. Convergence to the reference trajectory along these transversal directions ensures the local stability of the synchronous solution. Despite the stochastic and time-dependent nature of the broadcasting signal  $\varepsilon(k)$ , it only appears on the diagonal elements underlying the evolution of the error vector  $\xi(k)$ . Because  $\mu L$  is the only matrix in (2) that is not diagonal, we can diagonalize (2) with respect to the eigenspaces of the Laplacian matrix.

We obtain the stochastic master stability equation

$$\zeta(k+1) = \{DF[x(k)] - \mu\gamma - \varepsilon(k)\} \zeta(k), \tag{3}$$

where  $\gamma \in \{\gamma_1, \dots, \gamma_N\}$  and  $\zeta \in \mathbb{R}$  is a generic perturbation along the corresponding eigendirection of  $L$ . Notice that  $\gamma_1 = 0$  corresponds to the evolution of the error dynamics in the absence of a network. Finally, in order to simplify the analysis of the evolution of the variational equations, we re-scale the time variable with respect to the switching period

$$\tilde{\zeta}(k+1) = \prod_{i=0}^{m-1} \{DF[x(mk+i)] - \mu\gamma - \tilde{\varepsilon}(k)\} \tilde{\zeta}(k), \tag{4}$$

where  $\tilde{\zeta}(k) = \zeta(mk)$  and  $\tilde{\varepsilon}(k) = \varepsilon(mk)$ . This scalar equation provides the explicit dependence of the synchronization error on the network topology (via  $\mu\gamma$ ) and the strength of the broadcasted signal (via  $\varepsilon$ ). With this in mind, we continue by discussing the stability of the synchronization to the reference trajectory.

While there are many criteria that can be considered when determining stochastic stability of a synchronous solution, we use the lens of mean square stability for its practicality of implementation and inclusiveness with other criteria.<sup>30,31</sup>

**Definition 1.** The synchronous solution  $y_i(k) = x(k)$  for  $i = 1, 2, \dots, N$  in the stochastic system (1) is locally mean square asymptotically stable if  $\lim_{k \rightarrow \infty} \mathbf{E}[\tilde{\zeta}^2(k)] = 0$  for any  $\tilde{\zeta}(0)$  and  $\gamma \in \{\gamma_1, \dots, \gamma_N\}$  in (4), where  $\mathbf{E}[\cdot]$  denotes expectation with respect to the  $\sigma$ -algebra generated by the stochastic process underlying the switching.

Mean square stability of the stochastic system in (4), and by extension, synchronization in the original system (1), corresponds to studying the second moment of  $\tilde{\zeta}(k)$ . This is especially attractive because it reduces our study of a stochastic system to that of a deterministic one. Hence, we take the expectation of the square of the error in (4)

$$\begin{aligned} \mathbf{E}[\tilde{\zeta}^2(k+1)] &= \sum_{l=1}^n p_l \left( \prod_{i=0}^{m-1} \{DF[x(mk+i)] \right. \\ &\quad \left. - \mu\gamma - \varepsilon_l\} \right)^2 \mathbf{E}[\tilde{\zeta}^2(k)]. \end{aligned} \tag{5}$$

Reducing the stochastically switching system (1) to a deterministic system (5) allows for the use of standard tools from stability theory, such as Lyapunov exponents.<sup>32</sup> The Lyapunov exponent for (5) is computed as

$$\begin{aligned} \lambda &= \lim_{k \rightarrow \infty} \frac{1}{k} \ln \left[ \frac{\mathbf{E}[\tilde{\zeta}^2(k)]}{\tilde{\zeta}^2(0)} \right] \\ &= \lim_{j \rightarrow \infty} \frac{1}{j} \sum_{k=1}^j \ln \left\{ \mathbf{E}[\tilde{\zeta}^2(k+1)] \right\}. \end{aligned} \tag{6}$$

There are numerous pitfalls that can undermine the numerical computation of Lyapunov exponent from a time series, such as  $\mathbf{E}[\tilde{\zeta}^2]$  falling below numerical precision in a few time steps and incorrectly predicting stochastic synchronization for trajectories that would eventually diverge. With proper assumptions, one can use Birkoff’s ergodic theorem<sup>32</sup> to avoid these confounds and form the main analytical result of this paper.

**Proposition 1.** The synchronous solution  $x(k)$  of the stochastic system (1) is locally mean square asymptotically stable if

$$\lambda = \int_B \ln \left( \sum_{l=1}^n p_l \left\{ \prod_{i=0}^{m-1} [DF(t) - \mu\gamma - \varepsilon_l] \right\}^2 \right) \rho(t) dt, \tag{7}$$

is negative  $\forall \gamma \in \{\gamma_1, \dots, \gamma_N\}$ . Here,  $B$  is the region for which the invariant density  $\rho(t)$  of  $F$  is defined.

**Proof.** Assuming  $F$  is ergodic with invariant density  $\rho(t)$ , one can avoid computing the Lyapunov exponent from a time series using Birkoff’s ergodic theorem to replace the averaging over time with averaging over the state. This amounts to replacing the summation with integration in (6). Then, by virtue of (6) and the definition of a Lyapunov exponent, stability of the stochastic system reduces to monitoring the sign of this Lyapunov exponent.  $\square$

**Remark 1.** We reduce studying the stability of synchronization in (1) to monitoring the sign of the Lyapunov exponents in (7), with a different exponent for each eigenvalue  $\gamma$ . If each of these Lyapunov exponents is negative, the dynamics of

the network in the original system (1) converges to the dynamics of the reference trajectory. Furthermore, this allows the stability of stochastic synchronization to be studied explicitly in the network and broadcasting parameters  $\mu$ ,  $\{\gamma_1, \dots, \gamma_N\}$ ,  $\{\varepsilon_1, \dots, \varepsilon_n\}$ ,  $\{p_1, \dots, p_n\}$ , and  $m$ .

**Remark 2.** There are two notable consequences of the Laplacian spectrum on the stability conditions given by the sign of (7): (i)  $\mu\gamma = 0$  is always an eigenvalue, such that it is necessary that the nodes in the network pairwise synchronize to the reference node in the absence of a network topology and (ii) if the network is disconnected, fewer stability conditions need to be satisfied, whereby there will be repeated zero eigenvalues. In light of these consequences, a network is inherently resilient to broadcasting synchronization, in that it necessitates satisfying more stability conditions, and synchronization in the absence of a network is always one of the stability conditions.

### III. TENT MAPS

To explore some of the theoretical implications of the general stability criterion (7), we consider the broadcaster-network system (1) composed of chaotic tent maps. The chaotic tent map, described by the equation

$$x(k + 1) = F[x(k)] = \begin{cases} ax(k), & x(k) < 1/a, \\ a[1 - x(k)], & x(k) \geq 1/a \end{cases} \quad (8)$$

with parameter  $a = 2$ , is known to have a constant invariant density function  $\rho(t) = 1$ .<sup>33</sup> Therefore, the general criterion (7) can be written for controlled synchronization of chaotic tent maps in a compact form that depends only on the network and broadcasting parameters.

**Proposition 2 (Master stability function).** A stochastic system (1) of chaotic tent maps is locally mean square asymptotically stable if

$$\lambda = \frac{1}{2^m} \sum_{i=0}^m \binom{m}{i} \ln \left[ \sum_{l=1}^n p_l Y(i, m, \mu\gamma, \varepsilon_l) \right], \quad (9)$$

is less than zero, where  $Y(i, m, \mu\gamma, \varepsilon_l)$  is given by  $(2 + \mu\gamma + \varepsilon_l)^{2i} (2 - \mu\gamma - \varepsilon_l)^{2(m-i)}$  and  $\binom{m}{i} = \frac{m!}{(m-i)!i!}$ .

**Proof.** To derive the criterion (9), we employ ideas from our previous work<sup>24</sup> on synchronization of two stochastically coupled tent maps. Using the formula for the Lyapunov exponent for the general stochastic system (7), we substitute the invariant density  $\rho(t) = 1$  and region  $B = [0, 1]$  that the function  $F$  is defined on for the tent map. This yields the following equation:

$$\lambda = \int_0^1 \ln \left( \sum_{l=1}^n p_l \left\{ \prod_{i=0}^{m-1} [DF(t) - \mu\gamma - \varepsilon_l] \right\}^2 \right) dt. \quad (10)$$

Explicitly detailing the role of  $DF(t)$  for general values of  $m$  is trickier. Consider the case of fast switching, when  $m = 1$ . The product  $\prod_{i=0}^{m-1} [DF(t) - \mu\gamma - \varepsilon_l]$  only takes two values  $[2 + \mu\gamma + \varepsilon_l] = f_l^-$  and  $[2 - \mu\gamma - \varepsilon_l] = f_l^+$ , depending on whether or not we are taking the integral over the increasing, i.e.,  $[0, \frac{1}{2}]$ , or decreasing, i.e.,  $(\frac{1}{2}, 1]$ , branch of the tent map's domain, respectively. Splitting the integral in (10) and then

integrating, we obtain

$$\begin{aligned} \lambda &= \int_0^{\frac{1}{2}} \ln \left[ \sum_{l=1}^n p_l f_l^{+2} \right] dt + \int_0^{\frac{1}{2}} \ln \left[ \sum_{l=1}^n p_l f_l^{-2} \right] dt \\ &= \frac{1}{2} \ln \left[ \sum_{l=1}^n p_l f_l^{+2} \right] + \frac{1}{2} \ln \left[ \sum_{l=1}^n p_l f_l^{-2} \right]. \end{aligned} \quad (11)$$

For  $m = 2$ , which corresponds to two successive iterations of the tent map before switching states, a similar partitioning of the interval can be used, but this time into four distinct intervals:  $[0, \frac{1}{4}]$ ,  $(\frac{1}{4}, \frac{1}{2}]$ ,  $(\frac{1}{2}, \frac{3}{4}]$ , and  $(\frac{3}{4}, 1]$  which correspond to the four combinations of two consecutive iterations. We can perform a similar replacement to (11), but with four intervals

$$\begin{aligned} \lambda &= \int_0^{\frac{1}{4}} \ln \left[ \sum_{l=1}^n p_l f_l^{+4} \right] dt + \int_{\frac{1}{4}}^{\frac{1}{2}} \ln \left[ \sum_{l=1}^n p_l f_l^{+2} f_l^{-2} \right] dt \\ &\quad + \int_{\frac{1}{2}}^{\frac{3}{4}} \ln \left[ \sum_{l=1}^n p_l f_l^{-2} f_l^{+2} \right] dt + \int_{\frac{3}{4}}^1 \ln \left[ \sum_{l=1}^n p_l f_l^{-4} \right] dt \\ &= \frac{1}{4} \left( \ln \left[ \sum_{l=1}^n p_l f_l^{+4} \right] + 2 \ln \left[ \sum_{l=1}^n p_l f_l^{+2} f_l^{-2} \right] \right. \\ &\quad \left. + \ln \left[ \sum_{l=1}^n p_l f_l^{-4} \right] \right). \end{aligned} \quad (12)$$

This idea then naturally extends to  $2^m$  intervals of length  $\frac{1}{2^m}$ , in which the product  $\prod_{i=0}^{m-1} [DF(t) - \mu\gamma - \varepsilon_l]$  is the same. With the binomial expansion structure of this product, for the  $i$ th iteration, there are  $\binom{m}{i}$  intervals with the same product that can be collapsed. Hence, for tent maps, the general criterion (7) turns into (9).  $\square$

**Remark 3.** The closed-form analytical expression (9) for the Lyapunov exponents indicates the explicit dependence of the stability of controlled synchronization on the network coupling strength  $\mu$ , the eigenvalues of the Laplacian matrix for the network, the switching period  $m$ , the stochastically switching coupling strengths  $\{\varepsilon_1, \dots, \varepsilon_n\}$ , and their respective probabilities  $\{p_1, \dots, p_n\}$ . For controlled synchronization to be mean square stable, the Lyapunov exponent for any eigenvalue in the Laplacian spectrum must be negative.

To illustrate the power of our explicit criterion (9) for controlled synchronization and clearly demonstrate the emergence of windows of opportunity, we limit our attention to stochastic broadcasting between two coupling strengths  $\varepsilon_1$  (with probability  $p$ ) and  $\varepsilon_2$  (with probability  $1 - p$ ).

To choose the coupling strengths  $\varepsilon_1$  and  $\varepsilon_2$ , we consider two statically coupled tent maps (8),

$$\begin{aligned} x(k + 1) &= f[x(k)], \\ y(k + 1) &= f[y(k)] + \varepsilon[x(k) - y(k)]. \end{aligned} \quad (13)$$

The network (13) describes a pairwise, directed interaction between the dynamics of the broadcasting map  $x(k)$  and a single, isolated map  $y(k)$  from the network where the switching broadcasting coupling is replaced with a static connection of strength  $\varepsilon$ . The stability of synchronization in the static network (13) is controlled by the sign of the transversal

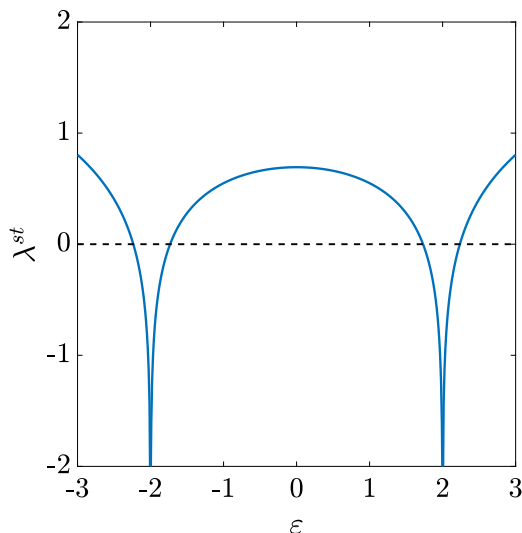


FIG. 2. Transversal Lyapunov exponent,  $\lambda^{st}$ , for the stability of synchronization in the static network of tent maps (13), calculated through (14) as a function of coupling  $\varepsilon$ . This diagram is used to choose the values of coupling  $\varepsilon_1 = -1.999$  (from a stability region) and  $\varepsilon_2 = -1.700$  (from an instability region) for the broadcasting node to switch its connections with each node of the network.

Lyapunov exponent<sup>33</sup>

$$\lambda^{st} = \ln |2 - \varepsilon| + \ln |2 + \varepsilon|. \tag{14}$$

Figure 2 indicates two disjoint regions given by  $\varepsilon \in [-\sqrt{5}, -\sqrt{2}]$  and  $\varepsilon \in [\sqrt{2}, \sqrt{5}]$  in which  $\lambda^{st} < 0$  and synchronization is stable.

Returning to the stochastically switching broadcaster-network system, we use the master stability function of Fig. 2 to choose  $\varepsilon_1 = -1.999$  from a stability region and  $\varepsilon_2 = -1.700$  from an instability region such that the connection from the broadcasting node to the network switches between the two values, where one value supports controlled synchronization and the other destabilizes it. In this way, the broadcaster sends two conflicting messages to the network to follow and not to follow its trajectory.

We pay particular attention to the case where the switching probability of the stabilizing coupling,  $\varepsilon_1$ , is higher ( $p > 1/2$ ). One’s intuition would suggest that fast-switching between the stable and unstable states of controlled synchronization with probability ( $p > 1/2$ ), that makes the system spend more time in the stable state, would favor the stability of synchronization. However, the master stability function of Fig. 3 calculated through the analytical expression for the Lyapunov exponent (9) shows that this is not the case. Our results reveal the presence of a stability zone (black area) which, in terms of the switching periods  $m$ , yields a window of opportunity when non-fast switching favors controlled synchronization, whereas fast or slow switching does not. The fact that slower switching at  $m > 25$  at the switching probability  $p = 0.9$  (see the transition from point A to B) desynchronizes the system is somewhat unexpected, as the system is likely to remain in the stable state, defined by  $\varepsilon_1$ , most of the time.

The exact cause of this effect remains to be studied; however, we hypothesize that this instability originates from

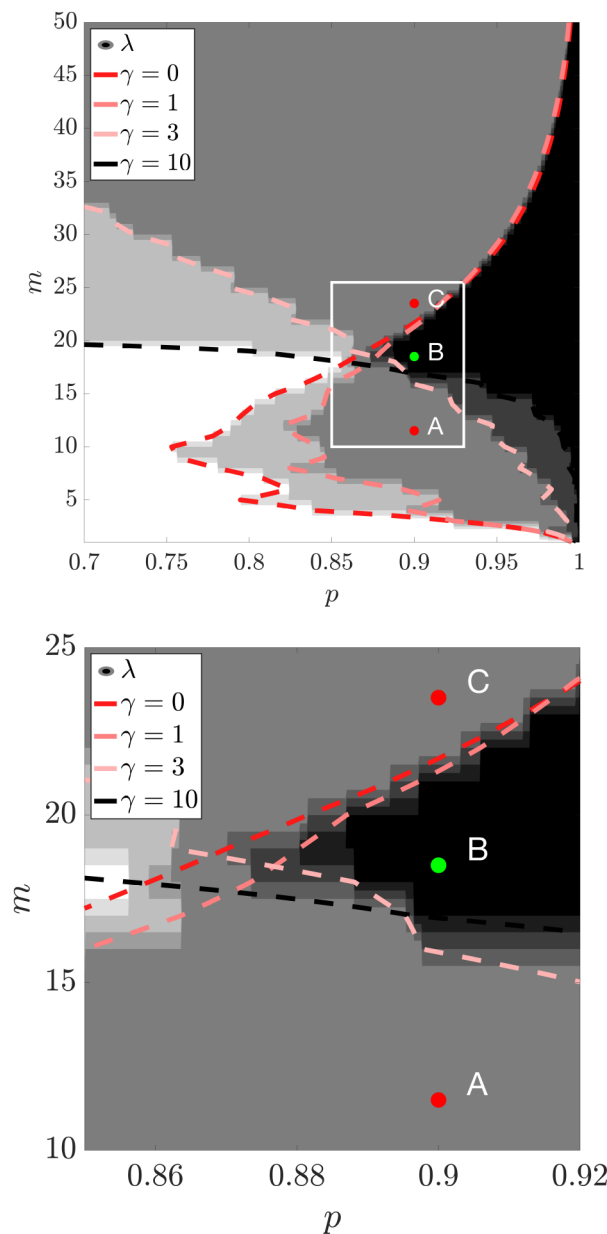


FIG. 3. Analytical calculation of the master stability function (9) for controlled synchronization of tent maps as a function of the switching probability  $p$  and switching period  $m$  for  $\varepsilon_1 = -1.999$ ,  $\varepsilon_2 = 1.700$ , and  $\mu = 0.01$ . (Top) The black region indicates the stability of controlled synchronization and the dashed lines represent the boundaries for the stability regions (gray areas) for various eigenvalues  $\gamma$  of a network’s Laplacian matrix. Notice that the size of the stability region is primarily controlled by only two curves, corresponding to  $\gamma = 0$  (red dashed) and  $\gamma = 10$  (black dashed), such that the addition of curves for eigenvalues  $\gamma \in (0, 10)$  only affects the small cusp part of the stability region (see the zoomed-in area). (Bottom) Zoom-in of the region marked by the white rectangle in (top). Points A, B, and C indicate pairs  $(p, m)$  for which synchronization is unstable, stable, and unstable, respectively, for different values of the switching period  $m$ . Note the window of favorable frequencies  $m$  which includes point B in the vertical direction from A to C. Remarkably, the size of the stability region remains persistent to changes of the intra-network coupling  $\mu$  (not shown), suggesting the existence of soft, lower and upper thresholds for favorable switching frequencies between  $m = 20$  and 30.

a large disparity between the time scale of weak convergence in the vicinity of the synchronization state during the (long) time lapse when the stabilizing coupling  $\varepsilon_1$  is on and the time scale of strong divergence from the synchronization

solution far away from it when the destabilizing coupling  $\varepsilon_2$  finally switches on. As a result, this unbalance between the convergence and divergence makes synchronization unstable.

The window of opportunity (black area) displayed in Fig. 3 appears as a result of intersections between the boundaries (dashed curves) of the stability zones (gray areas), where each boundary is calculated from the criterion (9) when the Lyapunov exponent is zero for the corresponding eigenvalue of the Laplacian matrix. The red curve for  $\gamma_1 = 0$  shows the stability region in the absence of a network, and is therefore a necessary condition for controlled synchronization in the presence of the network. In the general case of  $N$  distinct eigenvalues, there will be  $N$  curves. Each curve adds a constraint and, therefore, one would expect each eigenvalue  $\gamma_1, \dots, \gamma_N$  to play a role in reducing the size of the stability zone and shaping the window of opportunity as a function of network topology.

In contrast to these expectations, Fig. 3 provides a convincing argument that the stability zone is essentially defined by two curves, corresponding to the zero eigenvalue,  $\gamma_1$  (red dashed curve), and the largest eigenvalue,  $\gamma_N$  (black dashed curve). All the other curves offer a very minor contribution to shaping the stability region. As a consequence, windows of opportunity should be relatively robust to topological changes, preserving the maximum largest eigenvalue of the Laplacian spectrum. For example, the set of four distinct eigenvalues (0, 1, 3, 10) in Fig. 3 corresponds to a star network of 10 nodes with an additional edge connecting two outer nodes. In this case, the removal of the additional link reduces the spectrum to three distinct eigenvalues (0, 1, 10) and eliminates the curve for  $\gamma = 3$  which, however, does not essentially change the stability region. This observation suggests that the addition of an edge to a controlled network, which would be expected to help a network better shield from the external influence of the broadcasting node, might not necessarily improve network resilience to synchronization.

Similarly, the removal of an edge from an all-to-all network with two distinct eigenvalues (0,  $N$ ) changes the spectrum to (0,  $N - 1$ ,  $N$ ), which according to Fig. 3 does not significantly alter the stability region either. For general topologies, one may look at the degree distribution to gather insight on the largest eigenvalue. Combining the lower<sup>34</sup> and upper<sup>35</sup> bounds for a graph with at least one edge, one can estimate the largest eigenvalue  $\gamma_N$  as follows:

$$\max\{d_i\} + 1 \leq \gamma_N \leq \max\{d_i + d_j\}, \quad ij \in \mathcal{E}, \quad (15)$$

where  $d_i$  is the degree of node  $i = 1, \dots, N$ . Although these bounds are conservative, they indicate that the degree distribution plays a key role in defining  $\gamma_N$ .

For a fixed number of edges, networks with highly heterogeneous degree distributions, such as scale-free networks, may enhance resilience to broadcasting when compared to regular or random networks, with more homogeneous degree distributions. Our recent paper<sup>27</sup> contains a comparative study between a  $2K$ -nearest neighbor network, a scale-free network, and a random Erdős-Renyi graph with the same number of edges, indicating that the scale-free network tends to have larger values of  $\gamma_N$ .

In general, the entire spectrum of the Laplacian matrix may matter for the stability of controlled synchronization in a network of discrete-time oscillators. However, our analysis of coupled tent maps points to a simpler mechanism, whereby one can use the degree distribution for drawing conclusions on the switching periods that guarantee the success of the broadcaster to synchronize the network. Put simply, “you can run but you cannot hide”: the broadcaster will identify suitable switching rates to overcome the resilience of the network.

#### IV. CONCLUSIONS

While the study of stochastically switching networks has gained significant momentum, most analytical results have been obtained under the assumption that the characteristic time scales of the intrinsic oscillators and evolving connections are drastically different, enabling the use of averaging and perturbation methods. In regard to on-off stochastically switching systems, these assumptions typically yield two extremes, fast or slow (dwell-time<sup>36</sup>) switching, for which rigorous theory has been developed.<sup>18,19,21–23,37–42</sup> However, our understanding of dynamical networks with non-fast switching connections is elusive, and the problem of an analytical treatment of the dynamics and synchronization in non-fast switching network remains practically untouched.

In this paper, we sought to close this gap by creating an analytical approach to characterize the stability of network synchronization in stochastically switching networks of discrete-time oscillators as a function of network topology and switching period. We considered a special type of controlled synchronization induced in a static network of coupled maps through stochastic broadcasting from a single, external node. Extending our previous work on synchronization of two maps with non-fast switching connections<sup>24</sup> and broadcaster-induced synchronization in fast-switching networks,<sup>27</sup> we have established a rigorous toolbox for assessing the mean-square stability of controlled synchronization in broadcaster-network systems. Through a rigorous mathematical analysis of the transversal Lyapunov exponents, we have uncovered a complex interplay between a target network and the switching period of stochastic broadcasting. Stable synchronization in the network is possible, provided that the switching period falls into a window of opportunity.

Our approach is directly applicable to high-dimensional maps whose invariant density measure can be calculated explicitly. These systems include two-dimensional diffeomorphisms on tori such as Anosov maps,<sup>43</sup> for which the invariant density measure can be calculated analytically, and volume-preserving two-dimensional standard maps whose invariant density function can be assessed through computer-assisted calculations.<sup>44</sup> Although our work provides an unprecedented understanding of network synchronization beyond the fast switching limit, we have hardly scratched the surface of evolving dynamical networks theory. This work immediately raises the following questions: (1) What if the i.i.d. process underlying the switching was relaxed to be a more general Markov process? and (2) What if the underlying topology of the broadcasting was more complex? Both of these questions are of interest, but provide their own technical challenges

and require further study. We anticipate that combining our recent work on synchronization of two maps under Markovian switching with memory<sup>45</sup> with the approach developed in this paper should make progress toward unraveling a complex interplay between switching memory and network topology for controlled synchronization.

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