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Brief paper

Windows of opportunity for the stability of jump linear systems: Almost sure versus moment convergence☆

Maurizio Porfiri [a,*], Russell Jeter [b], Igor Belykh [b]

[a] Department of Mechanical and Aerospace Engineering, New York University Tandon School of Engineering, Brooklyn, NY, 11201, USA
[b] Department of Mathematics and Statistics and Neuroscience Institute, Georgia State University, Atlanta, GA 30303, USA

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ABSTRACT

In this paper, we examine the role of the switching period on the stochastic stability of jump linear systems. More specifically, we consider a jump linear system in which the state matrix switches every m time steps randomly within a finite set of realizations, without a memory of past switching instances. Through the computation of the Lyapunov exponents, we study δ-moment and almost sure stability of the system. For scalar systems, we demonstrate that almost sure stability is independent of m, while δ-moment stability can be modulated through the selection of the switching period. For higher-dimensional problems, we discover a richer influence of m on stochastic stability, quantified in an almost sure and a δ-moment sense. Through the detailed analysis of an archetypical two-dimensional problem, we illustrate the existence of disconnected windows of opportunity where the system is asymptotically stable. Outside of these windows, the system is unstable, even though it switches between two Schur-stable state matrices.

1. Introduction

Assessing stochastic stability is a critical problem in the study of jump linear systems, with important applications in modeling biological systems, formulating hybrid control algorithms, and designing power electronics, see, for example, Sun (2006). An excellent review of the history of research on stochastic stability of jump linear systems can be found in the work of Fang, Loparo, and Feng (1994). Rosembloom (1954) was the first to study moment stability of jump linear systems and Bellman (1960) was the first to tackle moment stability through Kronecker algebra. Bertram and Sarachik (1959) and Kats and Krasovskii (1960) put forward criteria based on Lyapunov's second method to investigate moment and almost sure stability. Building on these seminal papers from over sixty years ago, several breakthroughs have been made in the study of stochastic stability of jump linear systems, summarized in a number of comprehensive books, doctoral dissertations, and review papers (Costa, Fragoso, and Marques, 2006; Fang, 1994; Kozin, 1969; Kushner, 1971; Mariton, 1988). While there is no general consensus on the most desirable stability property for a jump linear system, almost sure stability seems to be the most useful criterion for practical applications and mean-square stability the simplest criterion to implement.

Here, we examine moment and almost sure stability of a specific class of jump linear systems in which the state matrix switches every $m \in \mathbb{Z}^+$ time steps, following an independent and identically distributed (i.i.d.) random process. As a result, in each time interval of length m, the state matrix is constant and the dynamics progresses analogously to a time-invariant system. Increasing m will simultaneously amplify the ability of Schur-stable matrices to steer the dynamics toward the origin and of unstable matrices to push the dynamics away from it, leading to a rich dependence of the stochastic stability of the jump linear system on m. As made clear in what follows, counterintuitive scenarios may emerge even in this seemingly trivial linear setting. For example, it is easy to predict that slow switching (large m) between two Schur-stable matrices will induce stable dynamics, but inferring stability for finite values of m is elusive. Stable dynamics may occur for fast switching ($m = 1$), and, surprisingly, disappear for intermediate switching.

Studying the stability of this class of systems bears several ramifications on our understanding of stochastic stability of dynamical systems. For example, the dynamics of the classical Kapitza's pendulum (Kapitza, 1951) can be associated with the stabilization of an unstable system via the cogent design of a stochastic, vibratory

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* Corresponding author.
E-mail address: mporfiri@nyu.edu (M. Porfiri).

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input. Similarly, the stability of high flapping frequency of hovering insects and flapping wing micro-air vehicles has been shown to be triggered by non-fast, higher order harmonics (Taha, Tahmasian, Woolsey, Nayfeh, & Hajj, 2015). In the context of synchronization of coupled systems, our previous work has unveiled a rich and often counterintuitive dependence of the stochastic stability of the synchronous solution on the switching period for scalar chaotic maps (Golovneva, Jeter, Belykh, & Porfiri, 2017; Jeter, Porfiri, & Belykh, 2018) and continuous-time oscillators (Jeter & Belykh, 2015).

More specifically, we have previously demonstrated the existence of multiple, disconnected intervals of the switching period where the synchronous solution is stable, termed "windows of opportunity." The characterization of windows of opportunity for jump linear systems is the main objective of this paper. By focusing on linear stochastic systems, we seek to gain insight into the determining factors for the occurrence of windows of opportunity, without potential confounds associated with nonlinearities. We focus on both moment and almost sure stability, bringing to light key differences between them for scalar and higher-dimensional systems.

The key technical contributions of our study entail the formulation of a mathematical framework to explain the emergence of windows of opportunity in the context of stochastic stability theory and the purposeful design of an exemplary two-dimensional problem that is amenable to analytical treatment. The mathematical framework includes the synergy between new analytical results, especially in the context of moment stability, and claims from the technical literature that are revisited toward an improved understanding of windows of opportunity. The two-dimensional example constitutes a transparent testbed to demonstrate the complexity of the phenomenon of windows of opportunity, shedding light on the interplay between the switching period and the internal system dynamics.

The main claims of our work are: (i) for scalar systems, windows of opportunity do not exist in an almost sure sense, but only in moment stability; (ii) for systems of higher dimension, windows of opportunity emerge from both moment and almost sure stability, although they may have different extent; and (iii) the topology of windows of opportunity for higher dimensional systems may be considerably richer than for scalar systems.

2. Mathematical preliminaries

Here, we briefly review key concepts on the stability of jump linear systems from the classical work of Fang, Loparo, and Feng (1995). Consider the linear discrete-time system

$$x_{k+1} = H(\sigma_k)x_k, \tag{1}$$

where $k \in \mathbb{N}$ is the discrete time variable, $x_k \in \mathbb{R}^n$ is the state variable with n being a positive integer defining the dimension of the system, H is a real valued matrix function in $\mathbb{R}^{n\times n}$, and σ_k is a finite-state i.i.d. random process taking values in $\{1,\dots,N\}$ with N being the number of state matrices and $p_1 \dots, p_N$ their respective probability to occur. For simplicity, we assume that the initial condition x_0 is a constant, nonrandom, vector. This setup is a specialization of the general framework considered by Fang et al. (1995), where σ_k could be a finite state, homogeneous Markov process.

Definition 2.1 from Fang et al. (1995) can be restated as follows:

Definition 1. Let $\Pr(\cdot)$ and $\mathsf{E}(\cdot)$ indicate, respectively, probability and expectation with regard to the sigma-algebra induced by the i.i.d. process, and let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ (for example, the Euclidean norm). The jump linear system in (1) is said to be

1. asymptotically δ-moment stable, if for any $x_0 \in \mathbb{R}^n$
$$\lim_{k\to\infty} \mathsf{E}\left(\|x_k\|^\delta\right) = 0; \tag{2}$$
2. exponentially δ-moment stable ($\delta > 0$), if for any $x_0 \in \mathbb{R}^n$, there exist $\alpha, \beta > 0$ such that
$$\mathsf{E}\left(\|x_k\|^\delta\right) < \alpha\|x_0\|^\delta e^{-\beta k}, \quad k \in \mathbb{N}; \quad \text{or} \tag{3}$$
3. almost sure asymptotically stable, if for any $x_o \in \mathbb{R}^n$,
$$\mathsf{P}\left(\lim_{k\to\infty} \|x_k\| = 0\right) = 1. \tag{4}$$

In the context of moment stability, we refer to $\delta = 2$ as mean-square stability.

By adapting the claims in Theorem 4.1, Proposition 4.3, and Lemma 4.8 from Fang et al. (1995),[1] we formulate the following set of relationships between these notions of stability.

Proposition 1. *Given the jump linear system in (1) and the stability notions in Definition 1, the following relationships hold.*

1. *(Theorem 4.1) Asymptotic and exponential δ-moment stability are equivalent.*
2. *(Proposition 4.3) Exponential δ-moment stability implies almost sure stability.*
3. *(Lemma 4.8) For any $0 < \delta_1 \le \delta_2$, asymptotic δ_2-moment stability implies δ_1-moment stability.*[2]

This Proposition supports our intuition that exponential and asymptotic moment stabilities are equivalent for jump linear systems, similar to classical time-invariant systems. Also, it clarifies that almost sure stability is the least conservative notion of stability and that the higher the value of δ is chosen, the more conservative the stability criterion is. Thus, almost sure stability should be regarded as the most reliable means to analyze and/or design jump linear systems, although it is generally difficult to tackle.

To assess the stability of (1), we may examine the sign of the top Lyapunov exponent associated with the δ-moment, sometimes referred to as the generalized Lyapunov exponent, or sample-path evolution. More specifically, if the limits exist (Fang & Loparo, 2002), we define the following quantities.

Definition 2. The top (or largest) sample-path Lyapunov exponent, λ, and the top δ-moment Lyapunov exponent, $g(\delta)$, of (1) are defined, respectively, as[3]

$$\lambda = \max_{x_0\neq 0}\lim_{k\to\infty}\frac{1}{k}\log\|x_k\| = \lim_{k\to\infty}\frac{1}{k}\log\|H(\sigma_{k-1})\cdots H(\sigma_0)\|, \tag{5a}$$

$$g(\delta) = \max_{x_0\neq 0}\lim_{k\to\infty}\frac{1}{k}\log\mathsf{E}\|x_k\|^\delta = \lim_{k\to\infty}\frac{1}{k}\log\mathsf{E}\|H(\sigma_{k-1})\cdots H(\sigma_0)\|^\delta, \tag{5b}$$

where the computation is independent of the norms used for vectors or matrices.

Remark. If λ ($g(\delta)$) is negative, the system is almost sure (δ-moment) asymptotically stable and it is unstable otherwise.[4]

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1 Some of these claims can also be found in Fang et al. (1994), albeit in a less general form.

2 Lemma 4.8 is an implementation of Jensen's inequality, and it specifically states that for any random variable ξ, the function $F(y) = \mathsf{E}(\|\xi\|^y)^{\frac{1}{y}}$ is nondecreasing in $\mathbb{R}^+$ whenever it is well defined. This is sufficient to prove our claim, which appears in a more general form as Theorem in 4.7 in Fang et al. (1995).

3 The sample path Lyapunov exponent should be interpreted in an almost sure sense (Fang et al., 1995).

4 These claims also include infinitely large values of the Lyapunov exponents that would imply convergence in one time step.

Remark. Computing the top δ-moment Lyapunov exponents for $\delta \in \mathbb{R}^+$ can be undertaken by using the notion of generalized spectral radius studied in a general setting by Ogura and Martin (2013). For the more common case of mean-square stability, it is easy to verify the following relationship:

$$g(2) = \log[\rho(\mathsf{E}(H(\sigma) \otimes H(\sigma)))] = \log\left[\rho\left(\sum_{j=1}^{N} p_j H(j) \otimes H(j)\right)\right], \tag{6}$$

where $\rho(\cdot)$ is the spectral radius of a matrix and $\otimes$ is the Kronecker product.[5]

Remark. Evaluating the top sample-path Lyapunov exponent is a much more challenging task (Tsitsiklis & Blondel, 1997). If a closed-form result is available for $g(\delta)$ and all the state matrices are invertible, then one may consider applying Proposition 2.3 from Fang and Loparo (2002), which posits that $\lambda = g'(0+)$, corresponding to the derivative of $g(\delta)$ from above. Alternatively, one may utilize the law of large numbers to compute

$$\lambda = \lim_{k\to\infty} \frac{1}{k}\mathsf{E}\log\|H(\sigma_{k-1})\cdots H(\sigma_0)\|, \tag{7}$$

almost surely, from Lemma 4.5 in Fang et al. (1995).

Very few closed-form results on sample-path Lyapunov exponents are presently available, typically restricted to scalar systems or commuting state matrices. The seminal paper by Pincus (1985) and later refinements by Lima and Rahibe (1994) have put forward analytical results for the case of binary switching between two 2×2 matrices with one of them being singular. More recently, tight bounds for binary switching between 2×2 shear hyperbolic matrices have been presented by Sturman and Thiffeault (2017), who also offer a meticulous overview of the state-of-the-art in the computation of Lyapunov exponents associated with random matrix products.

3. Windows of opportunity

Rather than switching between state matrices at every time step as in (1), we consider the more general case where the same state matrix is retained for m consecutive time steps. By scaling the time variable, our problem becomes

$$x_{k+1} = \underbrace{H(\sigma_k)\cdots H(\sigma_k)}_{m} x_k = H^m(\sigma_k)x_k, \tag{8}$$

where we take the mth power of each individual state matrix in one "scaled" time step.

Definition 3. Given the jump linear system in (1), we say that $\mathcal{W} \subset \mathbb{Z}^+$ is a window of opportunity for almost sure (δ-moment) stability if (8) is almost sure (δ-moment) asymptotically stable for any $m \in \mathcal{W}$.

The main objective of the rest of this paper is to determine the existence, extent, and topology of windows of opportunity as a function of the state matrices $H(1), \ldots, H(N)$ and their probability to occur $p_1, \ldots, p_N$. We begin with the study of scalar systems, for which we can precisely determine windows of opportunity in an almost sure ($\mathcal{W}_{\rm as}$) and δ-moment sense ($\mathcal{W}_{\delta-\rm m}$). Then, we turn to higher dimensional problems. First, we examine in detail the archetypical problem of switching between two generalized shear matrices in two dimensions, which offers surprising and counterintuitive evidence with respect to the claims made for scalar systems. Second, we establish conservative bounds for general higher-dimensional systems, which we illustrate on our archetypical problem.

4. Scalar systems

For a scalar system ($n = 1$), we can compute closed-form expressions for the almost sure and δ-moment Lyapunov exponents in (5a) and (5b), respectively, from which we can gather insight into the asymptotic stability of (8).

Proposition 2. *Consider the jump linear system in (1) for $n = 1$.*

1. *If $\sum_{j=1}^{N} p_j \log|H(j)| < 0$, then $\mathcal{W}_{\rm as} = \mathbb{Z}^+$, such that (8) is almost sure asymptotically stable for any choice of the switching period m.*
2. *If $\sum_{j=1}^{N} p_j \log|H(j)| > 0$, then $\mathcal{W}_{\rm as} = \varnothing$, such that it is not possible to find a value of m for which (8) is almost sure asymptotically stable.*

Proof. By adapting (7) to (8) and recalling that $n = 1$, we compute

$$\lambda(m) = m\sum_{j=1}^{N} p_j \log|H(j)|, \tag{9}$$

where we explicitly indicate the dependence on m. This expression proves that almost sure asymptotic stability of (8) is independent of m, such that the system is either asymptotically stable for any choice of m (case 1) or it is unfeasible to determine a value of m that guarantees asymptotic stability (case 2). □

Remark. We note that $\sum_{j=1}^{N} p_j \log|H(j)| < 0$ if all the individual realizations $H(1), \ldots, H(N)$ are Schur-stable, but also if there are some which are not Schur-stable, provided that their probability to occur is sufficiently small.

For δ-moment stability, we uncover a surprisingly different behavior through the following claim.

Proposition 3. *Consider the jump linear system in (1) for $n = 1$.*

1. *If all the individual realizations $H(1), \ldots, H(N)$ are Schur-stable, then $\mathcal{W}_{\delta-\rm m} = \mathbb{Z}^+$, such that (8) is δ-moment asymptotically stable for any choice of the switching period m.*[6]
2. *If some of the individual realizations $H(1), \ldots, H(N)$ are not Schur-stable, then there exists a single (potentially empty) window of opportunity $\mathcal{W}_{\delta-\rm m} = \{1, \ldots, \lfloor\overline{\mu}\rfloor\}$, with $\overline{\mu}$ being the nonzero solution of $\gamma(\overline{\mu}, \delta) = 0$ given by*

$$\gamma(\mu, \delta) = \log\left[\sum_{j=1}^{N} p_j |H(j)|^{\delta\mu}\right]. \tag{10}$$

Proof. By adapting (5b) to (8) and recalling that $n = 1$, we can compute the δ-moment Lyapunov exponent, $g(m, \delta)$, which is equal to $\gamma(m, \delta)$ in (10), where, again, we explicitly identify the dependence on m. Building on the arguments by Golovneva et al. (2017), we take the exponential of both sides of (10) and define $q(\mu) = \left[\sum_{j=1}^{N} p_j |H(j)|^{\delta m}\right]$. In case 1, $q(\mu) < 1$ for any $\mu > 0$, such that (8) is δ-moment asymptotically stable for any $m \in \mathbb{Z}^+$. In case 2, it is easy to verify that $q(\mu)$ is a concave function ($q''(\mu) \geq 0$ for any $\mu > 0$, where prime indicates derivative with respect to the argument) which takes values below 1 in the neighborhood of

[5] Eq. (6) can be found by writing the Lyapunov equation for the second moment matrix $\mathsf{E}\left[x_k x_k^{\rm T}\right]$ where T indicates matrix transposition, see, for example, Abaid & Porfiri (2011).

[6] This claim is true even if some of the realizations have unitary magnitude, provided there is at least one which is Schur-stable.

the origin ($q(0) = 1$ and $q'(0) = \sum_{j=1}^{N} p_j \log|H(j)| < 0$), although $\lim_{\mu\to\infty} q(\mu) = \infty$. Thus, (8) is δ-moment asymptotically stable for m ranging from 1 to $\overline{m} = \lfloor \overline{\mu} \rfloor$, such that $q(\overline{\mu}) = 1$ with $\overline{\mu} > 0$. This set might be empty if $\overline{m}$ is zero or one. □

Example. Consider the two-state process ($N = 2$) with $H(1) = \frac{5}{8}$ and $H(2) = \frac{9}{8}$ of equal probability $p_1 = p_2 = \frac{1}{2}$. The sample-path Lyapunov exponent in (9) is $\lambda(m) \simeq -0.176m$, such that the system is almost sure asymptotically stable for any switching period, that is, $\mathcal{W}_{\text{as}} = \mathbb{Z}^+$. The δ-moment Lyapunov exponent in (10) is $\gamma(m,\delta) = \log\left(2^{-3\delta m-1}5^{\delta m} + 2^{-3\delta m-1}9^{\delta m}\right)$, which becomes positive for sufficiently large values of m. For instance, the system is mean-square asymptotically stable only in the narrow window $\mathcal{W}_{2-\text{m}} = \{1, 2\}$.

Remark. Based on prior literature on the fast-switching stability of continuous systems (Belykh, Belykh, & Hasler, 2004; Porfiri, Stilwell, & Bollt, 2008; Stilwell, Bollt, & Roberson, 2006), one may be tempted to infer the stability of the jump linear system for small values of m (in particular $m = 1$) from the averaged system, whose state-matrix is $\mathrm{E}[H(\sigma_k)]$. However, this is incorrect for discrete-time systems, as pointed out by Golovneva et al. (2017) in the context of mean-square stability of scalar maps. Below, we expand on those arguments in the context of both almost sure and δ-moment stability, focusing on $m = 1$.

First, stability of the averaged system does not imply stability of the jump linear system, as shown through the following example.

Example. Consider the two-state process with $H(1) = 2$ and $H(2) = -2$ of equal probability $p_1 = p_2 = \frac{1}{2}$. The averaged system reaches the origin in a single time step, while the sample-path Lyapunov exponent in (9) with $m = 1$ is $\log 2$, such that the jump system is not almost sure (or δ-moment) asymptotically stable.

Second, the lack of stability of the averaged system does not prevent the jump system from being stable, as demonstrated through the following example.

Example. Consider the two-state process with $H(1) = \frac{15}{8}$ and $H(2) = \frac{3}{8}$ of equal probability $p_1 = p_2 = \frac{1}{2}$. The averaged system is unstable with Lyapunov exponent of $\log\frac{9}{8}$. On the other hand, the jump system with $m = 1$ is almost sure asymptotically stable – the sample-path Lyapunov exponent in (9) is equal to $\frac{1}{2}\log\frac{45}{64}$ – and δ-moment asymptotically stable for $\delta < 0.562$ – the δ-moment Lyapunov exponent, from (10), is equal to $\log\left[\frac{1}{2}\left(\frac{15}{8}\right)^{\delta} + \frac{1}{2}\left(\frac{3}{8}\right)^{\delta}\right]$.

5. Higher-dimensional systems

For n larger than one and state matrices that do not commute, we cannot expect an equivalent case to the scalar system analyzed above. More specifically, the notions that almost sure asymptotic stability is independent of m and that δ-moment stability is restricted to a single window in m that must start from $m = 1$ are both invalid.

5.1. The archetypical example of two generalized shear matrices in two dimensions

We illustrate the highly nontrivial nature of the problem through the analysis of a two-dimensional ($n = 2$) example. Specifically, we consider a two-state process

$$H(1) = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}, \quad H(2) = \begin{bmatrix} a & 0 \\ -1 & a \end{bmatrix}, \tag{11}$$

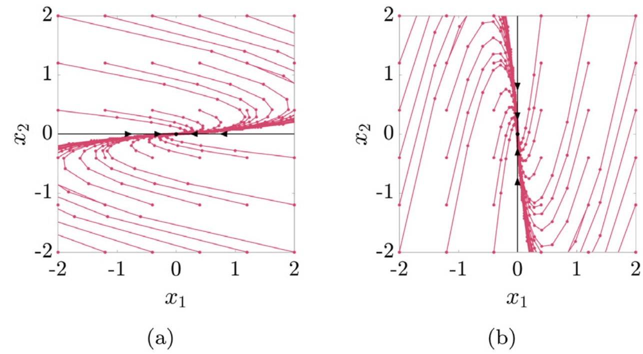

Fig. 1. Phase portraits for state matrices in (11) with $a = \frac{7}{10}$: (a) $H(1)$ and (b) $H(2)$. Temporally-adjacent states are connected for ease of illustration. Trajectories converge to the stable fixed point at the origin along the eigenvector (black line in each panel) in the direction of the black arrows.

where $0 < a < 1$ and $p_1 = p_2 = \frac{1}{2}$. In this case, each of the realization is Schur-stable as the spectral radius of each matrix is $a < 1$, but the coupled stochastic dynamics will reveal a complex dependence on m.

Fig. 1 illustrates the evolution associated with each state matrix for $a = \frac{7}{10}$. The system switches between two stable degenerate nodes with a repeated eigenvalue a and one linearly independent eigenvector. As a result, the trajectories of either system are tangent to the eigenvector and curve around to the opposite direction. Therefore, each trajectory can increase its relative distance from the origin before reaching the turning point where it starts approaching the stable fixed point. This property suggests a mechanism for the trajectory of the switching system to escape to infinity, for some value of the switching period.

We start by determining windows of opportunity in a mean-square sense through the following proposition.

Proposition 4. *Consider the two-dimensional jump linear system (1) with state matrices (11), $0 < a < 1$, and $p_1 = p_2 = \frac{1}{2}$.*

1. *For $a > \frac{1}{\sqrt{2}}$, there is a single window of opportunity for mean-square stability $\mathcal{W}_{2-\text{m}} = \{\lceil\overline{\mu}\rceil\ \lceil\overline{\mu}\rceil + 1, \ldots\}$, with $\overline{\mu}$ being the nonzero solution of $\gamma(\overline{\mu}, 2) = 0$ given by*

$$\gamma(\mu, 2) = \log\left[a^{2\mu}\left(1 + \frac{1}{2}\frac{\mu^2}{a^2}\right)\right]. \tag{12}$$

2. *For $a < 0.6434$, $\mathcal{W}_{2-\text{m}} = \mathbb{Z}^+$.*
3. *For $0.6435 < a < 0.6704$, $\mathcal{W}_{2-\text{m}} = \mathbb{Z}^+ \setminus \{2\}$.*
4. *For $0.6705 < a \leq 0.7001$, $\mathcal{W}_{2-\text{m}} = \mathbb{Z}^+ \setminus \{2, 3\}$.*
5. *For $0.7002 < a < \frac{1}{\sqrt{2}}$, $\mathcal{W}_{2-\text{m}} = \mathbb{Z}^+ \setminus \{2, 3, 4\}$.*

Proof. Our argument is based on the direct evaluation of the top second-moment Lyapunov exponent in (6), which is equivalent to the computation of the spectral radius of the following 4×4 matrix:

$$\frac{1}{2}(H^m(1) \otimes H^m(1) + H^m(2) \otimes H^m(2)) =$$

$$a^{2m}\begin{bmatrix} 1 & \frac{1}{2}\frac{m}{a} & \frac{1}{2}\frac{m}{a} & \frac{1}{2}\frac{m^2}{a^2} \\ -\frac{1}{2}\frac{m}{a} & 1 & 0 & \frac{1}{2}\frac{m}{a} \\ -\frac{1}{2}\frac{m}{a} & 0 & 1 & \frac{1}{2}\frac{m}{a} \\ \frac{1}{2}\frac{m^2}{a^2} & -\frac{1}{2}\frac{m}{a} & -\frac{1}{2}\frac{m}{a} & 1 \end{bmatrix}. \tag{13}$$

Table 1
Analysis of the problem in (11) with $p_1 = p_2 = \frac{1}{2}$, $a = \frac{7}{10}$. Each column shows results for a different value of the switching period m and each row depicts predictions of the top mean-square and sample path Lyapunov exponents, along with the claims about stability from the proposed sufficient conditions. The computation of sample path Lyapunov exponents is performed by drawing $\lfloor \frac{5000}{m} \rfloor$ realizations of the state matrix, taking their product, and evaluating the Lyapunov exponent through (5a) using the Euclidean norm. We average the last 10% of the samples to mitigate the effect of the initial transient. "N/A" indicates that the sufficient condition is not applicable, while "Yes" means that asymptotic stability can be inferred from the sufficient condition.

m	1	2	3	4	5	6	7	8	9	10
$g(m, 2)$	−0.010	0.199	0.181	−0.001	−0.289	−0.649	−1.062	−1.512	−1.993	−2.498
Proposition 5 - 1 norm	N/A	N/A	N/A	N/A	N/A	N/A	N/A	Yes	Yes	Yes
Proposition 5 - 2 norm	N/A	N/A	N/A	N/A	N/A	N/A	N/A	Yes	Yes	Yes
Proposition 5 - ∞ norm	N/A	N/A	N/A	N/A	N/A	N/A	N/A	Yes	Yes	Yes
$\lambda(m)$	−0.111	0.036	−0.080	−0.332	−0.529	−0.789	−1.049	−1.420	−1.702	−1.947
Proposition 6 - 1 norm	N/A	N/A	N/A	N/A	N/A	N/A	Yes	Yes	Yes	Yes
Proposition 6 - 2 norm	N/A	N/A	N/A	N/A	N/A	N/A	Yes	Yes	Yes	Yes
Proposition 6 - ∞ norm	N/A	N/A	N/A	N/A	N/A	N/A	Yes	Yes	Yes	Yes

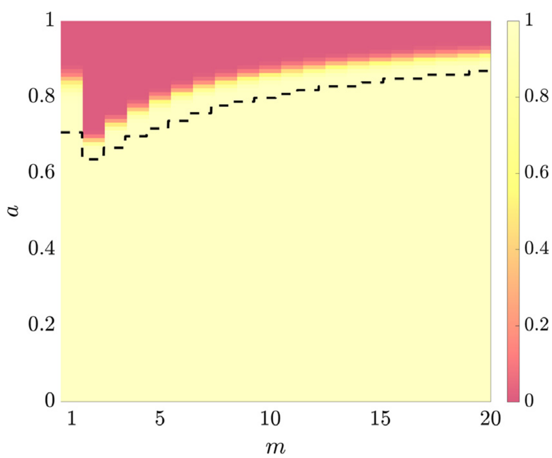

Fig. 2. Analysis of the problem in (11) with $p_1 = p_1 = \frac{1}{2}$. Probability that the top sample-path Lyapunov exponent, $\lambda(m)$, is negative as a function of the parameter a and switching period m. Probability is based on 1000 trials of 500 time steps, with light yellow indicating asymptotic stability (negative Lyapunov exponent) with probability 1 and pink indicating probability 0. The dashed black curve gives the boundary for which the Lyapunov exponent for mean-square stability changes sign; below the boundary, the system is mean-square asymptotically stable and it is unstable above it.

Through simple algebra, we can calculate the four eigenvalues of this matrix and determine a general form for the spectral radius, such that the top mean-square Lyapunov exponent of (8), $g(m, 2)$, is equal to $\gamma(m, 2)$ in (12). From the analysis of the function $\gamma(m, 2)$ we derive all the salient cases listed above. □

Remark. For $a = \frac{7}{10}$, we compare the claims on mean-square stability in Proposition 4 with almost sure stability results. This requires a computational approach, which yields the top sample path Lyapunov exponent. As shown in Table 1, we find that the system is almost sure asymptotically stable for every value of m, except for $m = 2$ when it loses stability, such that $\mathcal{W}_{\text{as}} = \mathbb{Z}^+ \setminus \{2\}$. For completeness, in Table 1, we also report the numerical values of the top mean-square Lyapunov exponent, for which we have a window of opportunity equal to $\mathcal{W}_{2-\text{m}} = \mathbb{Z}^+ \setminus \{2, 3\}$.

To shed more light on the effect of a on the stability of the system, in Fig. 2 we display results from a parametric analysis in which we systematically vary a from 0 to 1 and m from 1 to 20. Numerical results in Fig. 2 indicate that for small values of a, the system is almost sure asymptotically stable for any selection of m, while for values of a approaching 1, the system is unstable for any value of m up to 20. For values of a in the neighborhood of 0.8, we find the existence of two windows of opportunity, describing fast and slow-switching, and in the neighborhood of 0.9 stability is attained only through slow-switching. For completeness, we also present the stability bound associated with mean-square stability, derived by using Proposition 4. In agreement with Proposition 1, for all the values of a and m that guarantee mean-square stability, we determine that the system is stable for each of the trials.

Remark. By comparing (11) with a scalar problem, we evidence a dramatically different behavior. With respect to almost sure stability, in two dimensions, we demonstrate the existence of nontrivial windows of opportunity, in contrast to the scalar case where m plays no role. With respect to moment stability, we illustrate the possibility of disjoint windows of opportunities, as well as the possibility of windows of opportunity that do not include $m = 1$, in contrast with the scalar case where none of these scenarios is feasible. The latter opens the door to the possibility of controlling the stability of a jump linear system, by regulating its time spent in each of its possible states, see Table 1.

5.2. General claims

In general, it is difficult to provide necessary and sufficient conditions for stochastic stability of higher-dimensional systems due to the complexity of evaluating the spectrum of a large matrix associated with mean-square stability or the application of the law of large numbers for almost sure stability.

A trivial result is obtained if all the individual state matrices are Schur-stable. In this case, for sufficiently large values of m, each of the summands in (6) will asymptotically approach zero and the spectral radius of the matrix will tend to zero accordingly. As a consequence, the system will be mean-square stable for large switching periods.

However, stability may also be possible for unstable state matrices and a number of sufficient conditions might be established from the application of classical spectral bounding techniques, by extending the line of argument from Fang et al. (1994) to $m \neq 1$. For example, a sufficient condition for mean square stability can be derived by bounding the spectral radius of the matrix in (6) using classical norm bounds, as shown in what follows.

Proposition 5. *The jump linear system in (8) is mean-square asymptotically stable if for some p-norm, the following inequality holds*

$$\sum_{j=1}^{N} p_j \|H^m(j)\|^2 < 1. \tag{14}$$

Proof. For a p-norm, we have that $\rho(A) \leq \|A\|$ for any matrix $A \in \mathbb{R}^{n\times n}$. By applying the triangle inequality and recalling that the norm of the Kronecker product of two matrices is equal to the product of the norms, we prove our claim. □

Remark. Note that for $m = 1$ and the Euclidean norm, this is equivalent to the second statement in Theorem 2.5 from Fang et al. (1994). Also, the power m could be brought out of the norm leading to a more conservative bound, which is, however, simpler to implement.

For almost sure asymptotic stability, we can directly apply Theorem 2.2 from Fang et al. (1994), which for the case at hand will read as follows.

Proposition 6. *The jump linear system in (8) is almost sure asymptotically stable if for some matrix norm, satisfying the submultiplicative property, the following inequality holds*

$$\prod_{j=1}^{N} \|H^m(j)\|^{p_j} < 1. \tag{15}$$

Proof. The Proposition is a direct application of Young's inequality to the right hand side of (5a). □

Table 1 illustrates the application of these conservative bounds for our exemplary problem, for which all of these bounds take a compact form that is easy to check for different values of m. In this example, varying the norm does not influence the tightness of the lower bound, whereby for the three considered norms, we find that mean-square asymptotic stability is attained for m larger than 7, while almost sure asymptotic stability is reached for m larger than 6. While these predictions are on the same order of magnitude of exact computations, they do not assist in identifying the instability region for lower values of m, thereby challenging the identification of disjoint windows of opportunity.

6. Conclusions

In this note, we have studied the stochastic stability of a class of jump linear system, where the state matrix retains the same value for m consecutive time steps, before switching according to a finite-state i.i.d. process. Through a detailed analytical treatment, we have demonstrated a complex dependence of stochastic stability on the switching period m. Changing the dimensions of the system and the lens through which stability is examined has a remarkable effect on the existence, extent, and topology of windows of opportunity for the stability of the system.

For scalar systems, windows of opportunity in an almost sure sense are trivial, such that asymptotic stability is independent of the switching period: either a system is asymptotically stable for any value of m or it is always unstable. With respect to moment stability, finite windows of opportunity might appear if one of the realizations of the state matrices are Schur-stable, although these windows are constrained to have the form $\{1, \ldots, \overline{m}\}$. For higher dimensional systems, a completely different scenario that challenges our intuition emerges. Nontrivial windows of opportunity become feasible also for almost sure stability and their extent and topology radically change, encompassing infinite sets and disconnected regions.

Particularly surprising is the effect of the switching period on almost sure asymptotic stability. Our intuition suggests that increasing the switching period would automatically enhance almost sure stability, by reducing the range of matrix products on which we should enforce the top Lyapunov exponent to be negative. In other words, one may expect that if the system is almost sure asymptotically stable for a given m, then it should remain almost sure asymptotically stable for any larger, multiple, value of m. However, this is not the case of higher-dimensional systems, where we have shown that a system may be asymptotically stable for $m = 1$ but not for $m = 2$ and again be asymptotically stable for any larger switching period. It is tenable that for m larger than 2, the repetitive occurrence of multiple product of the same state matrices will cause a stabilizing counter-effect, against the destabilizing effect elicited by the second order powers of the state matrices.

Beyond the need for further research to illuminate the causes of windows of opportunity, especially in an almost sure sense, future work should seek to expand on the theoretical framework for the analysis and design of switched systems. For example, it is viable to incorporate memory effects through a Markovian switching process, contemplate the possibility of time-varying dynamics and switching periods, and include stochastic perturbations. With respect to design, future work should attempt at formulating mathematically-tractable criteria for the selection of switching periods to attain Lyapunov exponents within chosen performance bounds.

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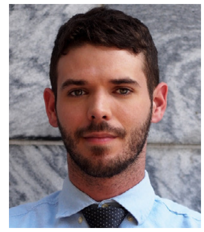
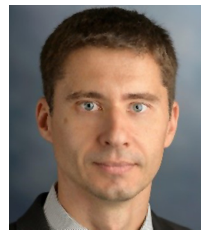
Dr. Maurizio Porfiri is a Professor in the Department of Mechanical and Aerospace Engineering at New York University Tandon School of Engineering. He received M.Sc. and Ph.D. degrees in Engineering Mechanics from Virginia Tech, in 2000 and 2006; a "Laurea" in Electrical Engineering (with honors) and a Ph.D. in Theoretical and Applied Mechanics from the University of Rome La Sapienza" and the University of Toulon (dual degree program), in 2001 and 2005, respectively. He is engaged in conducting and supervising research on dynamical systems theory, multiphysics modeling, and underwater robotics. Maurizio Porfiri is the author of more than 250 journal publications and the recipient of the National Science Foundation CAREER award. He has been included in the "Brilliant 10" list of Popular Science and his research featured in all the major media outlets, including CNN, NPR, Scientific American, and Discovery Channel. Other significant recognitions include invitations to the Frontiers of Engineering Symposium and the Japan–America Frontiers of Engineering Symposium organized by National Academy of Engineering; the Outstanding Young Alumnus award by the college of Engineering of Virginia Tech; the ASME Gary Anderson Early Achievement Award; the ASME DSCD Young Investigator Award; and the ASME C.D. Mote, Jr. Early Career Award.



Dr. Russell Jeter is a National Institute of Health (NIH) Postdoctoral Research Fellow in the Department of Biomedical Informatics at Emory University. He received his PhD in Mathematics (with a concentration in Bioinformatics) from Georgia State University in 2018. In addition to Dynamical Systems, his current research involves applying different types of Machine Learning to healthcare problems; in particular, utilizing healthcare data to develop methods for real-time computer-assisted diagnosis and treatment in critical care patients. His research has been featured in many major news outlets including: The New York Times, Physics World, Popular Science, Science News, and Russia News Today.



Dr. Igor Belykh is a Professor of Applied Mathematics with a joint appointment at Neuroscience Institute at Georgia State University, Atlanta, USA. He is also an Associate Member of the Center for Nonlinear Science at Georgia Institute of Technology. He received a Ph.D. in Applied Mathematics from the Andronov Chair of Nonlinear Dynamics and Automatic Control at the University of Nizhny Novgorod, Russia in 2000. He was a Postdoctoral Fellow at Swiss Federal Institute of Technology in Lausanne (EPFL) from 2001 to 2005. His research interests lie in dynamical systems and their applications in biology, neuroscience, and engineering. Dr. Belykh has served as an Associate Editor for the IEEE Transactions on Circuits and Systems II: Express Briefs and IEEE Transactions on Circuits and Systems I: Regular Papers, where he received the Best Associate Editor Award. He is on the Editorial Board of five international journals, including the SIAM Journal on Applied Dynamical Systems and the International Journal of Bifurcation and Chaos. His research has been featured in major news outlets including: The New York Times, Physics World, Popular Science, Science News, New Scientist, Russia News Today, and Welt Der Physik.