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Memory Matters in Synchronization of Stochastically Coupled Maps*

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Abstract. Synchronization of stochastically coupled chaotic oscillators is a topic of intensive research for its ubiquitous application across natural and technological systems. Several breakthroughs have been made over the last decade in understanding the underpinnings of stochastic synchronization. Yet, most of the literature has focused on memoryless switching, where the coupling between the oscillators intermittently changes independently of the switching history. Here, we analytically investigate the synchronization of two one-dimensional coupled nonlinear maps under Markovian switching. We linearize the system in the vicinity of the synchronous solution and examine the mean square asymptotic stability of the error dynamics. By leveraging state-of-the-art techniques in jump linear systems, fundamentals of ergodic theory, and perturbation analysis, we elucidate the potential of Markovian switching in manipulating the stability of synchronization. We focus on chaotic tent maps, for which we compute exact, closed-form expressions to measure the error dynamics. The hypothesis of memoryless switching has often been challenged in practical applications; this study makes a first, necessary step toward unraveling the role of switching memory in stochastic synchronization.

Key words. chaos, Lyapunov exponent, Markov process, mean square stability, switching

AMS subject classifications. 34D06, 34D08, 37H10, 93E15

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1. Introduction.

Examples of coupled dynamical systems include Internet routers, power grids, genetic networks, ecological networks, neuronal circuits, and communication/social networks [4, 49, 68]. A great deal of attention has been devoted to examining the relationship between the emergent behavior of the network and its intrinsic dynamical and topological features. Specifically, researchers have extensively studied the interplay between network topology and individual dynamics in the context of network synchronization; see, for example, [7, 11, 14, 15, 25, 50, 51]. Most studies have focused on networks whose connections are static, and only in the last decade have researchers considered networks with a topology that evolves in time based on a deterministic [5, 19, 23, 24, 27, 28, 30, 36, 39, 40, 41, 65, 66] or stochastic rule [6, 38, 42]. These networks belong to a wide class of evolving dynamical networks whose nonlinear dynamics and control are a hot research topic due to their potential application in a variety of key domains across science and engineering; see, for example, the recent reviews in [9, 25] and references therein.

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In many realistic networks, the dynamical systems only interact sporadically via on-off connections. Examples include stochastically switching engineering networks and circuits, such as power converters [70], and packet switched networks, such as the Internet [44]. Blinking networks were introduced in [10, 31, 45, 61, 62, 64, 67] to model this form of intermittent interactions, whose evolution could be tailored to support synchronization even though the network may be disconnected at any instant in time. These networks are generally composed of connections that switch on and off stochastically, at a frequency which is much larger than the characteristic frequency of the individual oscillators. In this "fast switching limit," it has been proved that synchronization in the blinking network emerges almost surely if synchronization in a static network, obtained by replacing the random variables by their mean, becomes stable [10, 31, 45, 61, 62, 63, 67].

Since their inception, we have examined several aspects of synchronization in fast switching blinking networks in both continuous- and discrete-time settings. Experiments on analog circuits to demonstrate the possibility of fast switching synchronization for periodically coupled units were presented in [57, 59], building on the theoretical insight in [60] on almost sure stability of synchronization and in [58] on the control of synchronization. In the discrete-time setting, our work has contributed to an improved understanding of mean square stability of consensus [1, 2, 46, 55] and synchronization of nonlinear maps [53, 54], and it has put forward criteria for the control of synchronization [3, 47, 48]. We have also investigated the effect of inherent noise in the maps, which could destroy synchronization [1] and potentially lead to a phase transition [56]. Beyond synchronization, a rigorous theory for the behavior of stochastically switching networks was developed in [32, 33]. In particular, this theory allows for deriving explicit bounds that relate the probability of converging toward an attractor of a multistable blinking network, the switching frequency, and the chosen initial conditions [8]. In a recent paper [29], we have made progress toward understanding synchronization in stochastically switching networks beyond the fast switching limit. We have demonstrated the central role of nonfast switching, which may provide opportunity for stochastic synchronization in a range of switching periods where fast switching fails to synchronize the maps, confirming previous numerical evidence in a continuous-time setting [37].

A common feature of all these efforts is that the blinking networks are a sequence of independent identically distributed (i.i.d.) random variables, without memory of prior switching history. From a practical point of view, switching memory may be critical to model collective dynamics in which the underlying network reflects the geographic arrangement of the coupled units. For example, in models of animal grouping, decision making is often based on local interactions among neighbors that change in time depending on their motion [72]. Similarly, epidemic spreading is determined by the physical contact between individuals, which is modulated by their health status and travel [71]. In the literature on time-continuous blinking models, the possibility of memory effects was explored in [63, 64], but ultimately these results are limited to the analysis of a fast switching limit. In this limit, the time scale of the units' motion is so fast that synchronization will be determined by an average network, computed as the long-run expectation of the dynamic network. Similar findings have been gathered through numerical simulations in [28]. Presently, a theory for synchronization in blinking networks where the units are coupled through a general Markovian process beyond fast switching is lacking.

Consensus problems in networks of linear time-invariant systems with Markovian switching have been recently studied in the technical literature, deriving necessary and sufficient conditions for stochastic convergence to a common state [43, 69, 73, 74]. In these problems, the maps are typically taken as the identity in one dimension and the switching network results in a nonexpansive process, in which the error dynamics is bound to never increase. The latter condition is related to the stochasticity of the state matrix, whose components are nonnegative and whose rows sum to one [61]. Here, we consider synchronization of nonlinear maps with arbitrary coupling gains, leading to nontrivial error dynamics, which, in principle, may increase in time or exhibit nonmonotonous decay. We focus on a pair of one-dimensional coupled maps, to gather analytical insight into the fundamental processes that shape stochastic synchronization under Markovian switching.

By linearizing the error dynamics in the vicinity of the synchronous solution, we derive a time-varying jump linear system whose stability controls the local synchronization of the coupled maps. Building on state-of-the-art techniques in jump linear systems [22], we formulate an ancillary deterministic problem for the mean square error dynamics, in the form of a linear N-dimensional time-varying system, with N being the number of states in the chain. The analysis of this deterministic system allows for examining the mean square stability of the error dynamics, without the need for intensive Monte Carlo simulations that would hamper the accurate evaluation of the stability of synchronization. Specializing the transition matrix of the Markov chain to i.i.d. switching sequences affords the recovery of classical results in the technical literature on static synchronization and stochastic synchronization [29]. Similarly, specializing the problem to synchronization about a fixed point reduces to the eigenvalue analysis underlying mean square stability of time-invariant jump linear systems, which is extensively studied in the control literature [22].

In the context of general Markov chains and time-varying dynamics, we establish an array of mathematical techniques to elucidate the stability of synchronization and pinpoint the added value of switching memory on synchronization. First, we establish upper and lower bounds for the Lyapunov exponent of the error dynamics, which translate into easy-to-apply conditions on the coupling gains and probability transition matrix to ensure or exclude synchronization. Next, we present a perturbative analysis, which enables the computation of the Lyapunov exponent for a Markovian switching proximal to an i.i.d. sequence. At the leading order, the Lyapunov exponent for Markovian switching is written as a linear combination of the Lyapunov exponent for the i.i.d. sequence and a perturbation, which depends on the memory of the chain. For chaotic systems, we demonstrate the application of Birkhoff's ergodic theorem [18] to compute all the bounds and the perturbative solution in terms of the invariant density of the synchronous solution. For chaotic tent maps, we establish closed-form results, affording critical insight into the physics of synchronization. We focus on a two-state Markov chain to ease the illustration of our analytical results; the generalization of our results to a multistate Markov chain is straightforward.

From the analysis of the upper and lower bounds for the Lyapunov exponent, we illustrate the existence of coupling gain values that always lead to synchronization and others that instead hinder synchronization for any selection of the probability transition matrix. We also offer evidence for the feasibility of manipulating synchronization through memory effects, by comparing the bounds for Markovian and i.i.d. switching. From the detailed analysis of i.i.d.

switching as a function of the switching probability and the coupling gains, we demonstrate a number of counterintuitive results. For example, we show the existence of values of the coupling gains that individually afford synchronization in the case of static coupling but could lead to unstable error dynamics for i.i.d. switching. Some of these surprising instances are resolved through Markovian switching, whereby we successfully demonstrate the possibility of changing the sign of the Lyapunov exponent of the error dynamics by varying the memory of the chain without changing the long-run average coupling.

The rest of paper is organized as follows. In section 2, we introduce the synchronization problem and present our theoretical framework. In section 3, we apply our findings to coupled chaotic maps, and in section 4, we summarize our main results and outline possible research directions. The appendix contains a comparison between Monte Carlo simulations and theoretical predictions on the mean square error dynamics.

2. Linear stability of synchronization under Markovian switching.

2.1. Problem statement. We investigate the stochastic synchronization of two one-dimensional maps with state variables $x_i \in \mathbb{R}$, $i \in \{1, 2\}$. We assume that the individual dynamics of each map evolves according to $x_i(k+1) = F(x_i(k))$, where $k \in \mathbb{Z}^+$ is the discrete time variable and $F\colon \mathbb{R} \to \mathbb{R}$ is a nonlinear scalar function that is differentiable almost everywhere [52].

The maps are linearly coupled such that

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} F(x_1(k)) + \varepsilon^1_{\theta(k)}(x_2(k) - x_1(k)) \\ F(x_2(k)) + \varepsilon^2_{\theta(k)}(x_1(k) - x_2(k)) \end{bmatrix} \tag{1}$$

with initial conditions x_{01} and x_{02}, respectively. Here, $\theta : \mathbb{Z}^+ \to \{1, \dots, N\}$ is a finite-state homogeneous Markov chain over a set of cardinality N with probability transition matrix $P \in \mathbb{R}^{N\times N}$, and $\varepsilon^1 : \{1, \dots, N\} \to \mathbb{R}$ and $\varepsilon^2 : \{1, \dots, N\} \to \mathbb{R}$ are two coupling gains, which as time progresses take values in $\{\varepsilon^1_1, \dots, \varepsilon^1_N\}$ and $\{\varepsilon^2_1, \dots, \varepsilon^2_N\}$, respectively. The initial condition for the Markov chain is θ_0, drawn from the initial probability distribution $p_0 \in \mathbb{R}^N$.

The maps synchronize at time-step k if their states are identical, that is, $x_1(k) = x_2(k)$. From (1), one may note that once the maps synchronize at a given time-step, they will stay synchronized. The common synchronous solution $s(k)$ is a solution of the individual dynamics, whereby $s(k+1) = F(s(k))$. The linear stability of synchronization can be investigated by linearizing (1) in the neighborhood of the synchronous solution, resulting in the following variational equation:

$$\xi(k+1) = \left[F'(s(k)) - d_{\theta(k)}\right] \xi(k). \tag{2}$$

Here, prime indicates differentiation; $\xi(k) = x_1(k) - x_2(k)$ is the synchronization error at time-step k; and $d_{\theta(k)} = \varepsilon^1_{\theta(k)} + \varepsilon^2_{\theta(k)}$ is the net coupling, which takes values in $\{d_1, \dots, d_N\}$. Equation (2) describes the linear transverse dynamics of the coupled maps, measured with respect to the difference between their states $\xi(k)$. This quantity is zero when the two oscillators are synchronized.

2.2. Mean square stochastic stability. Our treatment of (2) follows the general line of argument proposed in [20] and summarized in [22] for a finite-dimensional jump linear system, although the approach presented therein is made explicit only to time-invariant systems, for which the synchronous solution corresponds to a fixed point. We say that (2) is mean square asymptotically stable if for any initial condition $\xi_0 \neq 0$ and any probability distribution p_0

$$\lim_{k\to\infty} \mathrm{E}[\xi^2(k)] = 0, \tag{3}$$

where the expectation $\mathrm{E}[\cdot]$ is computed with respect to the probability space induced by the Markov chain.

In contrast to the special case of i.i.d. stochastic switching considered in [29], assessing the mean square stability of (2) involves the detailed treatment of the initial probability distribution. In other words, expectations shall be computed by accounting for both the stochastic nature of the switching and its initial distribution. To this end, it is convenient to introduce the indicator function $1_{\{\theta=i\}}$ for the ith state of the Markov chain, which is equal to 1 when $\theta = i$ and is zero otherwise. By using indicator functions, the second moment of $\xi(k)$ can be written as

$$\mathrm{E}[\xi^2(k)] = \sum_{i=1}^{N} q_i(k), \tag{4}$$

where

$$q_i(k) = \mathrm{E}[\xi^2(k)1_{\{\theta(k)=i\}}], \quad i = 1, \ldots, N. \tag{5}$$

The deterministic quantity (5) corresponds to the expectation of $\xi^2(k)$ conditional to the state of the Markov chain being equal to i at time k. By using (2), we establish a recursive relation to express $q_i(k)$ as a linear combination of $q_1(k), \ldots, q_N(k)$. Specifically, by substituting (2) into (5) at time $(k + 1)$, we obtain

$$q_i(k+1) = \mathrm{E}\left[r_{\theta(k)}(k)\xi^2(k)1_{\{\theta(k+1)=i\}}\right], \tag{6}$$

where, for brevity, we have introduced the time-varying function $r : \{1, \ldots, N\} \times \mathbb{Z}^+ \to \mathbb{R}$ that at the generic time-step k takes values in $\{r_1(k), \ldots, r_N(k)\}$ defined by

$$r_i(k) = \left[F'(s(k)) - d_i\right]^2, \quad i = 1, \ldots, N. \tag{7}$$

Similar to (4), the expectation on the right-hand side of (6) can be written as

$$\mathrm{E}[r_{\theta(k)}(k)\xi^2(k)1_{\{\theta(k+1)=i\}}] = \sum_{j=1}^{N} \mathrm{E}\left[r_{\theta(k)}(k)\xi^2(k)1_{\{\theta(k)=j\}}1_{\{\theta(k+1)=i\}}\right]. \tag{8}$$

By recalling that the ijth entry of the transition matrix P_{ij} is the probability that $\theta(k+1) = j$ given that $\theta(k) = i$, we finally obtain

$$q_i(k+1) = \sum_{j=1}^{N} r_j(k)P_{ji}q_j(k). \tag{9}$$

An equivalent result was derived in Proposition 3 of [20] for higher-order time-invariant systems.

We define the N-dimensional vector $q(k) = [q_1(k) \cdots q_N(k)]^{\mathrm{T}}$ for all times, where T indicates transposition. By using $q(k)$, the iteration in (9) can be compactly rewritten as

$$q(k+1) = L(k)q(k). \tag{10}$$

Here, we have introduced

$$L(k) = P^{\mathrm{T}}R(k) \tag{11}$$

and the diagonal matrix $R(k) = \mathrm{diag}[r(k)]$, with $r(k)$ being the N-dimensional vector whose components are given by (7). By construction, $L(k)$ is a nonnegative matrix, but it is in general not stochastic, due to the presence of $R(k)$. By iterating (10) from the initial time to time k, we ultimately find

$$q(k) = \xi_0^2 \left[\prod_{t=0}^{k-1} L(t)\right] p_0. \tag{12}$$

The second moment of $\xi(k)$ is simply obtained from (12) by left multiplying by the N-dimensional vector of all ones 1_N such that

$$\frac{\mathrm{E}[\xi^2(k)]}{\xi_0^2} = 1_N^{\mathrm{T}} \left[\prod_{t=0}^{k-1} L(t)\right] p_0. \tag{13}$$

The exponential growth rate of the mean square dynamics for the initial distribution p_0 can be quantified through the Lyapunov exponent. If it exists, the following limit measures the Lyapunov exponent for the mean square error dynamics:

$$\lambda(p_0) = \lim_{k\to\infty} \frac{1}{k} \ln \left[\frac{\mathrm{E}[\xi^2(k)]}{\xi_0^2}\right] = \lim_{k\to\infty} \frac{1}{k} \ln \left[1_N^{\mathrm{T}} \left[\prod_{t=0}^{k-1} L(t)\right] p_0\right]. \tag{14}$$

The analysis of this Lyapunov exponent is one of the main objectives of this study.

Under the premise of the existence of the limit in (14), the Lyapunov exponent can be used to ascertain the mean square asymptotic stability of the original system in (2). Specifically, negative values of the Lyapunov exponent for any selection of p_0 identify the region where the system is mean square asymptotically stable, while positive values are indicative of instability. In general, the limit in (14) may not exist and one should resort to the study of the upper and lower Lyapunov exponents computed as the supremum and infimum for the exponential rates of functions bounding the mean square dynamics from above and below; see, for example, [35]. In what follows, we hypothesize regularity conditions that enable the exact computation of the Lyapunov exponent through the limit, for every choice of p_0. Further, we exclude the possibility of singular dynamics (for which the error would go to zero in a single step), such that the computation of the Lyapunov exponent is independent of the selection of the initial time, taken as zero in our case.

2.3. Special cases. For time-invariant dynamics associated with synchronization about a fixed point, $L(k)$ is constant. In this case, mean square asymptotic stability reduces to enforcing the Schur stability of L, that is, ensuring that the spectral radius is less than 1. The equivalence between mean square asymptotic stability of the system and Schur stability of L is studied in detail in [21, 26] for finite-dimensional jump linear systems. While the sufficiency of Schur stability for mean square stability is trivial from (10) as originally posited in [26], its necessity requires some more thought. The proof relies on the arbitrariness of p_0 and the nonnegativeness of both p_0 and L as shown in [21]. By referring to the Euclidean norm, these facts imply $1_N^{\mathrm{T}} L^k p_0 \geq \| L^k p_0 \|_2$, such that $\lim_{k\to\infty} \| L^k p_0 \|_2 = 0$. Using Proposition 1 in [20], the necessity of Schur stability follows; the proposition relies on the decomposition of square matrices into positive-semidefinite matrices, which here simply entails writing a generic vector as the difference between two vectors with nonnegative entries.

Beyond time-invariant dynamics, in the special case of i.i.d. stochastic switching, we have that $P = 1_N p^{\mathrm{T}}$, where p is the N-dimensional vector whose ith component p_i is equal to the probability that $\theta(k) = i$. Equation (14) takes the form

$$\lambda^{\text{i.i.d.}}(p_0) = \lim_{k\to\infty} \frac{1}{k} \ln \left[1_N^{\mathrm{T}} \left[\prod_{t=0}^{k-1} p\, r(t)^{\mathrm{T}} \right] p_0 \right], \tag{15}$$

where we have used (7) and (11). By expanding the product in (15), we find

$$\lambda^{\text{i.i.d.}}(p_0) = \lim_{k\to\infty} \frac{1}{k} \ln \left[1_N^{\mathrm{T}} p \left[\prod_{t=1}^{k-1} r(t)^{\mathrm{T}} p \right] r(0)^{\mathrm{T}} p_0 \right], \tag{16}$$

which simplifies to

$$\lambda^{\text{i.i.d.}} = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \ln \left[r(t)^{\mathrm{T}} p \right], \tag{17}$$

where we have made it explicit that the Lyapunov exponent is independent of p_0; note that the summation above may start from 0 or 1 (or any finite value) without changing the computation. This expression is equivalent to equation (6) in [29] when the switching period is one, such that the coupling gain changes at every time-step.

If p is such that all its entries are zero except the ith, which is equal to one, then (17) further reduces to the deterministic Lyapunov exponent of the map for $\theta(k) = i$. Specifically,

$$\lambda_i^{\text{st}} = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \ln \left[r_i(t) \right], \tag{18}$$

where we have used the subscript "st" to make clear that it refers to the static case, in which switching is prevented.

For a chaotic system, one may apply Birkhoff's ergodic theorem [18] to replace the summation in either (17) or (18) with integration based on the knowledge of the invariant density of the map F, defined on a set B. The application of Birkhoff's theorem to evaluate static

Lyapunov exponents associated with the stability of synchronization was first proposed in [34], while the application to the study of i.i.d. switching was proposed in [29]. Once the invariant density $\rho : B \to \mathbb{R}^+$ is found analytically or numerically [13, 16], we may compute λ_i^{st} and $\lambda^{\mathrm{i.i.d.}}$ as follows:

$$\lambda_i^{\mathrm{st}} = \int_B \ln\left[y_i(z)\right]\rho(z)\mathrm{d}z, \tag{19a}$$

$$\lambda^{\mathrm{i.i.d.}} = \int_B \ln\left[p^{\mathrm{T}}y(z)\right]\rho(z)\mathrm{d}z, \tag{19b}$$

where

$$y_i(x) = (F'(x) - d_i)^2, \quad i = 1, \dots, N, \tag{20}$$

and $y(x) = [y_1(x) \cdots y_N(x)]^{\mathrm{T}}$.

2.4. Conservative bounds. First, we establish upper and lower bounds, independent of the transition matrix of the Markov chain, to offer insight into the role of the individual dynamics on the stochastic stability of the synchronous solution. To this end, we define

$$\underline{r}(k) = \min_{i=1,\dots N} r_i(k), \tag{21}$$

$$\overline{r}(k) = \max_{i=1,\dots N} r_i(k). \tag{22}$$

Using these quantities, the generic ijth entry of the matrix $L(t)$ in (10), $L_{ij}(t) = P_{ji}r_j(t)$, can be bounded as follows:

$$\underline{r}(t)P_{ji} \le L_{ij}(t) \le \overline{r}(t)P_{ji}. \tag{23}$$

We can then bound the ijth entry of the product of the $L(t)$'s in (14) to obtain

$$P_{ji}^k \prod_{t=0}^{k-1} \underline{r}(t) \le \left[\prod_{t=0}^{k-1} L(t)\right]_{ij} \le P_{ji}^k \prod_{t=0}^{k-1} \overline{r}(t). \tag{24}$$

By using these bounds, the argument of the logarithm in (14) satisfies

$$\prod_{t=0}^{k-1} \underline{r}(t) \le \sum_{i,j=1,\dots N} \left[\prod_{t=0}^{k-1} L(t)\right]_{ij} p_{0j} \le \prod_{t=0}^{k-1} \overline{r}(t), \tag{25}$$

where we have used the fact that the power of stochastic matrices is also a stochastic matrix. Hence, the Lyapunov exponent can be uniformly bounded in p_0 as follows:

$$\underline{\lambda} \le \lambda(p_0) \le \overline{\lambda} \tag{26}$$

with

$$\underline{\lambda} = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \ln \underline{r}(t), \tag{27}$$

$$\overline{\lambda} = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \ln \overline{r}(t). \tag{28}$$

In general, these quantities are not the deterministic Lyapunov exponents of the map. For chaotic systems, Birkhoff's ergodic theorem may be applied once again to evaluate these conservative bounds in terms of the invariant density $\rho(t)$, following the line of argument underlying (19). Specifically, we find

$$\underline{\lambda} = \int_B \ln\left[\underline{y}(z)\right]\rho(z)\mathrm{d}z, \tag{29}$$

$$\overline{\lambda} = \int_B \ln\left[\overline{y}(z)\right]\rho(z)\mathrm{d}z, \tag{30}$$

where

$$\underline{y}(x) = \min_{i=1,\ldots N} y_i(x), \tag{31}$$

$$\overline{y}(x) = \max_{i=1,\ldots N} y_i(x). \tag{32}$$

In order to account for the specific features of the transition matrix, an additional upper bound for the Lyapunov exponent is established based on the use of submultiplicative properties of induced matrix norms [12]. With respect to the ∞-norm, from (14) we obtain

$$1_N^{\mathrm{T}}\left[\prod_{t=0}^{k-1} L(t)\right] p_0 \leq \| 1_N^{\mathrm{T}} \|_\infty \left[\prod_{t=0}^{k-1} \| L(t) \|_\infty\right] \| p_0 \|_\infty \leq \| 1_N^{\mathrm{T}} \|_\infty \left[\prod_{t=0}^{k-1} r^\star(t)\right] \| p_0 \|_\infty, \tag{33}$$

where

$$r^\star(k) = \max_{i=1,\ldots,N} \sum_{j=1}^{N} P_{ji} r_j(k). \tag{34}$$

Thus, we can ultimately bound the Lyapunov exponent uniformly in p_0 through

$$\lambda(p_0) \leq \lambda^\star = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \ln r^\star(t). \tag{35}$$

For chaotic systems, Birkhoff's theorem may be leveraged once again to evaluate $\lambda^\star$, such that

$$\lambda^\star = \int_B \ln y^\star(z)\rho(z)\mathrm{d}z, \tag{36}$$

where

$$y^\star(x) = \max_{i=1,\ldots,N} \sum_{j=1}^{N} P_{ji} y_j(x). \tag{37}$$

2.5. Perturbation estimates.

An alternative approach to estimate the Lyapunov exponents entails the use of perturbation methods to simplify the matrix product. First, we hypothesize that the Markov chain is close to an i.i.d. process, whereby we assume

$$P = 1_N p^{\mathrm{T}} + \tilde{P}, \tag{38}$$

where $\|\tilde{P}\| = O(\epsilon)$, for some norm, with ϵ being a small positive number and $O(\cdot)$ being the Landau symbol. By replacing (38) in the outermost right-hand side of (14) and discarding terms of order higher than ϵ, we obtain

$$\lambda(p_0) \simeq \lim_{k\to\infty} \ln \left[\prod_{t=1}^{k-1} r(t)^{\mathrm{T}} p r(0)^{\mathrm{T}} p_0 + \left(\sum_{q=1}^{k-1} \left(r(q)^{\mathrm{T}} \tilde{P}^{\mathrm{T}} R(q-1) p \right) \prod_{t=1, t\neq q, q-1}^{k-1} r(t)^{\mathrm{T}} p r(0)^{\mathrm{T}} p_0 \right) \right]. \tag{39}$$

Next, we expand the logarithm as a MacLaurin series with respect to ϵ, leading to the following compact expression:

$$\lambda^{\epsilon} = \lambda^{\text{i.i.d.}} + \Delta\lambda, \tag{40}$$

where

$$\Delta\lambda = \lim_{k\to\infty} \frac{1}{k} \sum_{q=1}^{k-1} \frac{r(q)^{\mathrm{T}} \tilde{P}^{\mathrm{T}} R(q-1) p}{r(q)^{\mathrm{T}} p r(q-1)^{\mathrm{T}} p} \tag{41}$$

is the perturbation on the Lyapunov exponent associated with $\tilde{P}$. The approach could be potentially extended to higher-order perturbations, by including terms of higher order in ϵ. We comment that in the linearized expression, the dependence on p_0 is lost.

For a chaotic map, the application of Birkhoff's ergodic theorem leads to an elegant representation of $\Delta\lambda$, that is,

$$\Delta\lambda = \int_B \frac{\hat{y}(t)^{\mathrm{T}} \tilde{P}^{\mathrm{T}} Y(t) p}{\hat{y}(t)^{\mathrm{T}} p y(t)^{\mathrm{T}} p} \rho(t) \mathrm{d}t, \tag{42}$$

where $Y(t) = \operatorname{diag}[y(t)]$ and $\hat{y}(t) = [\hat{y}_1(t) \cdots \hat{y}_N(t)]^{\mathrm{T}}$ with

$$\hat{y}_i(t) = (F'(F(t)) - d_i)^2, \quad i = 1, \ldots, N. \tag{43}$$

Equation (43) mirrors the one-time-step lag in (41), such that the dynamics at time $q-1$ interacts with the dynamics at time q to modulate the linear stability of synchronization. Increasing the order of the perturbation analysis will lead to the presence of a higher-order interaction in the perturbation on the Lyapunov exponent, which should manifest in composite functions of order larger than one in (42).

For a given Markov chain, one may find the i.i.d. process about which to perform the expansion by solving a simple optimization problem. Specifically, given P, the corresponding rank one matrix $1_N p^{\mathrm{T}}$ about which to perform the expansion may be determined as

$\arg\min_p \|P - 1_N p^{\mathrm{T}}\|$ for a given norm and p varying such that its entries are all nonzero and have unit sum.

Alternatively, the i.i.d. process may be determined from the stationary limit of the Markov chain, if it exists, to establish a direct connection between the long-run behavior of the switching process and the stability of synchronization. The idea of linking synchronization of switching systems to the long-run behavior of the coupling dates back a decade. This approach was originally proposed in [63] to study synchronization of continuous-time chaotic oscillators coupled through Markovian discrete-time switching networks. Therein, it was shown that under fast switching conditions, the synchronization of the stochastically coupled maps could be inferred by studying synchronization over a static network, corresponding to the long-run average network.

3. Application to chaotic tent maps.

We demonstrate our analytical framework to study stochastic synchronization of two coupled chaotic tent maps, with parameter equal to 2, such that $F(x) = 2x$ for $x \in [0, 1/2]$ and $F(x) = -2x + 1$ for $x \in (1/2, 1]$. In this case, the invariant density is simply $\rho(x) = 1$ in $B = [0, 1]$ (see, for example, [16]), thereby simplifying the application of Birkhoff's theorem and ultimately leading to closed-form results.

We focus on a two-state Markov chain, $N = 2$, such that

$$P = \begin{bmatrix} u & 1-u \\ v & 1-v \end{bmatrix} \tag{44}$$

with $u, v \in (0, 1)$. Here, $1 - u$ is the probability of switching from state 1 to 2 and v is the probability of switching from 2 to 1. The stationary distribution of this ergodic Markov chain is $\pi = [\frac{v}{1-u+v}, 1 - \frac{v}{1-u+v}]^{\mathrm{T}}$, such that $P^{\mathrm{T}}\pi = \pi$. This distribution measures the average time spent in a given state, defined as the expected time the chain will take to return to that state; see, for example, [17]. Specifically, we have $\mathrm{E}\left[\inf\{k \in \mathbb{Z}^+ : \theta(k) = i\} | \theta_0 = i\right] = \frac{1}{\pi_i}$ with π_i being the ith component of π. The probability distribution from which the Markov chain is initiated is the uniform distribution, such that $p_0 = [1/2, 1/2]^{\mathrm{T}}$. Numerical computations were performed in Mathematica.

3.1. Understanding what Markovian switching may afford: The application of conservative bounds.

Paralleling the reasoning in [29], we consider three general cases as a function of the values of the coupling gains d_1 and d_2: Case I, Case II, and Case III. These cases are informed by the synchronization of statically coupled tent maps, which is controlled by the Lyapunov exponent in (19a). By replacing the equation of the tent map and its invariant density, we obtain

$$\lambda^{\mathrm{st}}(d^{\mathrm{st}}) = \ln|2 - d^{\mathrm{st}}| + \ln|2 + d^{\mathrm{st}}| \tag{45}$$

as a function of the static coupling gain d^{st}. This expression was first computed in [34]. For statically coupled tent maps, synchronization is attained in the following set composed of two disjoint intervals for the coupling gain d^{st}: $\mathcal{I}^{\mathrm{st}} = (-\sqrt{5}, -\sqrt{3}) \cup (\sqrt{3}, \sqrt{5})$. When $d^{\mathrm{st}} = \pm 2$, synchronization is attained in a single time-step, whereby the Lyapunov exponent goes to $-\infty$.

For Case I, neither of the two coupling gains supports synchronization of the statically coupled maps, that is, d_1 and d_2 do not belong to $\mathcal{I}^{\mathrm{st}}$. For Case II, one of the coupling gains supports synchronization and the other does not, that is, either d_1 or d_2 belongs to $\mathcal{I}^{\text{st}}$, while the other is in $\mathbb{R}/\mathcal{I}^{\text{st}}$. Finally, for Case III, both the coupling gains support synchronization, that is, both d_1 and d_2 are in $\mathcal{I}^{\text{st}}$.

The conservative bounds could be used to identify values of the coupling gains for which Markovian switching could be pursued to manipulate the stability of synchronization of the coupled maps. By applying once more Birkhoff's theorem in (29), we establish the following closed-form expression for the lower and upper bounds of the Lyapunov exponent:

$$\underline{\lambda} = \frac{1}{2}\ln[\min\{(2-d_1)^2,(2-d_2)^2\}] + \frac{1}{2}\ln[\min\{(2+d_1)^2,(2+d_2)^2\}], \tag{46}$$

$$\overline{\lambda} = \frac{1}{2}\ln[\max\{(2-d_1)^2,(2-d_2)^2\}] + \frac{1}{2}\ln[\max\{(2+d_1)^2,(2+d_2)^2\}]. \tag{47}$$

When $\underline{\lambda} \geq 0$, no Markovian switching could be designed to achieve synchronization, and, on the other hand, when $\overline{\lambda} \leq 0$, synchronization is possible for any selection of the switching signal. From Figure 1(a), we evince that there is a wide region of the d_1d_2 space in which synchronization is not possible. By construction, this region is a subset of Case III described earlier, in which none of the two coupling gains would support synchronization for statically coupled maps. On the other hand, Figure 1(b) demonstrates that synchronization is feasible for any switching signal in two small portions, which nearly cover $[\sqrt{3},\sqrt{5}]\times[\sqrt{3},\sqrt{5}]$ and $[-\sqrt{5},-\sqrt{3}]\times[-\sqrt{5},-\sqrt{3}]$ pertaining to Case III. For completeness, we report a zoomed-in view of one of these portions in Figure 1(d), along with a zoomed-in view of Figure 1(a) in the same region as Figure 1(c).

By applying Birkhoff's theorem again, we derive a closed-form result for the other conservative upper bound in (36),

$$\begin{aligned}\lambda^\star = {}&\frac{1}{2}\ln[\max\{u(2-d_1)^2+v(2-d_2)^2,(1-u)(2-d_1)^2+(1-v)(2-d_2)^2\}]\\ &+\frac{1}{2}\ln[\max\{u(2+d_1)^2+v(2+d_2)^2,(1-u)(2+d_1)^2+(1-v)(2+d_2)^2\}].\end{aligned} \tag{48}$$

Figure 2 shows a zoomed-in view of the d_1d_2 space for two selected values of u and v. By comparing Figure 2 with Figure 1(d), we confirm the possibility of leveraging Markovian switching to synchronize coupled maps in Cases II and III. Specifically, by changing the transition probability matrix, we may enlarge the synchronization region within Case III as compared to Figure 1(d), and we may further extend beyond this region to Case II, where one of the couplings does not support synchronization. In Figure 2(a), we show i.i.d. switching with $p_1 = u = v = 0.2$ and Markovian switching with $u = 0.6$ and $v = 0.1$ as an illustration; both instances have the same stationary distribution with $\pi_1 = 0.2$.

3.2. What i.i.d. switching can and cannot do.

By applying Birkhoff's theorem in (19b), we evaluate the Lyapunov exponent for two coupled maps with i.i.d. switching between two coupling gains d_1 and d_2 with probability p_1 and $1-p_1$:

$$\begin{aligned}\lambda^{\text{i.i.d.}}(d_1,d_2,p_1) = {}&\frac{1}{2}\ln[p_1^2(4-d_1^2)^2+(1-p_1)^2(4-d_2^2)^2+p_1(1-p_1)((2-d_1)^2(2+d_2)^2\\ &+(2-d_2)^2(2+d_1)^2)].\end{aligned} \tag{49}$$

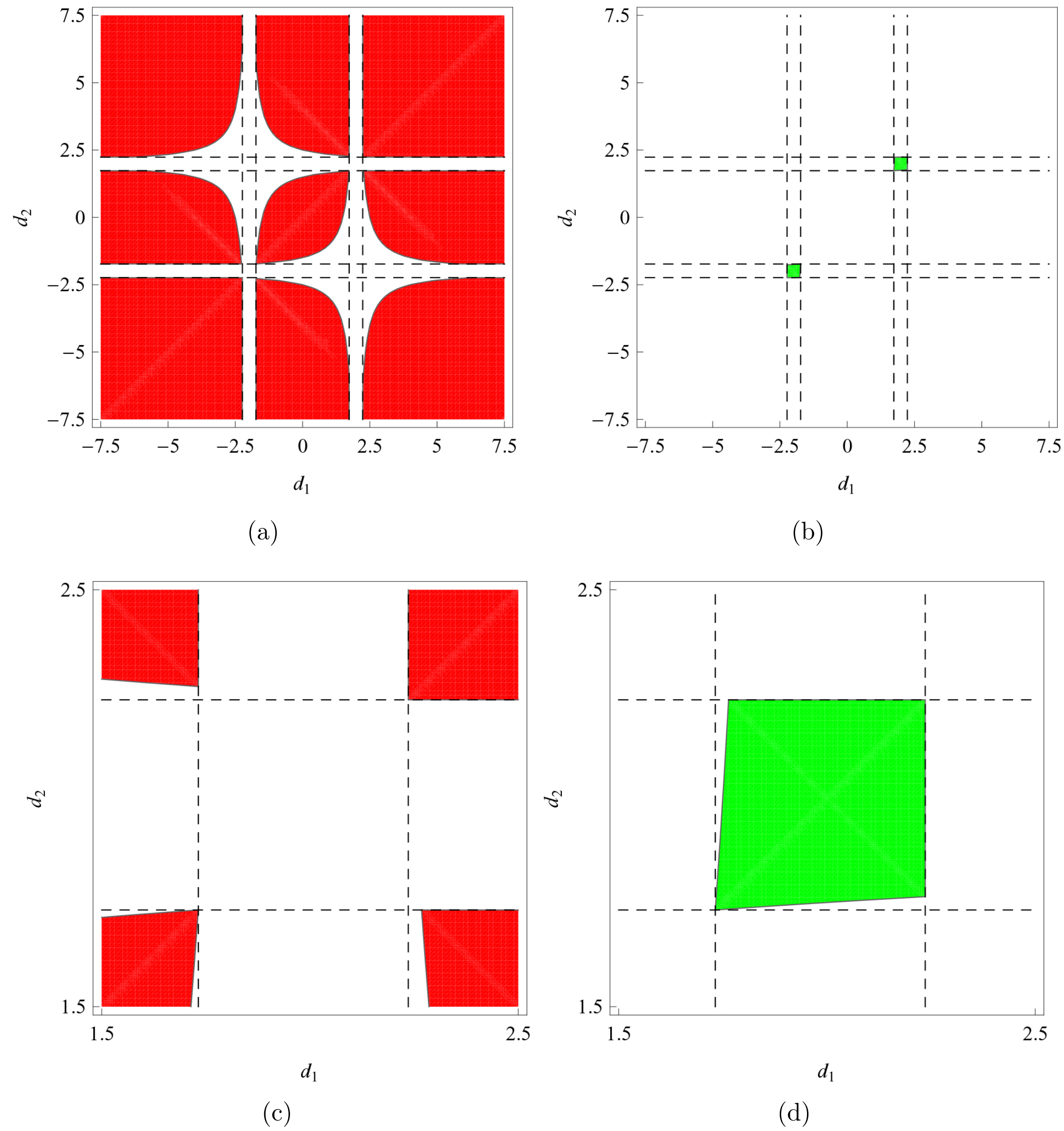


Figure 1. *Predictions of the conservative bounds on the synchronization of coupled chaotic tent maps with Markovian switching. With respect to the bounds in* (46)*, red identifies a positive value of* $\underline{\lambda}$*, green a negative value of* $\overline{\lambda}$*, and gray zero values of either of them. The white regions correspond to negative values of* $\underline{\lambda}$ *and positive values of* $\overline{\lambda}$*, for which no inference can be made on the synchronization of the stochastically coupled maps. The dashed lines at* $\pm\sqrt{3}$ *and* $\pm\sqrt{5}$ *correspond to the critical values of* d_1 *and* d_2 *for the synchronization of statically coupled maps.* (a)–(b) $\underline{\lambda}$ *and* $\overline{\lambda}$ *in a broad range of variation of* d_1 *and* d_2*.* (c)–(d) *Zoomed-in views.*

Although this expression can be recovered from (26) in [29], a detailed analysis of stochastic synchronization for i.i.d. switching is currently lacking. In what follows, we seek to address this gap to offer a basis on which to elucidate the role of Markovian switching.

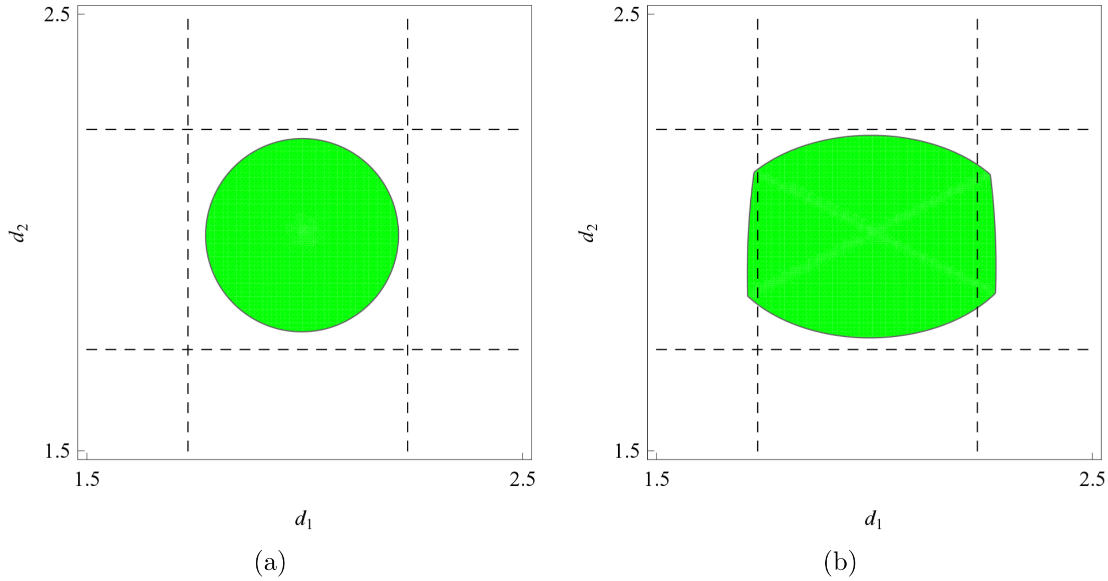


Figure 2. *Predictions of the conservative upper bound based on the ∞-norm bound on the synchronization of coupled chaotic tent maps with Markovian switching. With respect to the bound in* (48)*, green corresponds to a negative value of $\lambda^\star$ and gray to zero. The white regions indicate positive values, for which no inference can be made on the synchronization of the stochastically coupled maps. The dashed lines at $\pm\sqrt{3}$ and $\pm\sqrt{5}$ correspond to the critical values of d_1 and d_2 for the synchronization of statically coupled maps.* (a) *Zoomed-in view for $u = 0.2$ and $v = 0.2$, and* (b) *Zoomed-in view for $u = 0.6$ and $v = 0.1$.*

If neither of the two coupling gains supports synchronization of the statically coupled maps (Case I), then stochastic synchronization is not feasible for any choice of p_1. This claim is verified by evaluating (49) for every pair of coupling gains and confirming that there is not a value of p_1 for which $\lambda^{\text{i.i.d.}}(d_1, d_2, p_1)$ is negative. This step is specifically addressed by checking that the region in which the curvature of the argument of the logarithm in (49) is positive (existence of a local minimum) does not intersect with the region in which the extremum of the argument of the logarithm in (49) is between 0 and 1.

If instead one of the coupling gains supports synchronization (Case II), then stochastic synchronization is possible for some values of p_1. Assuming that $d_2 \in \mathcal{I}^{\text{st}}$, without lack of generality, for $p_1 = 0$ the maps do not synchronize and as p_1 increases, there is a critical value above which synchronization is always attained. The critical value of the probability is simply computed by solving the second-order equation $\lambda^{\text{i.i.d.}}(d_1, d_2, p_1) = 1$ for p_1.

If both of the coupling gains support synchronization (Case III), we may always find values for p_1 which leads to synchronization similar to Case II. Surprisingly, even if individually d_1 and d_2 may lead to the stability of the synchronous solution, one may find values of p_1 such that switching between them will cause a highly unstable error dynamics. If d_1 and d_2 have opposite signs, the maps synchronize only in the vicinity of $p_1 = 0$ and $p_1 = 1$, whereby a maximum for the Lyapunov exponent always exists between 0 and 1. This scenario is also found for d_1 and d_2 with the same sign, but close to the stability limits, that is, in the vicinity of the following points in the $d_1 d_2$ space: $\pm(\sqrt{3}, \sqrt{5})$ and $\pm(\sqrt{5}, \sqrt{3})$. This behavior

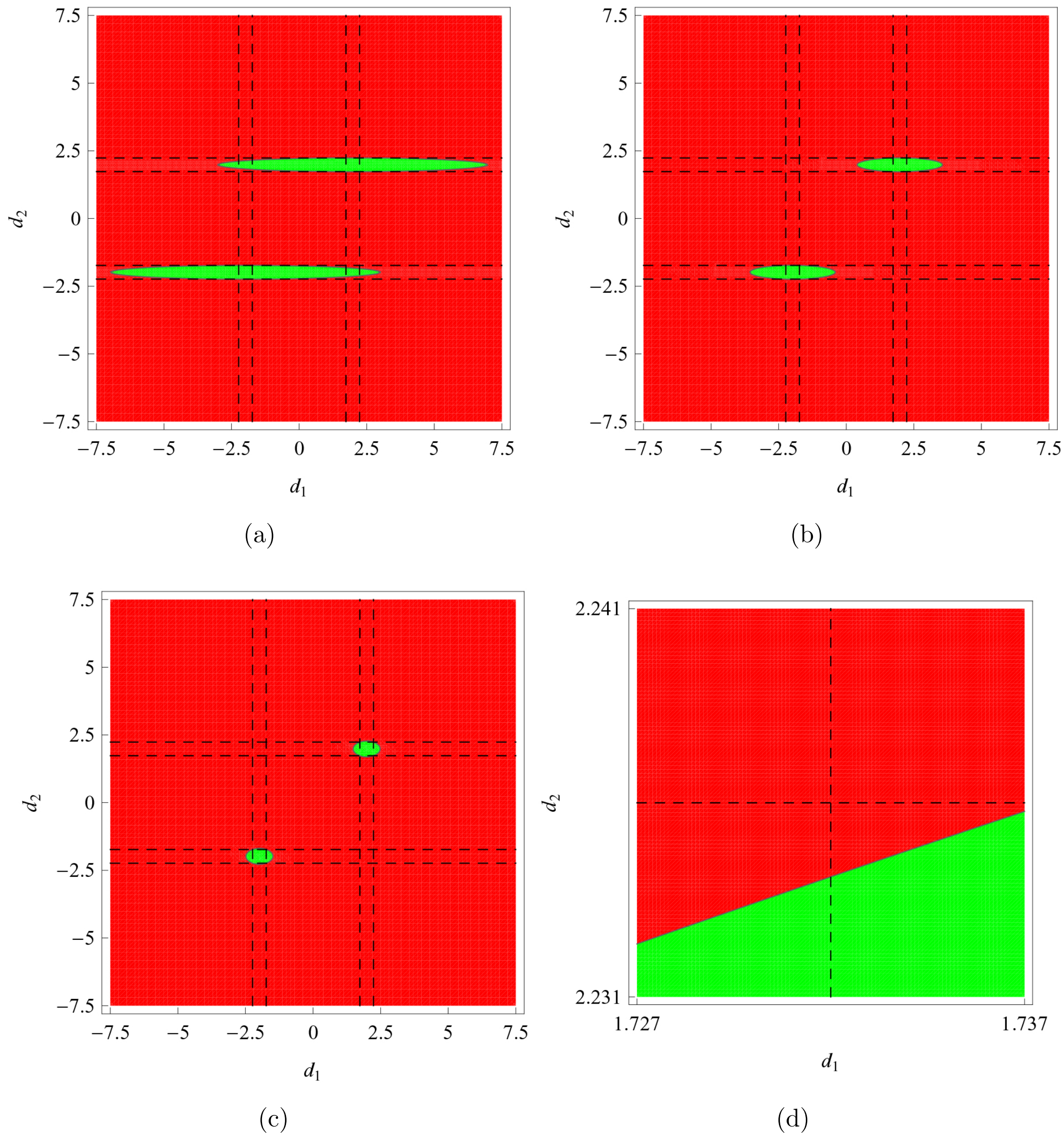


Figure 3. *Synchronization of coupled chaotic tent maps with i.i.d. switching. With respect to the Lyapunov exponent in* (49)*, red identifies a positive value, green a negative value, and gray zero. The dashed lines at* $\pm\sqrt{3}$ *and* $\pm\sqrt{5}$ *correspond to the critical values of* d_1 *and* d_2 *for the synchronization of statically coupled maps.* (a) $p_1 = 0.0025$, (b) $p_1 = 0.025$, (c) $p_1 = 0.25$, *and* (d) $p_1 = 0.25$ *(zoomed-in view).*

is compatible with the prediction from the upper bound in Figure 1, which fails to predict synchronization of stochastically coupled maps in these regions.

Figure 3 shows $\lambda^{\text{i.i.d.}}(d_1, d_2, p_1)$ for three values of p_1 as functions of d_1 and d_2, in the same parameter space explored in Figure 1. As p_1 increases, we observe that the oblate regions, where synchronization is attained, shrink toward the two squares $(\sqrt{3}, \sqrt{5}) \times (\sqrt{3}, \sqrt{5})$ and

$(-\sqrt{5},-\sqrt{3}) \times (-\sqrt{5},-\sqrt{3})$ constituting part of Case III. The rest of Case III comprises the two squares $(\sqrt{3},\sqrt{5}) \times (-\sqrt{5},-\sqrt{3})$ and $(-\sqrt{5},-\sqrt{3}) \times (\sqrt{3},\sqrt{5})$, which could support synchronization only for p_1 less than approximately 0.0451. Although Figures 3(a), (b), and (c) may suggest that synchronization is always attained within $(\sqrt{3},\sqrt{5}) \times (\sqrt{3},\sqrt{5})$ and $(-\sqrt{5},-\sqrt{3}) \times (-\sqrt{5},-\sqrt{3})$, a closer view in Figure 3(d) justifies our previous claim that that synchronization is lost for d_1 and d_2 close to some stability boundaries.

Case II synchronization is instead always visible, with the oblate regions consistently extending beyond $[\sqrt{3},\sqrt{5}] \times [\sqrt{3},\sqrt{5}]$ and $[-\sqrt{5},-\sqrt{3}] \times [-\sqrt{5},\sqrt{3}]$. In fact, as p_1 decreases the probability that the net coupling is equal to d_1 decreases and synchronization is dominated by d_2. As d_1 plays a secondary role in synchronization, we confirm the possibility of Case II synchronization. In the limit of $p_1 \to 0$, the oblate regions will become horizontal bands. If p_1 is increased above 0.5, the same behavior is observed, with the oblate regions turning vertical.

To offer more context into the complexity induced by switching, even in the simple i.i.d. case, we compare (49) with the Lyapunov exponent obtained by evaluating (45) in correspondence to the expected coupling $\mathrm{E}[d] = p_1 d_1 + (1-p_1)d_2$. Figure 4 illustrates the prediction on synchronization which would be garnered by looking at the Lyapunov exponent of the average switching for $p_1 = 0.25$. By comparing Figure 4 with Figures 3(c) and (d), we evince that (i) synchronization of the average system does not imply synchronization of the stochastically coupled maps (see the extent of the green bands in Figure 4, which is far beyond the oblate regions in Figure 3) and (ii) lack of synchronization of the average system does not imply lack of synchronization of the stochastically coupled maps (note that the transition in Figure 3(d) is abolished in Figure 4, where a zoomed-in view would just be a green box).

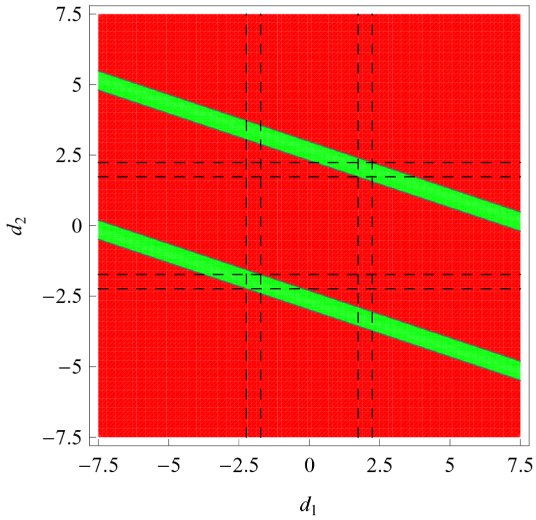


Figure 4. *Synchronization of statically coupled chaotic tent maps with a net coupling* $\mathrm{E}[d] = p_1 d_1 + (1-p_1)d_2$ *and* $p_1 = 0.25$. *With respect to the Lyapunov exponent in* (45) *evaluated in correspondence to the expected coupling, red identifies a positive value, green a negative value, and gray zero. The dashed lines at* $\pm\sqrt{3}$ *and* $\pm\sqrt{5}$ *correspond to the critical values of* d_1 *and* d_2 *for the synchronization of statically coupled maps.*

3.3. How Markovian switching may help. Using the stationary distribution, we can write the transition matrix as

$$P = \begin{bmatrix} \pi_1 & 1-\pi_1 \\ \pi_1 & 1-\pi_1 \end{bmatrix} + \delta \begin{bmatrix} -1+\pi_1 & 1-\pi_1 \\ \pi_1 & -\pi_1 \end{bmatrix}, \tag{50}$$

where we introduced $\delta = v - u$. A positive value of δ implies that it is more likely to switch coupling gain than to retain it, while a negative value indicates that it is more likely to keep the same coupling gain than to switch. By construction, irrespective of the value of δ, the long-run average of the coupling is $\pi_1 d_1 + (1-\pi_1)d_2$, which only depends on the stationary distribution.

By specializing (42) to the tent map with a uniform invariant density, we determine the following expression for the perturbation of the Lyapunov exponent:

(51)

$$\Delta\lambda = -\,\delta\frac{(1-\pi_1)\pi_1\left(4d_1^2-16d_1d_2+12d_2^2+d_1^2d_2^2-d_2^4+(d_1-d_2)^2(d_1+d_2-4)(d_1+d_2+4)\pi_1\right)^2}{(4+(d_2-4)d_2(1-\pi_1)+(d_1-4)d_1\pi_1)^2(4+d_2(4+d_2)(1-\pi_1)+d_1(4+d_1)\pi_1)^2}.$$

To evaluate the integral in (42), we have considered four different intervals $(0,1/4)$, $(1/4,1/2)$, $(1/2,3/4)$, and $(3/4,1)$ corresponding to the four possible combinations of $F'(x)$ and $F'(F(x))$.

The perturbation on the Lyapunov exponent is linear in δ by construction, such that the Lyapunov exponent would reduce to the prediction of i.i.d. switching for $u = v$. The perturbation may have a dominant role on the Lyapunov exponent, potentially altering the synchronization of the coupled maps. In principle, changing the value of δ could be used to lower the Lyapunov exponent of maps that would not synchronize under i.i.d. switching to attain synchronization for the same expected coupling gain. Similarly, changing δ could increase the Lyapunov exponent of maps that are synchronized under i.i.d. coupling, potentially destroying synchronization.

Figure 5 displays the Lyapunov exponent for Markovian switching computed using the complete expression in (14), against the results from perturbation analysis in (41) and application of Birkhoff's theorem in (51), corresponding to Case II ($d_1 = -1.6$ and $d_2 = 1.9$). The numerical computation underlying (14) and (41) is based on a time series of 10000 time-steps for $s(k)$, initialized at 0.1. The parameter of the tent map is set to $2 - 10^{-10}$ to avoid the generation of periodic orbits from rational initial conditions, and the initial probability distribution is artificially magnified to maintain the error dynamics above the minimum machine precision. The Lyapunov exponent is then evaluated by averaging the last 100 samples of the long time series. When plotting the Lyapunov exponent from both perturbation analysis in (41) and the application of Birkhoff's theorem in (51), we use $\lambda^{\text{i.i.d.}}$ from (49).

We show two different values of π_1, for which i.i.d. switching supports ($\pi_1 = 0.0035$) or does not support ($\pi_1 = 0.005$) synchronization in Figures 5(a) and (b), respectively. Several comments should be drawn from the analysis. First, we confirm that changing δ allows us to manipulate the stability of synchronization, without changing the long-run average of the coupling gain. Specifically, in Figure 5(a), increasing δ above approximately 0.002 will cause

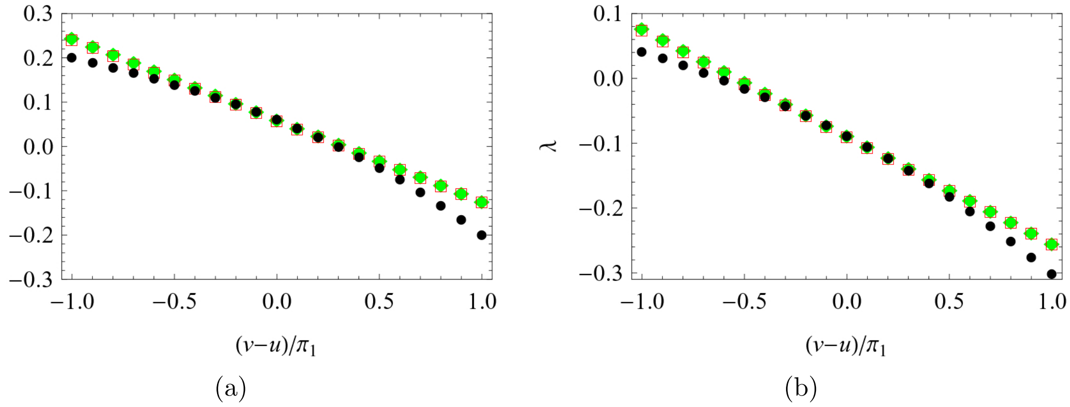


Figure 5. *Synchronization of coupled chaotic tent maps under Markovian switching for* $d_1 = -1.6$ *and* $d_2 = 1.9$*. Black dots are predictions from* (14)*, open red squares results from perturbation analysis in* (41)*, and green diamonds exact findings from the application of Birkhoff's theorem in* (51)*.* (a) $\pi_1 = 0.005$ *and* (b) $\pi_1 = 0.0035$*.*

the chaotic maps to synchronize, while in Figure 5(b), decreasing δ below approximately -0.0021 will destroy synchronization. Second, the application of Birkhoffs theorem yields excellent agreement with respect to the numerical computation in (41), confirming the validity of using the uniform invariant density as a proxy for the stochastically coupled maps. Third, the linear perturbation analysis is in very good agreement with (14), for a broad range of variation of δ, with noticeable differences only for δ close to π_1. As δ approaches π_1, u tends to $\pi_1^2 \approx 0$ and v to $\pi_1 + \pi_1^2 \approx \pi_1$, making the Markov chain significantly different from an i.i.d. process. As δ approaches $-\pi_1$, u tends to $2\pi_1 - \pi_1^2 \approx 2\pi_1$ and v to $\pi_1 - \pi_1^2 \approx \pi_1$, and the agreement is in fact better.

An alternative illustration of the findings in Figure 5 could be garnered by examining synchronization of coupled chaotic maps as a function of both u and v, through (40) with the i.i.d. Lyapunov exponent and the linear perturbation given by the closed-form expressions in (49) and (51). In Figure 6(a), we present the sign of the Lyapunov exponent of the coupled maps for the same instance of coupling gains considered in Figure 5, by systematically varying u and v. Each white curve represents a value of the stationary distribution π_1, giving evidence that synchronization could be promoted or hampered without changing the stationary distribution. In Figure 6(b), we present a similar diagram for $d_1 = -1.9$ and $d_2 = 1.9$, corresponding to Case III. We again confirm the possibility of changing the stability of synchronization, without the need of varying the long-run average the coupling gain.

Figure 6(c) also refers to a Case I selection of the coupling gains ($d_1 = 1.731$ and $d_2 = 2.235$), for which i.i.d. switching was not successful in achieving synchronization for $p_1 = 0.25$, as shown in Figure 3(d). Surprisingly, the stability boundary seems to coincide with a level line for π_1, such that synchronization cannot be obtained without changing the long-run average of the coupling gain from the value at $\pi_1 = 0.25$. An equivalent behavior was retrieved in every attempt to elicit synchronization in Case I, where none of the coupling gains would individually lead to synchronization, similar to [29], where increasing the switching period for i.i.d. switching was also found to not influence synchronization in Case I.

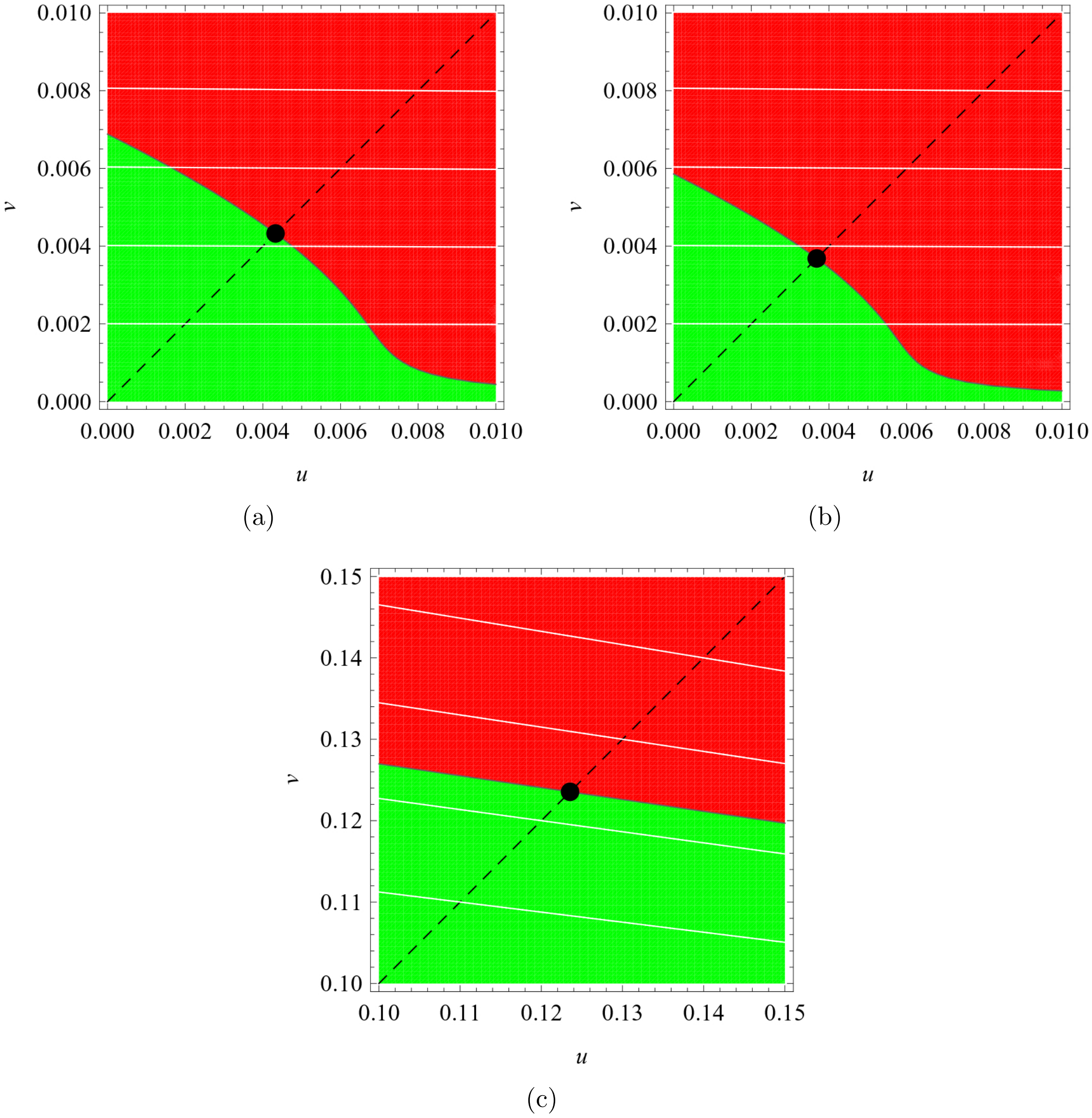


Figure 6. *Synchronization of coupled chaotic tent maps with Markovian switching. With respect to the Lyapunov exponent obtained by summing* (49) *and* (51)*, red identifies a positive value, green a negative value, and gray zero. The dashed line corresponds to the bisectrix, pertaining to i.i.d. switching. The dot refers to the value of* $u = v$ *needed for synchronization under i.i.d. switching, obtained by setting* (49) *to zero. The white lines are level lines for* π_1*, identifying pairs of* u *and* v *leading to the same value of the long-run expected coupling.* (a) $d_1 = -1.6$ *and* $d_2 = 1.9$*;* (b) $d_1 = -1.9$ *and* $d_2 = 1.9$*; and* (c) $d_1 = 1.731$ *and* $d_2 = 2.235$.

4. Conclusions.

The hypothesis of i.i.d. switching has often been questioned in the literature on stochastic synchronization, whereby collective dynamics of many realistic networks seem to be influenced by some memory of the network evolution. For example, infectious contacts supporting the spread of epidemics may take place only when individuals are in close physical proximity: these contacts should not be adequately described through an i.i.d.

sequence. People who are in physical proximity at a given time are likely to contact again in the near future, suggesting that a more general Markovian process should be preferred to i.i.d. switching. Similarly, information sharing between animals is unlikely to be well approximated by an i.i.d. sequence, whereby individuals will tend to consistently interact with others in their vicinity, rather than with randomly chosen group members as an i.i.d. process would dictate. This paper makes a first, necessary step toward unveiling the effect of switching memory on synchronization.

To disentangle the role of the network topology, we have focused on a simple dyadic interaction between two nonlinear maps in a discrete-time setting. The maps are coupled through a Markovian switching process, which relates the coupling at the present time to the coupling at the previous time-step. By leveraging state-of-the-art tools in jump linear systems [22], we have transformed the analysis of the one-dimensional stochastic error dynamics into a higher-order deterministic system which is amenable to analytical treatment. We have established an equation for the Lyapunov exponent of the mean square error dynamics, in terms of the transition probability matrix, the evolution of the synchronous solution, and values of the coupling gains. This approach is not prone to the numerical confounds associated with the generation of a large statistical population, which may hinder the accurate evaluation of the Lyapunov exponent through Monte Carlo simulations—often the only available tool to investigate stochastic stability.

We have presented an in-depth, rigorous analysis of the Lyapunov exponent by exploring a number of relevant aspects. First, we have demonstrated the possibility of recovering classical results for i.i.d. switching and static coupling by simply specializing the transition probability matrix and the synchronous solution [21, 26, 29]. For the general case of Markovian switching and time-varying synchronous solutions, we have derived upper and lower bounds for the Lyapunov exponent, which have helped clarifying the potential of Markovian switching to influence the stability of synchronization. Through perturbation theory, we have established a first-order approximation of the Lyapunov exponent with respect to synchronization under an associated i.i.d. switching process. For chaotic maps, the application of Birkhoff's ergodic theorem [18] allows for the evaluation of all the bounds and the perturbative solution from the invariant density of the synchronous solution.

To demonstrate our approach, we have focused on two chaotic tent maps coupled by a two-state Markov chain, for which we have presented closed-form expressions for all of the bounds and the perturbative solution. Through the application of the conservative bounds, we have shown that (i) there are pairs of coupling gains which support synchronization for any choice of the transition probability matrix and pairs for which synchronization is always unstable, and (ii) introducing memory through the Markov process could change the stability of the synchronous solution with respect to i.i.d. switching. To further delve into the latter claim, we have systematically contrasted the stability of synchronization for Markovian to i.i.d. switching. From the analysis of i.i.d. switching, we have unveiled surprising results that challenge the use of averaged network models in the study of stochastic synchronization in a discrete-time setting. For example, we have demonstrated the possibility of unstable synchronization for coupling gains that would individually support synchronization and, at the same time, would lead to an average coupling that also supports synchronization. In discrete-time, the switching frequency is bounded from above, whereby a single time-step could produce a large variation in the state of the system. This in contrast with continuous-time synchronization for which the switching frequency could be made arbitrarily large, thereby causing the average system to closely describe the stochastic dynamics [10, 31, 45, 61, 62, 63, 67].

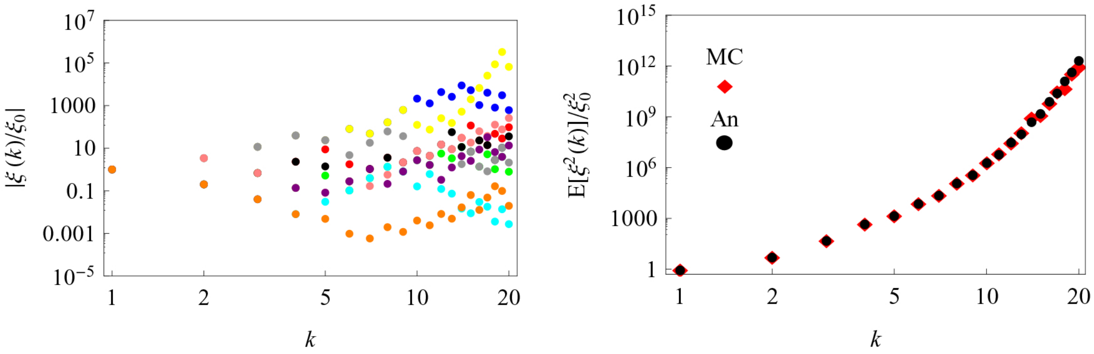
Our results on coupled chaotic tent maps demonstrate that memory has a direct impact on the stability of synchronization. Specifically, we have analytically shown that the sign of the Lyapunov exponent of the mean square error dynamics may be changed by varying the memory of the Markov chain without altering the long-run average coupling. The extent of memory effects may shape the stability of synchronization, without changing the times that the switching process yields one coupling gain rather than the other one. For example, we have demonstrated the existence of coupling gains that do not support synchronization under i.i.d. switching for a given switching probability but yield stable synchronization for sufficiently strong memory. This is also in contrast with the continuous-time literature [63, 64], which has suggested that the ergodic limit of the Markovian switching network could be used to infer stochastic synchronization under fast switching. In this paper, we have focused on stochastic switching governed by a first-order Markov chain where the probability of a state depends only on the probability of the previous state. An extension of our results to higher-order Markov chains which take into account multistep memory of the switching sequence will require the use of higher-order matrices. Likely, such an extension is amenable to an equivalent mathematical treatment; however, this study is beyond the scope of this paper and will be reported elsewhere.

We expect that the study of nonfast switching of coupled maps in [29] to be integrated with Markovian switching considered in this paper to investigate the interplay between memory and switching frequency. Such an interplay is likely to modulate the occurrence and shape of "windows of opportunities," defined in [29, 37] as nonfast ranges of switching frequencies where synchronization is stable. The extension of the proposed analytical approach to larger networks is the subject of ongoing research, which seeks to combine the eigenvalue analysis on ancillary higher-order algebraic problems put forward in [1, 2, 46, 53, 54, 55] with the current approach to tackle Markovian switching. The proposed framework also promises to allow analytical treatment of the stochastic convergence to nonsynchronous attractors for multistable networks of nonlinear maps, which we have addressed in a continuous-time setting for i.i.d. switching in [8, 32, 33].

Appendix. Verification of analytical findings through Monte Carlo simulations. To demonstrate the validity of our approach and illustrate its computational power, we compare analytical findings on the second moment of the error dynamics with Monte Carlo simulations. First, we run the individual tent map to generate a baseline synchronous solution with initial condition equal to 0.1 over 20 time-steps.

We select the following numerical values for the system parameters: $d_1 = -1.4$, $d_2 = 1.8$, $u = 0.8$, and $v = 0.4$. For the Monte Carlo analysis, we run the error dynamics (2) for 1000000 realizations of the Markov chain using a unitary initial condition, and at each time-step we average the square of the error of all the realizations to estimate $\mathrm{E}[\xi^2(k)]$ for $k = 1, \ldots, 20$. Within our theoretical approach, we simply evaluate (13) once for all.

Figure 7(a) displays five realizations of (2), while Figure 7(b) demonstrates the accuracy of our analytical predictions from (13) with respect to Monte Carlo simulations. Even if the



(a) (b)

Figure 7. *Error dynamics for coupled chaotic tent maps with* $d_1 = -1.4$*,* $d_2 = 1.8$*,* $u = 0.8$*, and* $v = 0.4$*.* (a) *Five realizations from Monte Carlo simulations and* (b) *comparison of Monte Carlo simulations (MC) for* 1000000 *realizations and analytical predictions (An).*

simulations are run for only 20 time-steps, 1000000 realizations seem to be barely sufficient to mitigate the variability of the error dynamics toward an accurate statical evaluation of the expectation of the square of the error. Some of the realizations show a moderate growth within a factor of 10, while others display a violent change of five orders of magnitude. Increasing the numbers of time-steps would hamper the accurate evaluation of the mean square dynamics from Monte Carlo simulations, thereby prohibiting the evaluation of the Lyapunov exponent from Monte Carlo simulations. An equivalent scenario will appear when considering stable error dynamics, for which the error may reach below small values below numerical precision in a few steps, thereby hindering the evaluation of the Lyapunov exponent from the mean square dynamics.

Our theoretical approach simply entails the evaluation of a sequence of matrix products in (13), which can be easily performed over long time series to accurately evaluate the Lyapunov exponent of the mean square dynamics. The use of perturbation analysis and the application of Birkhoff's theorem further simplify the analysis, leading to closed-form expressions.

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