Synchronization in On-Off Stochastic Networks: Windows of Opportunity

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***Abstract*—We study dynamical networks whose topology and intrinsic parameters stochastically change, on a time scale that ranges from fast to slow. When switching is fast, the stochastic network synchronizes as long as synchronization in the averaged network, obtained by replacing the random variables by their mean, becomes stable. We apply a recently developed general theory of blinking systems to prove global stability of synchronization in the fast switching limit. We use a network of Lorenz systems to derive explicit probabilistic bounds on the switching frequency sufficient for the network to synchronize almost surely and globally. Going beyond fast switching, we consider networks of Rössler and Duffing oscillators and reveal unexpected windows of intermediate switching frequencies in which synchronization in the switching network becomes stable even though it is unstable in the averaged/fast-switching network.**

***Index Terms*—Nonlinear oscillators, stochastic processes, synchronization.**

I. Introduction

Examples of dynamical networks include Internet routers, power grids, genetic networks, ecological networks, neuronal networks, and communication/social networks [1]–[3]. A great deal of attention has been focused on examining the interconnectedness of the dynamical properties of the individual nodes and those of the network topology [1]–[4]. Specifically, researchers have studied the interplay between these network characteristics as they apply to causing synchronization within the network, as synchronization is a key property among both biological and technological networks (see, for example, [5]–[17]). Most studies have looked at networks whose connections are static; only very recently researchers have considered networks with topology that evolves in time based on a deterministic or stochastic rule [18]–[49]. These networks belong to a wide class of *evolving dynamical networks* whose nonlinear dynamics and control are a hot research topic due to their potential in a variety of emerging applications (see a review paper [50] and the references therein).

In many realistic networks, collections of nodes that are organized into a network interact only sporadically via on-off connections. Examples include stochastically switched engineering networks and circuits, such as power converters [51], [52] and packet switched networks such as the Internet. Blinking networks that were introduced in [20], [21], are particularly relevant models for such sporadically interacting systems. These networks are composed of oscillators with connections that switch on and off randomly; the switching time is fast, with respect to the characteristic time of the individual oscillator. Different types of synchronization in blinking networks were studied in [20], [28], [30]–[34]. Beyond network synchronization, rigorous theory for the behavior of stochastically switching networks that blink rapidly was developed in [42], [43]. In particular, this theory allows for deriving explicit bounds that connect the probability of converging towards an attractor of a multistable blinking network, the switching frequency, and the chosen initial conditions [44].

The purpose of this paper is two-fold. First, we review and promote the general theory of blinking networks [42], [43] and show how its general theorems on probabilistic convergence can be applied to synchronization in dynamical networks of concrete periodic or chaotic oscillators (Sections II and III). In particular, we consider a network of chaotic Lorenz systems with both stochastically switching connections *and* intrinsic parameters and derive bounds on the switching frequency and the time sufficient to converge globally and surely to approximate synchronization (Section III). Remarkably, these bounds are explicit in the parameters of the blinking network. We also demonstrate how the probability of stable synchronization scales with the switching frequency beyond the conservative bounds. Second, we focus on synchronization of networks with non-fast switching connections (Section IV). We show the advantages of slower switching over fast switching by means of prototypical examples such as networks of i) chaotic Rössler systems; ii) non-autonomous Duffing oscillators in which slow switching provides opportunities for network synchronization while fast switching does not. More specifically, the network switches between topologies where synchronization is unstable, with its averaged network also being unstable for synchronization. Yet there is a window of "opportunity" in which an intermediate, not-fast switching frequency causes synchronization in the unstable network to stabilize. We also reveal this unexpected behavior in periodically switched dynamical networks with additional windows of opportunity.

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II. The Stochastic Network Model

Introduced in [20], the blinking network consists N oscillators interconnected pairwise via a stochastic communication network:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}_i\left(\mathbf{x}_i, \hat{s_i}(t)\right) + \varepsilon \sum_{j=1}^{N} s_{ij}(t) P(\mathbf{x}_j - \mathbf{x}_i), \tag{1}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^d$ is the state of oscillator i, $\mathbf{F}_i : \mathbb{R}^d \to \mathbb{R}^d$ describes the oscillators' individual dynamics, $\varepsilon > 0$ is the coupling strength. The $d \times d$ matrix P determines which variables couple the oscillators, $s_{ij}(t)$ are the elements of the time-varying connectivity (Laplacian) matrix $G(t)$. To define the functions s_{ij} we divide the time axis into intervals of length τ; we then assume $s_{ij}(t)$ to be binary signals that take the value 1 with probability p and the value 0 with probability $1-p$ in each time interval. The existence of an edge from vertex i to vertex j is determined randomly and independently of other edges with probability $p_{ij} \in [0, 1]$. Expressed in words, every switch in the network is operated independently, according to a similar probability law, and each switch opens and closes in different time intervals independently. All possible edges $s_{ij} = s_{ji}$ are allowed to switch on and off so that the communication network $G(t)$ is constant during each time interval $[k\tau, (k+1)\tau)$ and represents an Erdös-Rényi graph of N vertices. Fig. 1 gives an example of a "blinking" graph. The generalization of all the results of the paper to more complex switching topologies [1], [2] and directed graphs [30] is straightforward. In the model (1), we have also introduced stochasticity into one of the intrinsic system parameters, such that it switches between two values according to the same stochastic rule $\hat{s}_i(t)$ (though with an independent switching frequency and sequence). This stochastic intrinsic parameter accounts for internal fluctuations in the circuit. As a result, the randomly switching oscillators are non-identical and complete synchronization between the oscillators is impossible. Thus, we have chosen to tackle a more difficult problem of proving the stability of approximate synchronization in this switching network with stochastic intrinsic parameters, rather than considering identical oscillators. However, all the results of this paper are directly applicable to networks with identical oscillators. This can be done by setting the mismatch parameter $a = 0$ in the network of Lorenz oscillators (2) (Section III) and applying more restrictive Theorem 11.1 from [43].

The switching network (1) is a relevant model for stochastically changing networks such as information processing cellular neural networks [21], epidemiological networks [28], [48], and mobile ad-hoc networks [47]. For example, independent and identically distributed (i.i.d.) stochastic switching of packet networks communicating through the Internet comes from the fact that network links have to share the available communication time slots with many other packets belonging to other communication processes and the congestion of the links by the other packets can also occur independently. As far as network synchronization is concerned, local computer clocks, that are required to be synchronized throughout the network, are a representative example. Clock synchronization is achieved by sending information about each other computer's time as packets through the communication network [20]. The local clocks are typically implemented by an uncompensated quartz oscillator. As a result, the clocks can be unstable/inaccurate and need to receive synchronizing signals, that aim to reduce the timing errors. These signals must be sufficiently frequent to guarantee sufficient precision of synchronization between the clocks. At the same time, the communication network must not be overloaded by the administrative signals. This is a compromise between the precision of synchronization and the traffic load on the network. Remarkably, this blinking network administration can provide precise functioning of a network composing of imprecise elements. It also indicates the importance of optimal switching frequencies that ensure this compromise.

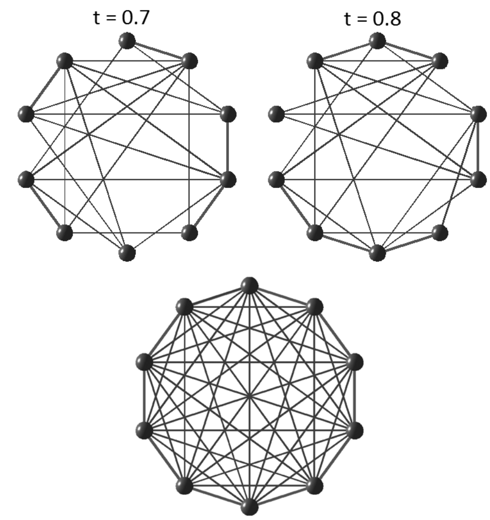

Fig. 1. (Top) Two subsequent instances of the switching network. Probability of switchings $p = 0.5$, the switching time step $\tau = 0.1$. (bottom): The corresponding averaged network where the switching connections of strength ε are replaced with static all-to-all connections of strength $p\varepsilon$, representing their mean value.

In the following, we will explore the dependence of network synchronization on different stochastic switching rates.

III. Fast Switching: Lorenz Oscillators

If switching is fast compared to the oscillator's intrinsic time scale, it is natural to expect the switching system (1) to follow the averaged system, which is obtained from taking the expectation of all of the stochastic variables $s_{ij}(t)$ and $\hat{s}_i(t)$. This amounts to replacing the non-zero entries of $G(i \neq j)$ and the intrinsic parameter $\hat{s_i}(t)$ with the switching probability p. We denote this averaged system Φ, as in [42]–[44]. Conceptually, for the coupling, this equates to connections between nodes that are always present, but that are weaker than "on" connections (as a connection in Φ has coupling $p\varepsilon$ and an "on" connection in $\mathbf{F}$ has coupling ε) (see Fig. 1).

The relation between the dynamics of the stochastically blinking network and its averaged analog is a non-trivial problem and a substantial contribution to its solution has been made in the previous work [20], [21], [28], [29], [42]–[44]. While averaging is a classical technique in the study of nonlinear oscillators, averaging for blinking systems needs some special mathematical techniques for obtaining rigorous convergence proofs. Such techniques have been developed for synchronization of blinking networks of chaotic dynamical systems [20] and for the convergence of the blinking network to an attractor [21], [42]–[44]. It was proven in different contexts

[20], [28], that switching networks (1) of coupled identical oscillators can synchronize even if the network is insufficiently coupled to support synchronization at every instant of time.

A complete rigorous theory for the behavior of stochastically switching networks that blink rapidly was developed very recently in [42], [43]. There are four distinct classes of switching dynamical networks. Two properties differentiate them: single or multiple attractors of the averaged system and their invariance or non-invariance under the dynamics of the switching system. In particular, in the most general case where the averaged network is multistable, the trajectory of the switching network may converge to the desired attractor with high probability or may escape towards another attractor with small probability. This general theory [42], [43] allows deriving explicit bounds for these probabilities (see its application to a multistable switching oscillator in [44]).

A. Rigorous Bounds

In this section, we apply this rigorous theory to network synchronization and demonstrate what bounds on the dynamics of the switching network must be satisfied to guarantee the convergence to stable synchrony. We use the Lorenz oscillator as an individual unit, comprising the network (1). The network (1) of x-coupled Lorenz oscillators with stochastic coupling and a stochastic intrinsic parameter is given as follows:

$$\begin{aligned} \dot{x}_i &= \sigma(y_i - x_i) + \varepsilon \sum_{j=1}^{N} s_{ij}(t)(x_j - x_i) \\ \dot{y}_i &= x_i \left([1 + a\hat{s}_i(t)]\, r - z_i \right) - y_i \\ \dot{z}_i &= x_i y_i - b z_i, \end{aligned} \tag{2}$$

where $i = 1, 2, \ldots, N$ and a is an additional positive parameter, forcing the main parameter r to switch between two values r and $(1+a)r$. The corresponding averaged system Φ reads:

$$\begin{aligned} \dot{x}_i &= \sigma(y_i - x_i) + \varepsilon p \sum_{j=1}^{N} (x_j - x_i) \\ \dot{y}_i &= x_i (r^* - z_i) - y_i \\ \dot{z}_i &= x_i y_i - b z_i, \end{aligned} \tag{3}$$

where $r^* = (1+pa)r$. Synchronization in the averaged network (3) with all-to-all connections of strength $p\varepsilon$ is defined by the invariant manifold $M = \{x_1 = x_2 = \ldots = x_N, y_1 = y_2 = \ldots = y_N, z_1 = z_2 = \ldots = z_N\}$. It is important to emphasize that this synchronization solution *does not* exist in the stochastically switching network (2) due to the presence of the intrinsic parameter a, bringing stochastically changing parameter mismatch into the system. Therefore, the trajectory of the switching network cannot converge to the completely synchronized solution, and it can only get to its δ-neighborhood, corresponding to approximate δ-synchronization. The faster the switching, the smaller the synchronization mismatch (error) δ is. In analogy with the terminology introduced in [42]–[44], we also call the δ-synchronization solution "ghost"-synchronization. Our goal is to prove the global stability of δ-synchronization and demonstrate that the set of on-off switching sequences that fail to trigger δ-synchronization has probability zero as long as synchronization in the averaged network is globally stable and switching is fast enough. We also aim at deriving the probabilistic bounds on the switching frequency and δ that can be calculated explicitly via the parameters of the switching network (2).

The proof involves two steps. In the first step, one constructs a Lyapunov function for the difference (transverse) oscillators' variables in the averaged network (3) and proves that it decreases to zero. Therefore, one proves that complete synchronization is globally stable in the averaged network. In the second step, one uses the same Lyapunov function for the convergence to the δ-neighborhood of the ghost synchronized solution in the switching network (2). Due to the stochastic nature of the switching, this Lyapunov function may increase temporarily, but the general tendency is to decrease. Switching is a stochastic process, therefore, the convergence properties also have a probabilistic flavor. This can be expressed by showing that after a certain time the Lyapunov function decreases with high probability as long as the switching frequency is sufficiently fast.

We define the Lyapunov function as:

$$W = \frac{1}{2} \sum_{i=1}^{N-1} \left\{ X_{i,i+1}^2 + Y_{i,i+1}^2 + Z_{i,i+1}^2 \right\}, \tag{4}$$

where $X_{i,i+1} = x_i - x_{i+1}$, $Y_{i,i+1} = y_i - y_{i+1}$, $Z_{i,i+1} = z_i - z_{i+1}$ are the $N-1$ difference variables. We notice that both the averaged system (3) and the stochastic system (2) are assumed to have the same Lyapunov function, so we will differentiate between the two of them using W_Φ and W_F for the averaged and stochastic systems, respectively. Before we can begin to understand the behavior of the stochastic network, it is important to understand the dynamics of the static averaged network. Global stability of complete synchronization in networks of x-coupled Lorenz systems was extensively studied, for example, in [7]. Based on the previous analysis [7], we can formulate the following statement.

Proposition 1: Synchronization in the averaged all-to-all x-coupled Lorenz network (3) with coupling strength $p\varepsilon$ and the averaged intrinsic parameter $r^* = (1+pa)r$ is globally stable if

$$p\varepsilon > p\varepsilon^* = \left(\frac{1}{N}\right) \left(\frac{b(b+1)(r^* + \sigma)^2}{16(b-1)} - \sigma \right). \tag{5}$$

Proof: The direct application of the bound on the coupling strength sufficient for global synchronization in the x-coupled network of Lorenz oscillators with arbitrary topologies (see [7, App. A]) to the averaged network (3) yields the condition (5).□

This bound can be generalized to more complex averaged network topologies, corresponding to networks with given switching connections that are not all-to-all. For example, the bound for a $2K$-nearest neighbor network of x-coupled Lorenz oscillators can be obtained by replacing the $1/N$ in (5) with $(1/N)(N/2K)^3(1 + (65K/4N))$, using the Connection Graph method [7]. In the case of directed networks, one can use the Generalized Connection graph [10].

The theory for the behavior of stochastic on-off systems [42] and [43] involves placing upper bounds on the first and second time derivatives of the Lyapunov function W, calculated along solutions of the averaged and switching systems. To facilitate cross-paper reading, we shall use the same notations for those bounds:

$$\begin{aligned} B_{W\Phi} &= \max_{\mathbf{x}\in\mathbf{R}} |D_\Phi W(\mathbf{x})| \\ LB_{W\Phi} &= \max_{\mathbf{x}\in\mathbf{R}} \left|D_\Phi^2 W(\mathbf{x})\right| \\ B_{WF} &= \max_{\mathbf{s}} \max_{\mathbf{x}\in\mathbf{R}} |D_F W(\mathbf{x},\mathbf{s})| \\ LB_{WF} &= \max_{\mathbf{s},\bar{\mathbf{s}}} \max_{\mathbf{x}\in\mathbf{R}} \left|D_F^2 W(\mathbf{x},\bar{\mathbf{s}},\mathbf{s})\right|, \end{aligned} \tag{6}$$

where $\mathbf{x}$ is the vector composed of $\mathbf{x}_i = \{x_i, y_i, z_i\}$, $i = 1,\ldots,N$, $\mathbf{R}$ represents the systems' absorbing domain, vector $\mathbf{s}$ indicates a set of stochastic sequences corresponding to $N(N-1)/2$ stochastic connections and N intrinsic parameter switchings; similarly, $\bar{\mathbf{s}}$ corresponds to a set of another realizations of the stochastic switching sequences.

Before giving all of the explicit bounds (6) for the N-node networks (2) and (3), we will walk the reader through the process of obtaining B_{WF} and LB_{WF} for the two-node network (2) with $\hat{s}_1 = \hat{s}_2 = s(t)$. We shall use the notation $X = x_1 - x_2$, $Y = y_1 - y_2$, and $Z = z_1 - z_2$. We start with the first time derivative of the Lyapunov function (4) along the solutions of the stochastic two-node system (2):

$$D_F W(\mathbf{x},\mathbf{s}) = X\dot{X} + Y\dot{Y} + Z\dot{Z}, \tag{7}$$

where the derivatives of the difference variables are governed by the following equations, obtained from the system (2) with $N = 2$ [7], [53]:

$$\begin{cases} \dot{X} = \sigma(Y-X) - 2s_{12}(t)\varepsilon X \\ \dot{Y} = \left[(1+as(t))\,r - U^{(z)}\right]X - Y - U^{(x)}Z \\ \dot{Z} = U^{(y)}X + U^{(x)}Y - bZ, \quad i,j = 1,\ldots,N, \end{cases} \tag{8}$$

where $U^{(\xi)} = (\xi_1+\xi_2)/2$ for $\xi = x,y,z$ are the corresponding sum variables. For simplicity, we assume that the coupling and intrinsic stochastic parameters are governed by the same stochastic sequence $s_{12}(t) = s(t)$. The generalization to different sequences is straightforward. We can rewrite (7) as a quadratic form

$$D_F W(\mathbf{x},\mathbf{s}) = -\Big[(2s(t)\varepsilon+\sigma)X^2 + \Big[U^{(z)} - (1+as(t))r + \sigma\Big]XY + Y^2 - U^{(y)}XZ + bZ^2\Big]. \tag{9}$$

To bound the absolute value of this derivative by its maximum and get B_{WF}, we will maximize each term in the quadratic form (9). It is well-known that the individual Lorenz system has bounds that guarantee that its trajectories will remain in the absorbing domain. These bounds can be found, for example, in [7] and are $|x,y| < |\varphi|$ and $|z| < |\varphi| + r + \sigma$, where $\varphi = b(r+\sigma)/2\sqrt{b-1}$. Therefore, for example, $X = x_1 - x_2 = \varphi - (-\varphi) = 2\varphi$ is the upper bound for the difference X. As we are maximizing the quadratic form term by term, we can guarantee that it is maximized if $s(t) = 1$. While in practice, this will lead us to a more conservative result, it is much more analytically tractable. We show this term by term maximization

$$\begin{aligned} \max_{\mathbf{x}\in\mathbf{R}} &|D_F W(\mathbf{x},\mathbf{s})| \\ &= \left|4(2\varepsilon+\sigma)\varphi^2 + [\varphi + r + \sigma + ((1+a)r - \sigma)]\,\varphi^2 \right. \\ &\quad \left. + 4\varphi^2 + 4\varphi^3 + 4b\varphi^2\right| \\ &= |4(2\varepsilon+\sigma) + 5\varphi + (2+a)r + 4(b+1)|\,\varphi^2 \\ &= B_{WF}. \end{aligned} \tag{10}$$

To get LB_{WF}, we follow a similar procedure, but using the second derivative of the Lyapunov function. We can jump right to this:

$$\begin{aligned} D_F^2 W(\mathbf{x},\mathbf{s}) = &\dot{X}\,[\sigma(Y-X) - 2s(t)\varepsilon X] \\ &+ X\left[\sigma(\dot{Y}-\dot{X}) - 2s(t)\varepsilon\dot{X}\right] \\ &+ \dot{Y}\left[\left([1+as(t)]\,r - U^{(z)}\right)X - Y - U^{(x)}Z\right] \\ &+ Y\left[\left((1+as(t))\,r - U^{(z)}\right)\dot{X} - X\dot{U}^{(z)} \right. \\ &\qquad \left. -\dot{Y} - U^{(x)}\dot{Z} - Z\dot{U}^{(x)}\right] \\ &+ \dot{Z}\left[U^{(y)}X + U^{(x)}Y - bZ\right] \\ &+ Z\left[U^{(y)}\dot{X} + X\dot{U}^{(y)} + U^{(x)}\dot{Y} + Y\dot{U}^{(x)} - b\dot{Z}\right]. \end{aligned} \tag{11}$$

Of course, the stochastic sequences in these new derivatives are different from the stochastic sequences in the previous derivatives. To make this clear, we denote these new stochastic variables $\bar{\mathbf{s}}$, which means $D^2W(\mathbf{x},\mathbf{s},\bar{\mathbf{s}})$ is a function of the state variables and two different stochastic sequences. We place the bound on $LB_{W\Phi}$ in much the same way that we placed the bound on $B_{W\Phi}$. When finding the maximum value of the function $D^2W(\mathbf{x},\mathbf{s},\bar{\mathbf{s}})$, we must consider not only the optimal values of the state variables and the stochastic variables from the first derivative $D_F W$ (i.e., whether the function is maximized for $s = 0$ or $s = 1$), but we must also find the optimal values including the stochastic variables introduced by the second derivative $D_F^2 W$. These calculations are somewhat tedious, and we have to skip most of intermediate steps due to the space constraint. However, the reader should be able to reproduce the results, following these steps. Using the ideas above [term-by-term maximization of the derivatives and conservative (optimal) choices for the stochastic variables (0 vs. 1)], we get the following bounds for the original N-node networks (2) and (3):

$$\begin{aligned} B_{W\Phi} = &(N-1)\varphi^2\,|4(Np\varepsilon+\sigma) + 5\varphi + (2+ap)r + 4(b+1)|, \\ LB_{W\Phi} = &(N-1)4\varphi\,[11\varphi^2 + 2\varphi\,(6r + 3apr + 8\sigma \\ &+ 3b + 3p\varepsilon(N-1) + 7) + 2 + 2b^2 + (2p\varepsilon(N-1))^2 \\ &+ r(6 + 3ap + 4r + 4apr + a^2p^2r + b + 5ap\sigma + 10\sigma), \\ &+ 4\sigma + 10\sigma^2 + \sigma b + p\varepsilon(N-1)\,((2+ap)r + 12\sigma)] \\ B_{WF} = &(N-1)\varphi^2\,|4(N\varepsilon+\sigma) + 5\varphi + (2+a)r + 4(b+1)|, \\ LB_{WF} = &(N-1)4\varphi\,[11\varphi^2 + 2\varphi\,(6r + 3ar + 8\sigma + 3b \\ &+ 3\varepsilon(N-1) + 7) + 2 + 2b^2 + (2\varepsilon(N-1))^2 \\ &+ r(6 + 3a + 4r + 4ar + a^2r + b + 5a\sigma + 10\sigma) \\ &+ 4\sigma + 10\sigma^2 + \sigma b + \varepsilon(N-1)\,((2+)r + 12\sigma)]\,. \end{aligned} \tag{12}$$

This completes the first step towards formulating the stability criterion.

The second step is to define the size of the δ-neighborhood of the ghost synchronization solution of the stochastic system (2). This is done by choosing a level curve of the Lyapunov function W_Φ for the averaged system (3)

$$V_0 : W_\Phi = \frac{1}{2}\sum_{i=1}^{N-1}\left\{\delta^2_{X_{i,i+1}} + \delta^2_{Y_{i,i+1}} + \delta^2_{Z_{i,i+1}}\right\}. \tag{13}$$

We let $\delta = \max\{\delta_{X_{12}}, \dots, \delta_{Z_{N-1,N}}\}$, which gives us the level $V_0 : W_\Phi \le (3/2)(N-1)\delta^2$. We define another level, V_1 as absorbing domain of the Lyapunov function, which we obtain by replacing each difference variable with its maximum value, subject to the constraints on $\mathbf{x}$. We get

$$V_1 : W_\Phi \le \frac{1}{2}(N-1)\left(8\varphi^2 + 4(\varphi + r + \sigma)^2\right). \tag{14}$$

These level curves let us define the following quantity:

$$\gamma = \min_{\mathbf{x}\in\mathbf{R}, V_0 \le W_\Phi \le V_1} |DW_\Phi(\mathbf{x})|.$$

Notice that the derivative of the Lyapunov function DW_Φ for the averaged system is similar to the quadratic form (9) where the stochastic variables $s(t)$ are replaced with probability p. Under the condition that the coupling strength ε exceeds the threshold (5), we can get the bound on γ:

$$\gamma = (N-1)|Np\varepsilon + \sigma - 2\varphi - apr + 1 + b|\delta^2, \tag{15}$$

where the level V_0, corresponding to the δ-synchronization yields the minimum value γ on the interval $V_0 \le W_\Phi \le V_1$. To derive the bound (15), we have used term-by-term mimimization/maximization and replaced the difference variables with δ and set $U^{(z)} = -(\varphi + r + \sigma)$ and $U^{(y)} = \varphi$.

We also define the following constants [42], [43] that use the bounds from (6), which help make the statement of the following theorem more manageable and explicit:

$$\begin{aligned} c &= \frac{1}{64(LB_{WF} + LB_{W\Phi})B^2_{WF}} \\ D &= 8(LB_{WF} + LB_{W\Phi}) \\ U_0 &= \left\{\mathbf{x} | W_\Phi(\mathbf{x}) < V_0 + \frac{4\gamma^2}{D}\right\}, \end{aligned} \tag{16}$$

where U_0 is a neighborhood of the synchronization solution of the averaged system (3) and is slightly larger that the level curve V_0, corresponding to the δ-synchronization.

Theorem 1: Assume that the coupling strength ε is strong enough, i.e., greater than the bound given in (5), such that synchronization in the averaged network (3) is globally stable. Then, the following properties hold:

- If the switching time τ is small enough to satisfy

$$\tau < \frac{c\gamma^3}{\ln\left[D\frac{(V_1 - V_0)}{\gamma^2}\right]}, \tag{17}$$

  then the trajectory of the switching system (2) almost surely reaches the U_0-neighborhood of the ghost-synchronization solution, in finite time. Therefore, the switching network (2) converges to approximate synchronization globally and almost surely.

- Assume that (17) holds and that $W(x(0)) \le V_1$ (i.e., the initial condition is chosen in the attracting domain). Let $P_1(n)$, $\forall n \in \mathbf{N}$, be the probability that it takes time $2n((V_1 - V_0)/\gamma)$ to reach U_0. Then

$$P_1(n) \le \exp\left(-n\left(\frac{c\gamma^3}{\tau} - \ln\left[D\frac{V_1 - V_0}{\gamma^2}\right]\right)\right). \tag{18}$$

Proof: This theorem is a shortened version of the general theorem, Theorem 9.1, given in [43] and applied to the Lyapunov function (4) and its bounds (12)–(16) for the global stability of approximate synchronization in the stochastic network (2). The complete statement and proof of Theorem 9.1 can be found in [43]. □

Remark 1: The difference between the δ- and U_0-neighborhoods is a technicality, coming from the proof of Theorem 9.1. The desired precision of the approximate synchronization in the switching network is defined by U_0 and can be explicitly calculated through δ. As δ is typically chosen arbitrarily, we generally see the U_0-synchronization as the δ-synchronization.

Remark 2: The probability bound (18) depends on the number of discrete steps n as this is the probability that the system will take *at least* time $2n((V_1 - V_0)/\gamma)$ to converge to the U_0-neighborhood. Notice that this time depends on how far from the neighborhood of the ghost synchronization solution lies from the border of the attracting domain, expressed via V_1.

In order to make the theorem more explicit, we shall consider what this looks like when the attractor is in a typical chaotic regime, i.e., let $r = 30$, $a = 2/30$, $b = 8/3$, and $\sigma = 10$. Also, let $\delta = 0.1$. Doing this, we get

$$\begin{aligned} c =& 2187\Big[274877906944000(N-1)^2\cdot(175+80\sqrt{15}+6N\varepsilon)^2 \\ &\times\Big(242021+52704\sqrt{15}+9(173+34\sqrt{15})p+18p^2 \\ &+ 18\varepsilon^2(N-1)^2(1+p^2) + 9\varepsilon(N-1) \\ &\times\Big(91+32\sqrt{15}+(90+32\sqrt{15})p+p^2\Big)\Big)\Big]^{-1}, \\ D =& \frac{40960}{27}\cdot\Big(242021 + 52704\sqrt{15} + 9(173 + 64\sqrt{15})p \\ &+18p^2+18\varepsilon^2(N-1)^2(1+p^2)+9\varepsilon(N-1) \\ &\times\Big(91 + 32\sqrt{15} + (90 + 32\sqrt{15})p + p^2\Big)\Big), \\ \gamma =& \left|3.33\times10^{-3}(N-1)(-2.0687\times10^3+(-6+3\varepsilon N)p\right|. \end{aligned}$$

To give the reader an idea of the magnitude of these numbers, we set $N = 2$, $\varepsilon = 1$ and $p = 0.5$, which gives us:

$$c \approx 7.14722\times10^{-23},\ D \approx 6.84128\times10^8,\ \gamma \approx 6.895\times10^{-1}.$$

These values determine the size of the neighborhood $U_0 = \{\mathbf{x}|W(\mathbf{x}) < 0.05\}$. Turning our attention back to (17), we use the values that we have computed to get the explicit bound on τ for the two node stochastic network (2). Thus, if:

$$\tau < 7.5615 \times 10^{-25}, \tag{19}$$

then the probability (18) is bounded by:

$$P_1(n) \le \exp\left(\frac{n\left(-2.34354 \times 10^{-23} + 30.993\tau\right)}{\tau}\right).$$

Of course, given the bound that we have on τ, it is evident that the probability (18) converges to 0 very quickly if τ is this small.

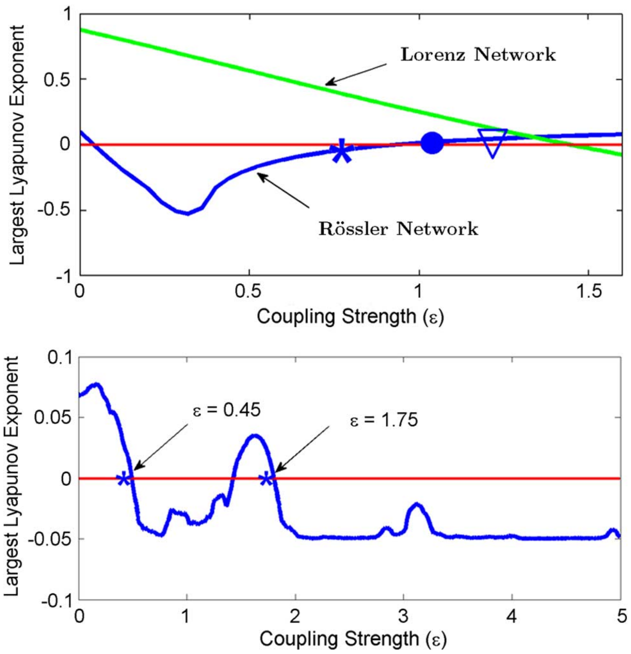

Fig. 2. (Color online) (top) Transversal stability of synchronization in the averaged ten-node, x-coupled Lorenz network (green) and the averaged ten-node, x-coupled Rössler system (blue), expressed via the Lyapunov exponent. For the Lorenz network, when $\varepsilon > \varepsilon^* \approx 1.5$, the system tends to synchrony, whereas for the Rössler system, there is a synchronization window for $\varepsilon^- < \varepsilon < \varepsilon^+$ the system exhibits synchronous behavior. The values of ε used in Fig. 5 are marked with $*$, $\bullet$, and ∇ on the blue curve. (bottom) Transversal stability of synchronization in the two-node, Duffing network (21). There are two synchronization windows, for $\varepsilon_1 < \varepsilon < \varepsilon_2$ and $\varepsilon_3 < \varepsilon$. The values $\varepsilon = 0.45$ and $\varepsilon = 1.75$ are used for the switching networks in Fig. 9.

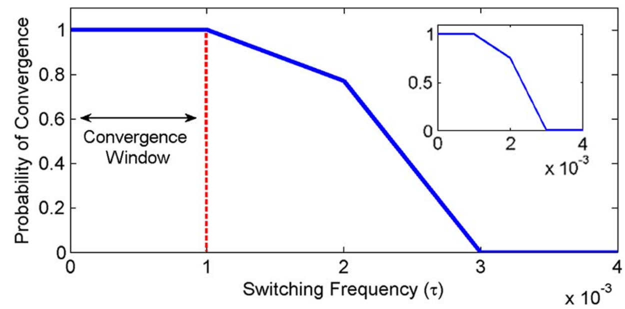

Fig. 3. Probability of δ-synchronization in the stochastic ten-node Lorenz network (2) for small values of τ. For $\tau = 0.001$, and $\varepsilon = 3$, there is convergence with probability 1, which rapidly decays as τ increases. Nearly identical convergence probabilities for different choices of the integration steps. (Main graph): Integration step $dt = 0.001$. (Insert): $dt = 0.0005$. Probability calculations are based on 100 trials.

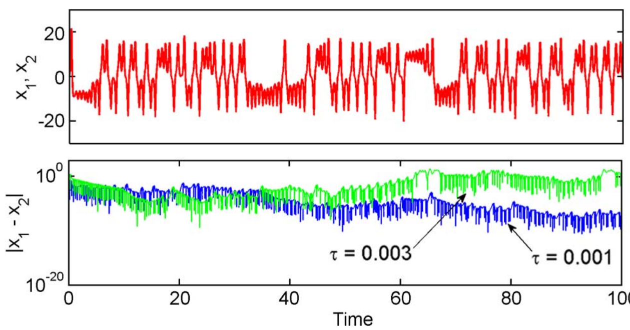

Fig. 4. (Color online) (top) Traces for x_1 (blue) and x_2 (red) in the stochastic Lorenz network for $\tau = 0.001$. The two traces practically coincide, showing δ-synchronization. (bottom): The synchronization error corresponding to two switching frequencies, $\tau = 0.003$ (green) and $\tau = 0.001$ (blue). The blue curve oscillates about a small value, close to 10^{-10}, because the stochasticity in the parameter r prevents the stochastic network from converging to complete synchronization. This mimics the expected behavior from Fig. 3.

This probability guarantees a few things that are not immediately clear. It shows that if the switching rate is fast enough, the stochastic system will not just asymptotically converge to the U_0-neighborhood, but converge in a finite amount of time, with the faster switching is, the shorter the time limit guaranteeing convergence is. This leads to an additional implication of this probability bound: it ensures that there is a window for values of τ in which the stochastic system will converge within some given finite time, i.e., for $0 < \tau \leq \tau^*$. As we emphasized, the explicit bound given in (19) is very conservative; however, it rigorously proves that synchronization can be stably achieved in the switching network (2) even if the probability of switching p is small and the network is disconnected most of the time. It also suggests that the probability of not-converging to δ-synchronization within the finite time [see (18)] decreases at least exponentially fast as τ decreases.

In the following subsection, we present numerically calculated bounds in order to isolate the real window of switching periods τ, corresponding to δ-synchronization.

B. Numerical Results

As an example, we consider a ten-node network of coupled Lorenz oscillators (2). Hereafter, the intrinsic parameters and the switching probability are chosen and fixed as follows: $r = 30$, $a = 2/30$, $b = 8/3$, $\sigma = 10$, and $p = 1/2$. The coupling strength $\varepsilon = 3$ is chosen such that synchronization in the averaged system with $p\varepsilon = 1.5$ is stable [see Fig. 2 (top)].

Considering small values of τ, Fig. 3 shows that for $\varepsilon/2 \approx \varepsilon^*$, the system synchronizes with probability 1 for $\tau = 0.001$, but the probability decays rapidly for bigger values of τ, until $\tau = 0.003$, where there is no longer synchrony (also see Fig. 4. The window for convergence, $0 < \tau \leq \tau^*$ is guaranteed by the probability bound given by (18), and while this window is expected, the extent of it shown in Fig. 3 is not. Fig. 3 also indicates that the integration step of the oscillators does not essentially play a role. While choosing a smaller integration step allows one to choose smaller switching periods τ, closer to instantaneous switching, the convergence probability remains the same and equal to one.

There are circumstances for which not converging to the averaged system is favorable, and the present theory is not able to make definitive claims about the behavior of the stochastic system *beyond fast switching*. This leads us to explore the effects of not-fast-switching (where τ is not necessarily small) on the dynamics of the switching network.

IV. Beyond Fast Switching: Windows of Opportunity

We consider switching networks (1) of Rössler and Duffing oscillators. These networks are known to exhibit synchronization properties distinct from those of the Lorenz network considered in Section III. While the Lorenz network belongs to Class I, x-coupled Rössler networks and networks of coupled Duffing-type oscillators [58] are in Class II [14]. Class I networks synchronize when the coupling exceeds a threshold value, and synchronization remains stable up to infinitely strong coupling (see Fig. 2 for the unbounded stability window for synchronization in the Lorenz networks, to the right from the threshold coupling value). Class II networks have a bounded range of coupling where synchronization remains stable (see a well in the stability diagrams of the Rössler and Duffing networks in Fig. 2 where synchronization loses its stability when the coupling increases). These differences between Class I and Class II networks generate distinct windows of switching frequencies, yielding stable synchrony in the corresponding switching networks.

It is important to emphasize that Class II networks such as x-coupled Rössler networks can only exhibit local synchronization within the stability well and global synchronization cannot be achieved/proved for any coupling. This is due to so-called shortwave length bifurcations [55] and the persistent presence of saddle fixed points, lying outside of the synchronization manifold in the phase space of the x-coupled Rössler system [56]. Hence, there are trajectories that escape to infinity, and the existence of the global absorbing domain cannot be proved. As a result, the above rigorous theory of global convergence to synchronization in fast switching networks cannot be applied to the Class II networks considered below.

A. x-Coupled Rössler Oscillators

We begin with a switching network of ten x-coupled Rössler oscillators:

$$\begin{aligned}\dot{x}_i &= -(y_i+z_i)+\sum_{j=1}^{N}\varepsilon_{ij}s_{ij}(x_j-x_i)\\ \dot{y}_i &= x_i+ay_i\\ \dot{z}_i &= b+z_i(x_i-c).\end{aligned} \tag{20}$$

Hereafter, the intrinsic parameters are chosen and fixed as follows: $a = 0.2$, $b = 0.2$, $c = 7$. To simplify multi-trial simulations, we did not include a stochastic intrinsic parameter into this system; however, all the reported windows of switching are robustly seen when it is present [54]. The averaged network is an all-to-all network, similar to that of the Lorenz network. As discussed above, synchronization in a network of x-coupled Rössler systems is known [5] to destabilize after a critical coupling strength ε^*, which depends on the eigenvalues of the connectivity matrix G. We choose the coupling strengths in the stochastic network such that the coupling in the averaged network is defined by one of the three values, marked in Fig. 2. In particular, for $\varepsilon = 1$, synchronization in the averaged network is unstable. As a result, synchronization in the fast-switching network is also unstable. Surprisingly, there is a window of intermediate switching frequencies for which synchronization becomes stable (see Fig. 5). In fact, the stochastic network switches between topologies whose large proportion does not support synchronization or is simply disconnected.

To better isolate the above effect and gain insight into what happens when switching between a connected network in which the synchronous solution is unstable, and a completely disconnected network in which the nodes' trajectories behave independently of one another, we consider a two-node Rössler network (20). Figs. 6 and 7 demonstrate the emergence of synchrony windows for various intermediate values of τ for which the fast-switching network does not support synchronization. In essence, the system is switching between two unstable systems, and yet when the switching period τ is in a favorable range, the system stabilizes. Fig. 7 also indicates the role of switching probability p and shows that the largest synchrony window is achieved at probability $p = 1/2$. This probability corresponds to the most probable (frequent) re-switching for

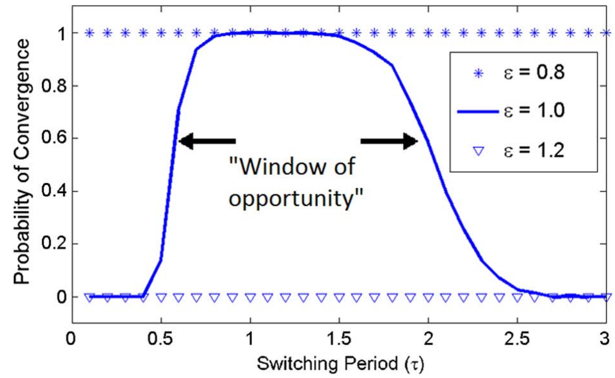

Fig. 5. Probability of synchrony in the ten-node stochastic Rössler network with differing coupling strengths, showing the effects of varying τ. These are coupling strengths for which synchronization in the averaged system is stable ($*$), weakly unstable ($\bullet$), and strongly unstable (∇), respectively (cf. Fig. 2 for the values marked with $*$, $\bullet$, and ∇). The bell-shaped curve corresponds to an optimal range of non-fast switching $0.6 < \tau < 2.2$ ("the window of opportunity"), where synchronization in the stochastic network becomes stable with high probability, whereas synchronization in the corresponding averaged system is unstable ($\varepsilon = 1$). Switching probability $p = 1/2$. Probability calculations are based on 1000 trials.

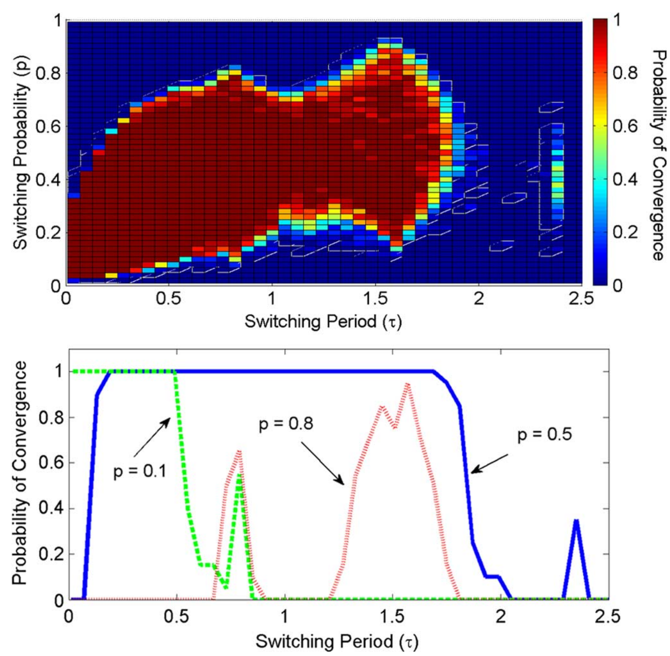

Fig. 6. (Color online) (top) Probability of synchrony in the two-node Rössler network (20) as a function of the switching probability p and switching period τ. Red (lighter) colors correspond to higher probability of convergence (with dark red at probability 1) and blue (darker) colors correspond to lower probabilities (dark blue at probability 0). The coupling strength of the connection is fixed at $\varepsilon = 7$. As p increases, $p\varepsilon$, the effective coupling in the averaged/fast-switching network progresses through the window of synchrony indicated in Fig. 2 (top). For the two-node network this interval is $p\varepsilon \in [0.08\ 2.2]$, yielding the stability range $p \in [0.011\ 0.31]$ (the red interval on the y-axis) for $\epsilon = 7$ and small τ. (bottom) Slices from the top Figure for different p. Notice the emergence of "windows of opportunity" for not-fast switching periods at which synchrony in the fast-switching network is unstable. Switching probability around $p = 1/2$ yields the largest stability window. Probability calculations are based on 100 trials.

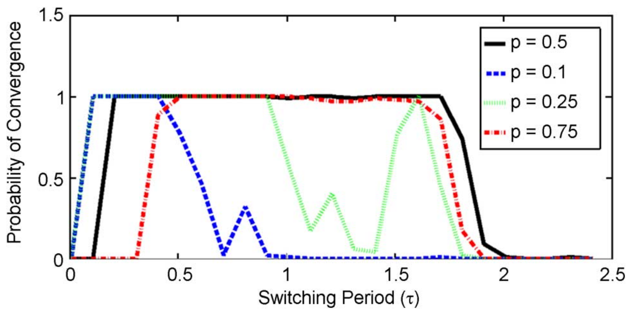

Fig. 7. (Color online) Role of switching probability p in the two-node Rössler network. $p\varepsilon = 3.5$ is chosen in the instability window in the fast switching network and kept constant when changing p. There is a window of opportunity for each chosen value of p ($p = 0.1$ (blue, dashed line), $p = 0.25$ (green, dotted line), $p = 0.5$ (black, solid line) and $p = 0.75$ (red, dashed and dotted line)). Notice the largest window for $p = 1/2$, indicating a positive role of more probable re-switching between the unstable systems.

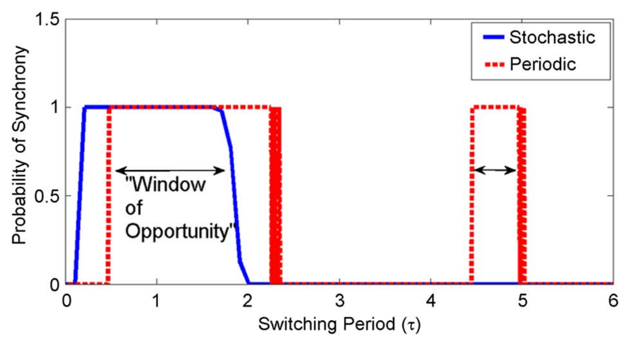

Fig. 8. (Color online) Comparison of the probability of synchrony in the two-node Rössler network in which switching is stochastic (blue, solid curve) and periodic (red, dashed curve) when switching is slow. The networks switch between two couplings strengths: $\varepsilon = 0$ and $\varepsilon = 7$, both corresponding to unstable synchrony. The averaged network with $\varepsilon = 3.5$ is also unstable for synchronization. There is substantial overlap between the two switching frequency windows that induce synchrony. Notice the second (right) stability window for periodic switching where stochastic switching fails to stabilize synchronization. Probability calculations are based on 1000 trials.

a fixed switching period τ and indicates a stabilizing effect of re-switching.

In an attempt to better understand the underlying phenomenon, we consider what happens in this connected/disconnected network when the switching is periodic instead of stochastic. We find that for periodic switching (the dashed curve in Fig. 8) there is a window of similar length to when switching is stochastic, and for similar values of τ. Furthermore, when switching is periodic, we have an additional, though smaller, window for $\tau \in [4.5, 5]$.

The appearance of additional windows of opportunity in the periodically switching network and the overall better stability of synchronization, compared to the stochastic network can be explained by an analogy to the dynamics of Kapitza's pendulum. Kapitza's pendulum is a rigid pendulum in which the pivot point vibrates in a vertical direction, up and down [57]. The unusual property of Kapitza's pendulum is that the vibrating suspension can stabilize the pendulum in an upright vertical position which corresponds to an otherwise unstable equilibrium in the absence of suspension vibrations. This effect is also known as dynamic stabilization. Loosely speaking, the switching network of Rössler oscillators performs a similar function as it switches between two unstable networks, yet it dynamically stabilizes the

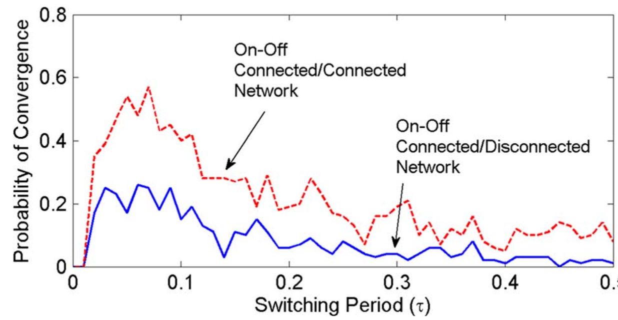

Fig. 9. (Color online) Probability of synchrony in the Duffing network. Switching probability: $p = 1/2$, number of trials: 100. (Blue solid line) On-off connected/disconnected network, switching between $\varepsilon = 0$ and $\varepsilon = 1.75$ (depicted from the instability window shown in Fig. 2 (bottom)). (Red dashed line) On-off connected/connected network, switching between $\varepsilon = 0.45$ and $\varepsilon = 1.75$, corresponding to two instability zones in Fig. 2. In either case, the network switches between two configurations that do not support stable synchronization. Synchrony in the fast switching/averaged network is unstable, yet there are windows of opportunity, yielding stable synchrony for intermediate τ.

synchronous state. In this context, the stabilizing oscillations around the unstable equilibrium in Kapitza's pendulum may be viewed as switchings in our network. While both periodic and stochastic vibrations of the suspension can stabilize Kapitza's pendulum, the periodic forcing clearly has an advantage over the stochastic perturbation as the latter can generate a sequence of "unfortunate" pushes in one direction, letting the pendulum pass the point of no return and fall down. The same is likely to happen in our network of Rössler systems where periodic switching provides better dynamic stabilization of the unstable synchronous state and has wider ranges of favorable switching frequencies.

B. *Duffing Network*

To demonstrate that the emergence of windows of opportunity is a general phenomenon in Class II networks, we consider a network of driven Duffing oscillators:

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -x_i^3 - hy_i + q\sin(\eta t) + \varepsilon(x_j - x_i) \end{aligned} \tag{21}$$

for $i, j = 1, 2$, $i \neq j$. This network has a multiple window diagram for stability of synchronization [see Fig. 2 (bottom)] and belongs to Class II networks. The intrinsic parameters are chosen and fixed as follows: $\eta = 1$, $h = 0.1$, and $q = 5.6$ [58]. Fig. 9 demonstrates the emergence of windows of opportunity for stable synchronization in an intermediate range of switching periods.

V. Conclusions

We have studied the stability of synchronization in dynamical networks with both stochastically switching connections and intrinsic parameters. In particular, we have proved that approximate synchronization in the fast-switching network of Lorenz oscillators can be achieved globally and almost surely, provided that the corresponding averaged network synchronizes globally. For the first time, we have given the *explicit* bounds on the switching frequency sufficient for global synchronization and the probability that synchronization is achieved within some given finite time. By this work, we also promote the general rigorous theory of stochastically blinking networks [42]–[44] which allows one to derive explicit bounds for probabilistic convergence to an attractor (or even a ghost-attractor) of any blinking system in question, provided that a Lyapunov function for the convergence in the averaged system can be constructed.

An important research problem is to develop a rigorous theory for understanding dynamical networks *beyond fast switching*. In our paper, we have used examples to show that connections, that are only present with some probability p in a complex network, can stabilize synchronization even in a normally unstable regime. We have explored the possibilities when the time scale for the stochastic process does not approach zero, and showed that not-fast switching can be favorable, compared to fast switching, when one does not want to follow the dynamics of the averaged system. We have also shown that switching cannot be too slow, as this can make the system even more unpredictable. This gives the impression that there is some window for each system for which we have a sense of "controlled unpredictability." Moreover, this phenomenon also seemed to pop up in [44] when we were analyzing an entirely different system whose parameters blink stochastically (i.e., not a network with stochastic connections). We named this controlled unpredictability, "windows of opportunity," to further emphasize that there seem to consistently be favorable conditions in which the stochastic and deterministic parameters match up appropriately, and allow the system to behave favorably, against all odds.

While the numerical results for the non-fast switching networks reveal unexpected windows of opportunity and give us plenty of insight as to the effects of stochastic coupling in dynamical networks (and how stochastic connections can actually be favorable to static ones), we are currently working on developing an analytical approach to this problem. The effect of non-fast switching between stable two-dimensional nonlinear [59] and linear [60] systems has previously been analytically explored. An interesting observation is that switching at *exponential times* between two stable linear degenerate nodes can cause the trajectory to escape to infinity when the switching is not fast [60]. The highly nonlinear dynamical effects in multidimensional stochastic networks with non-fast switching reported in our paper call for analytical approaches and techniques and represent a subject of future study.

Acknowledgment

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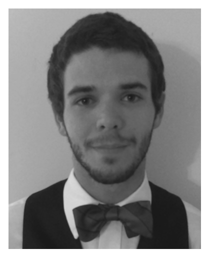

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